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Fourth moment structure of the BL-GARCH(p,q,d) models

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Abstract In this article, we explore a broad class of Bilinear-GARCH processes, abbreviated as BL-GARCH. Our exclusive focus lies on examining the moment structure of the BL-GARCH model. Specifically, we investigate both sufficient and necessary conditions for the existence of the unconditional fourth moment of the BL-GARCH(p, q, d) process. Additionally, we derive the autocorrelation function for the squared process.

1 Introduction

The bilinear-GARCH model, introduced by Storti and Vitale [[20], [21]] and further developed by Ghezal et al. [[8], [9], [12]], is designed to capture asymmetry, allowing it to account for the leverage effect. This effect, a negative correlation between return shocks and subsequent shocks in volatility patterns for financial time series, has garnered attention across various models such as the generalized quadratic ARCH (GQARCH, [19]), threshold GARCH (TGARCH, [[18], [22], [14]]), GJR-GARCH [15], logGARCH [[10], [11]], exponential GARCH (EGARCH, [17]), and bilinear (BL, [[4], [5], [6], [7], [13]]) models. A $BL-GARCH$ model $(X_t)_{t\in\mathbb{Z}}$, $\mathbb{Z} := \{0, \pm1, \pm2, ...\}$ is defined on a probability space $(\Omega, \mathcal{A}, P)$ and satisfies

$$X_t = \sqrt{\sigma_t} e_t. \tag{1.1}$$

Here, the innovation process $(e_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence with zero mean and unit variance defined on the same probability space $(\Omega, \mathcal{A}, P)$ and σ_t (volatility) is a general function depending on the past observations and their realizations, given by:

$$\sigma_t = a_0 + \sum_{i=1}^{q} a_i X_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j} + \sum_{k=1}^{d} c_k X_{t-k} \sqrt{\sigma_{t-k}}. \tag{1.2}$$

Here, $d = \min(p, q)$, so $(a_i)_{0\le i\le q}$, $(b_j)_{1\le j\le p}$ are non-negative coefficients with $a_0 > 0$, $(c_k)_{1\le k\le d}$ have values in $(-\infty, +\infty)$. If $c_k = 0$, $k = 1, ..., d$, it reduces to the standard $GARCH(p, q)$ model. The positivity of the coefficients $(a_i)_{0\le i\le q}$, $(b_j)_{1\le j\le p}$ ensures the positivity of σ_t in $GARCH(p, q)$. A sufficient condition for $\sigma_t > 0$ is given by the following assumption

[A.0] $c_k^2 \le 4a_k b_k$ for $k = 1, ..., d$ (see Storti and Vitale [20]).

Despite the growing literature on bilinear$-GARCH$ models, the fourth-moment structures have not, to our knowledge, been provided thus far. Cavicchioli derived these moments for a range of Markov switching multivariate and univariate $GARCH$ models (see., [1], [2]). In this article, we present the sufficient and necessary condition for the existence of the unconditional fourth moment of the $BL-GARCH$ process and provide an expression for the moment itself. Section 3 covers the derivation of the autocorrelation function for the squared process. Section 4 serves as the conclusion for the articles.
Some notations are used throughout the article:

- $I_{(n)}$ is the $n \times n$ identity matrix.
- $O_{(n,m)}$ denotes an $n \times m$ matrix with zero entries. For simplicity, we set $O_{(n)} := O_{(n,n)}$ and $\underline{O}_{(n)} := O_{(n,1)}$.
- The spectral radius of a square matrix M is denoted by $\rho(M)$.
- $\underline{e}_p := (1, ..., 1, 0, ..., 0)'$ is an $r \times 1$ vector with the first p components equal to 1 and the last $r - p$ components equal to 0. $\widetilde{\underline{e}}_p$ is a $(r+1) \times 1$ vector with the $p - th$ component equal to 1 and otherwise equal to 0, $\underline{1}_r$ denotes the vector of order $r \times 1$ whose entries are all ones.
- $\underline{\gamma}_R(t) := (\gamma_1(t), ..., \gamma_r(t))'$ and $\underline{\delta}_R$ is its expectation. $\underline{\sigma}_R := (\sigma_{t-1}, ..., \sigma_{t-r})'$ is an $r \times 1$ vector with the index set $R = \{1, ..., r\}$. Meanwhile, $\underline{\gamma}_{R\backslash\{h\}}(t)$ is a $(r-1) \times 1$ subvector of $\underline{\gamma}_R(t)$ obtained by excluding the element $\gamma_h(t)$ from $\underline{\gamma}_R$ and defining the other index set $R_h := \{h+1, ..., h+r\}$.
- $\odot$ denotes the usual Hadamard product of matrices. If $(M_i, i \in I \subset \mathbb{Z})$ is a sequence of $n \times n$ matrices, for any integer l and j, $\prod_{i=l}^{j} M_i = M_l M_{l+1} ... M_j$ if $l \leq j$, and $I_{(n)}$ otherwise.
- Finally, we define some matrices used here, for $h \leq r$,

$$\Gamma_j := \begin{pmatrix} \underline{\gamma}'_{(R-1)\backslash\{h\}}(t-j+1) & \gamma_r(t-j+1) \\ I_{(r-2)} & \underline{O}_{(r-2)} \end{pmatrix}_{(r-1)\times(r-1)} \quad \text{for } j = 2, ..., h;$$

for $j = 1, ..., r-2$,

$$S_{r-j}(\gamma_{j+1}(t-i+2)) := \begin{pmatrix} O_{(j-1,r-j-1)} & \underline{O}_{(j-1)} \\ \underline{\gamma}'_{(R-1)\backslash\{1,...,j\}}(t-i+2) & \gamma_r(t-i+2) \\ \gamma_j(t-i+2) I_{(r-j-1)} & \underline{O}_{(r-j-1)} \end{pmatrix}_{(r-1)\times(r-j)};$$

if $j = r-1$,

$$S_1(\gamma_r(t-i+2)) := \begin{pmatrix} \underline{O}'_{(r-2)} & \gamma_r(t-i+2) \end{pmatrix}', \widetilde{r} = \sum_{j=1}^{r-1}(r-j);$$

$$\Gamma_{h+1} := \begin{pmatrix} S_{r-1}(\gamma_2(t-h)) & \vdots & S_{r-2}(\gamma_3(t-h)) & \vdots & \ldots & \vdots & S_1(\gamma_r(t-h)) \end{pmatrix}_{(r-1)\times\widetilde{r}};$$

where, for $i > h+1$

$$\Gamma_i := \begin{pmatrix} S_{r-1}(\gamma_2(t-i+1)) & \vdots & S_{r-2}(\gamma_3(t-i+1)) & \ldots & S_1(\gamma_r(t-i+1)) \\ \ldots & & \ldots & \ldots & \ldots \\ I_{(r-2)} \vdots \underline{O}_{(r-2)} & \vdots & O_{(r-2)} & \ldots & 0 \\ O_{(r-3,r-1)} & \vdots & I_{(r-3)} \vdots \underline{O}_{(r-3)} & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & 0 \\ \underline{O}'_{(r-1)} & \vdots & \underline{O}'_{(r-2)} & \ldots & \end{pmatrix}_{\widetilde{r}\times\widetilde{r}}.$$

For $h > r$,

$$\widetilde{\Gamma}_j := \begin{pmatrix} 1 & \underline{O}'_{(r-1)} & 0 \\ a_0 & \underline{\gamma}'_{(R-1)}(t-j+1) & \gamma_r(t-j+1) \\ \underline{O}_{(r-1)} & I_{(r-1)} & \underline{O}_{(r-1)} \end{pmatrix}_{(r+1)\times(r+1)} \quad \text{for } j = 2, ..., h,$$

and

$$\widetilde{\Gamma}_j = \Gamma_j \text{ for } j > h+1,$$

$$\widetilde{\Gamma}_{h+1} := \begin{pmatrix} O_{(2,r-1)} & \vdots & O_{(2,r-2)} & \vdots & \ldots & \vdots & \underline{O}_{(2)} \\ S_{r-1}(\gamma_2(t-h)) & \vdots & S_{r-2}(\gamma_3(t-h)) & \vdots & \ldots & \vdots & S_1(\gamma_r(t-h)) \end{pmatrix}_{(r+1)\times\widetilde{r}}.$$

2 Condition for the existence of the fourth moment

In this section, we derive the sufficient and necessary condition for the existence of the unconditional fourth moment of the $BL-GARCH\,(r,r,r)$ process and provide an expression for the moment itself. In what follows, we shall assume, without loss of generality, that $a_q>0$, $b_p>0$, $c_d\neq 0$ and $r=\max\,(p,q)$, Equation (1.2) can be expressed in the following representation

$$\sigma_t=a_0+\sum_{i=1}^{r}\gamma_i\,(t)\,\sigma_{t-i}, \tag{2.1}$$

where $\gamma_i\,(t)=b_i+c_ie_{t-i}+a_ie_{t-i}^2$ for $i=1,...,r$, with $(\gamma_i\,(t))$ being a sequence of i.i.d. random variables such that $\gamma_i\,(t)$ is independent of σ_{t-i}. Now, we consider the following assumption:

[A.1] The $BL-GARCH$ model has a finite fourth moment and let $\kappa=E\left\{e_t^4\right\}$.

From the assumption $[A.1]$, implying the existence of the second moment of (X_t), it follows that $\sum\limits_{i=1}^{r}\delta_i^{(1)}<1$, where $\delta_i^{(1)}=E\left\{\gamma_i\,(t)\right\}=a_i+b_i$, $i=1,...,r$. Now, let's look at the unconditional fourth moment of (X_t), in the form of the square Equation (2.1), and then take the unconditional expectations. This gives:

$$E\left\{\sigma_t^2\right\}=a_0^2+2a_0\delta_1E\left\{\sigma_t\right\}+\delta_2E\left\{\sigma_t^2\right\}+2\sum_{i<j}E\left\{\gamma_i\,(t)\,\gamma_j\,(t)\,\sigma_{t-i}\sigma_{t-j}\right\}, \tag{2.2}$$

where $\delta_k=\sum\limits_{i=1}^{r}\delta_i^{(k)}$, $k=1,2$ and $\delta_i^{(2)}=E\left\{\gamma_i^2\,(t)\right\}=a_i^2\kappa+b_i^2+c_i^2+2a_ib_i$, $i=1,...,r$, and $E\left\{\sigma_t\right\}=\frac{a_0}{1-\delta_1}$. In the next step, $E\left\{\sigma_t^2\right\}$ can be determined by Equation (2.2) if we find a comfortable expression for $E\left\{\gamma_i\,(t)\,\gamma_j\,(t)\,\sigma_{t-i}\sigma_{t-j}\right\}$ as functions of $E\left\{\sigma_t\right\}$ and $E\left\{\sigma_t^2\right\}$. In the following lemma, we shall confront this problem by searching for a suitable expression for $\sigma_t\sigma_{t-h}$, $h\geq 1$

Lemma 2.1. *For any $l\geq h+1$, $\sigma_t\sigma_{t-h}$ can be expressed as combinations of the terms of σ_{t-i}, σ_{t-j}^2 and $\sigma_{t-i}\sigma_{t-j}$ such that*

$$\sigma_t\sigma_{t-h}=\left(\Sigma_0^{(1)}+\Sigma_0^{(2)}\right)+\sum_{i=h+1}^{l-1}\left(\Sigma_i^{(1)}+\Sigma_i^{(2)}\right)+\Sigma_l, \tag{2.3}$$

where, if $h\leq r$,

$$\Sigma_0^{(1)}\ :\ =a_0\left\{\left(1+\underline{\gamma}'_{R\backslash\{h\}}\,(t)\sum_{i=1}^{h-1}\left\{\prod_{j=2}^{i}\Gamma_j\right\}\underline{e}_1\right)\sigma_{t-h}+\underline{\gamma}'_{R\backslash\{h\}}\,(t)\left\{\prod_{j=2}^{h}\Gamma_j\right\}\underline{\sigma}_{R_h\backslash\{h+r\}}\right\},$$

$$\Sigma_0^{(2)}\ :\ =\left(\gamma_h\,(t)+\underline{\gamma}'_{R\backslash\{h\}}\,(t)\sum_{i=1}^{h-1}\left\{\prod_{j=2}^{i}\Gamma_j\right\}\underline{e}_1\gamma_{h-i}\,(t-i)\right)\sigma_{t-h}^2$$
$$+\underline{\gamma}'_{R\backslash\{h\}}\,(t)\left\{\prod_{j=2}^{h}\Gamma_j\right\}\underline{\sigma}_{R_h\backslash\{h+r\}}^{(1)},$$

$$\Sigma_i^{(1)}\ :\ =a_0\underline{\gamma}'_{R\backslash\{h\}}\,(t)\left\{\prod_{j=2}^{i}\Gamma_j\right\}\underline{\sigma}_i^{(1)}\,(t),$$

$$\Sigma_i^{(2)}\ :\ =\underline{\gamma}'_{R\backslash\{h\}}\,(t)\left\{\prod_{j=2}^{i}\Gamma_j\right\}\underline{\sigma}_i^{(2)}\,(t),$$

$$\Sigma_l\ :\ =\underline{\gamma}'_{R\backslash\{h\}}\,(t)\left\{\prod_{j=2}^{l}\Gamma_j\right\}\underline{\sigma}_l\,(t),$$

and if $h > r$,

$$\begin{aligned}
\Sigma_0^{(1)} &: = \underline{\widetilde{\gamma}}_R' (t) \left\{ \prod_{j=2}^{h} \widetilde{\Gamma}_j \right\} \underline{\widetilde{\sigma}}_t^{(1)},\ \Sigma_0^{(2)} := \underline{\widetilde{\gamma}}_R' (t) \left\{ \prod_{j=2}^{h} \widetilde{\Gamma}_j \right\} \underline{\widetilde{\sigma}}_t^{(2)}, \\
\Sigma_i^{(1)} &: = a_0 \underline{\widetilde{\gamma}}_R' (t) \left\{ \prod_{j=2}^{i} \widetilde{\Gamma}_j \right\} \underline{\sigma}_i^{(1)} (t),\ \Sigma_i^{(2)} := \underline{\widetilde{\gamma}}_R' (t) \left\{ \prod_{j=2}^{i} \widetilde{\Gamma}_j \right\} \underline{\sigma}_i^{(2)} (t), \\
\Sigma_l &: = \underline{\widetilde{\gamma}}_R' (t) \left\{ \prod_{j=2}^{l} \widetilde{\Gamma}_j \right\} \underline{\sigma}_l (t).
\end{aligned}$$

Proof. First step: if $h \leq r$, we can use Equation (2.1) to find the following

$$\sigma_t \sigma_{t-h} = a_0 \sigma_{t-h} + \gamma_h (t) \sigma_{t-h}^2 + a_0 \underline{\gamma}_{R\setminus\{h\}}' (t) \underline{\sigma}_{R\setminus\{h\}} \sigma_{t-h}. \tag{2.4}$$

Again, applying the Equation (2.1) to σ_{t-1} and continuing the iteration, we find

$$\begin{aligned}
\sigma_t \sigma_{t-h} \;=\;& a_0 \sigma_{t-h} + \gamma_h (t) \sigma_{t-h}^2 + a_0 \underline{\gamma}_{R\setminus\{h\}}' (t) \sum_{i=1}^{n-1} \left\{ \prod_{j=2}^{i} \Gamma_j \right\} \underline{e}_1 \sigma_{t-h} \\
&+ \underline{\gamma}_{R\setminus\{h\}}' (t) \sum_{i=1}^{n-1} \left\{ \prod_{j=2}^{i} \Gamma_j \right\} \underline{e}_1 \gamma_{h-i} (t-i) \sigma_{t-h}^2 \\
&+ a_0 \underline{\gamma}_{R\setminus\{h\}}' (t) \left\{ \prod_{j=2}^{h} \Gamma_j \right\} \underline{\sigma}_{R_h\setminus\{h+r\}} + \underline{\gamma}_{R\setminus\{h\}}' (t) \left\{ \prod_{j=2}^{h} \Gamma_j \right\} \underline{\sigma}_{R_h\setminus\{h+r\}}^{(1)} \\
&+ \underline{\gamma}_{R\setminus\{h\}}' (t) \left\{ \prod_{j=2}^{h+1} \Gamma_j \right\} \underline{\sigma}_{h+1} (t),
\end{aligned}$$

where:

$$\begin{aligned}
\underline{\sigma}_{R_h\setminus\{h+r\}}^{(1)} &: = \underline{\gamma}_{(R-1)} (t-h) \odot \underline{\sigma}_{R_h\setminus\{h+r\}}^{\odot 2},\ \underline{\sigma}_i^{(1)} (t) := \left(\underline{\sigma}_{R_i\setminus\{i+r\}}', \underline{O}_{(\widetilde{r}-r+1)}' \right)', \\
\underline{\sigma}_i^{(2)} (t) &: = \left(\left(\underline{\sigma}_{R_i\setminus\{i+r\}}^{(1)} \right)', \underline{O}_{(\widetilde{r}-r+1)}' \right)', \\
\underline{\sigma}_l (t) &: = \left(\sigma_{t-l} \underline{\sigma}_{R_l\setminus\{l+r\}}', \sigma_{t-l-1} \underline{\sigma}_{R_l\setminus\{l+1,l+r\}}', \ldots, \sigma_{t-l-r+2} \sigma_{t-l-r+1} \right)_{\widetilde{r}\times 1}'.
\end{aligned}$$

As $i > h+1$, using Equation (2.1) again for σ_{t-i} in $\underline{\sigma}_i (t)$, we have

$$\underline{\sigma}_i (t) = a_0 \underline{\sigma}_i^{(1)} (t) + \underline{\sigma}_i^{(2)} (t) + \Gamma_{i+1} \underline{\sigma}_{i+1} (t). \tag{2.5}$$

Finally, by the recursion of Equation (2.4) and using Equation (2.5), we arrive at Equation (2.3).
Second step: if $h > r$, here we use the same method above to get the Equation (2.3) with

$$\begin{aligned}
\underline{\widetilde{\sigma}}_t^{(1)} &: = \left(\sigma_{t-h}, 0, a_0 \underline{\sigma}_{R_h\setminus\{h+r\}}' \right)_{(r+1)\times 1}',\ \underline{\widetilde{\sigma}}_t^{(2)} = \left(0, \sigma_{t-h}^2, \underline{\sigma}_{R_h\setminus\{h+r\}}^{(1)} \right)_{(r+1)\times 1}', \\
\underline{\widetilde{\gamma}}_R (t) &: = \left(a_0, \underline{\gamma}_R' (t) \right)_{(r+1)\times 1}'.
\end{aligned}$$

This completes the proof. □

Currently, we insist on the necessity of $E\{\gamma_i (t) \gamma_j (t) \sigma_{t-i} \sigma_{t-j}\}$ with $1 \leq i < j \leq r$. For this, and the application of Lemma 2.1 by exchanging $t-i$ for t and $j-i$ for h, we have

$$E\left\{\gamma_i(t)\gamma_j(t)\sigma_{t-i}\sigma_{t-j}\right\} = E\left\{\gamma_i(t)\gamma_j(t)\Sigma_0^{(1)}\right\} + E\left\{\gamma_i(t)\gamma_j(t)\Sigma_0^{(2)}\right\} + E\left\{\gamma_i(t)\gamma_j(t)\Sigma_l\right\} \tag{2.6}$$

$$+ \sum_{u=j-i+1}^{l-1}\left(E\left\{\gamma_i(t)\gamma_j(t)\Sigma_u^{(1)}\right\} + E\left\{\gamma_i(t)\gamma_j(t)\Sigma_u^{(2)}\right\}\right).$$

Suppose that the process began at some finite value infinitely many periods ago. It has been shown that the limit of Equation (2.6) exists and is independent of t as $l \to +\infty$ iff

$$\lambda_{(1)} := \rho(\Lambda) < 1, \tag{2.7}$$

where $\Lambda := E\{\Gamma_l\}$ for $l > j-i+1$. For this purpose, we have the following results

Lemma 2.2. *For $1 \le i < j \le r$, $E\{\gamma_i(t)\gamma_j(t)\Sigma_l\} \underset{l\uparrow\infty}{\longrightarrow} 0$ iff the Condition (2.7) is satisfied.*

Proof. Using Lemma 2.1, we find

$$\begin{aligned} E\{\gamma_i(t)\gamma_j(t)\Sigma_l\} &= E\left\{\gamma_i(t)\gamma_j(t)\underline{\gamma}'_{R\backslash\{j-i\}}(t)\left\{\prod_{u=2}^{l}\Gamma_u\right\}\underline{\sigma}_l(t)\right\} \text{ for } l \ge r+j-i \\ &= \delta_i^{(1)}\delta_j^{(1)}E\left\{\underline{\gamma}'_{R\backslash\{j-i\}}(t)\left\{\prod_{u=2}^{j-i+1}\Gamma_u\right\}\right\}E\left\{\prod_{u=j-i+2}^{l-r+1}\Gamma_u\right\} \\ &\quad \times E\left\{\left\{\prod_{u=l-r+2}^{l}\Gamma_u\right\}\underline{\sigma}_l(t)\right\}. \end{aligned}$$

We note that in the recent equality, for any $l \ge r+j-i$, $E\left\{\prod_{u=j-i+2}^{l-r+1}\Gamma_u\right\} = \Lambda^{l-(r+j-i)}$ because $E\{\Gamma_u\Gamma_v\} = E\{\Gamma_u\}E\{\Gamma_v\}$ for $u,v > j-i+1$ and $u \ne v$, while the third expectation $E\left\{\left\{\prod_{u=l-r+2}^{l}\Gamma_u\right\}\underline{\sigma}_l(t)\right\}$ is not a function of l. Using these notes, the proof is complete. □

Lemma 2.3. *For $1 \le i < j \le r$,*

$$\lim_{l\to+\infty}\sum_{u=j-i+1}^{l-1}E\left\{\gamma_i(t)\gamma_j(t)\Sigma_u^{(1)}\right\} = a_0\delta_i^{(1)}\delta_j^{(1)}\underline{\delta}'_{R\backslash\{j-i\}}\left\{\prod_{u=2}^{j-i+1}\Lambda_u\right\}\left(I_{\widetilde{r}}-\Lambda\right)^{-1}\underline{e}_{(r-1)}E\{\sigma_t\}, \tag{2.8}$$

where $\Lambda_u := E\{\Gamma_u\}$ for $2 \le u \le j-i+1$. You can find the Equation (2.8) iff the Condition (2.7) is realized.

Proof. We know that

$$\begin{aligned} \sum_{u=j-i+1}^{l-1}E\left\{\gamma_i(t)\gamma_j(t)\Sigma_u^{(1)}\right\} &= a_0\delta_i^{(1)}\delta_j^{(1)}E\left\{\underline{\gamma}'_{R\backslash\{j-i\}}(t)\right\}E\left\{\prod_{u=2}^{j-i+1}\Gamma_u\right\} \\ &\quad \times\left(\sum_{u=j-i+1}^{l-1}\Lambda^{u-(j-i+1)}\right)E\left\{\underline{\sigma}_i^{(1)}(t)\right\}. \end{aligned}$$

Since the $\sum_{u=j-i+1}^{l-1}\Lambda^{u-(j-i+1)} \underset{l\uparrow+\infty}{\longrightarrow} \left(I_{\widetilde{r}}-\Lambda\right)^{-1}$ iff the Condition (2.7) is realized. □

Lemma 2.4. *For* $1 \leq i < j \leq r$,

$$\lim_{l \to +\infty} \sum_{u=j-i+1}^{l-1} E\left\{\gamma_i(t)\gamma_j(t)\Sigma_u^{(2)}\right\} = \delta_i^{(1)}\delta_j^{(1)}\left\{\sum_{u=1}^{r-1} \underline{\delta}_{R\setminus\{j-i\}}^{(2;j-i+u)} + \underline{\delta}'_{R\setminus\{j-i\}}\left\{\prod_{u=2}^{j-i+1}\Lambda_u\right\}\right. \quad (2.9)$$
$$\left.\times\left(I_{\widetilde{r}}-\Lambda\right)^{-1}\underline{\delta}_{(j-i+r+1)}^{(j-i+2r-1)}\right\}E\left\{\sigma_t^2\right\},$$

where $\underline{\delta}_{R\setminus\{j-i\}}^{(u-i-1;u)} := E\left\{\underline{\gamma}'_{R\setminus\{j-i\}}(t)\left\{\prod_{v=0}^{i+1}\Gamma_{u-(i+1)+v}\right\}\left(\underline{\gamma}'_{(R-1)}(t-u-i+1), \underline{O}'\right)'\right\}$ *for* $i < u \leq r$,

$\underline{\delta}_{(u-v)}^{(u)} := E\left\{\left\{\prod_{l=0}^{v}\Gamma_{u-v+l}\right\}\left(\underline{\gamma}'_{(R-1)}(t-u-i+1), \underline{O}'\right)'\right\}$ *for* $v \leq r-2$, $u > j-i+2r-1$.

You can find the Equation (2.9) iff the Condition (2.7) is valid.

Proof. In the proof of Lemma 2.4, we must evaluate $E\left\{\gamma_i(t)\gamma_j(t)\sum_{u=j-i+1}^{l-1}\Sigma_u^{(2)}\right\}$, but the task is not an easy due to $\underline{\sigma}_i^{(2)}(t)$ not being stochastically independent of the matrix product $\prod_{j=i-r+2}^{i}\Gamma_j$. Now, assume that $k \geq j-i+r+1$, we have:

$$\begin{aligned}
E\left\{\gamma_i(t)\gamma_j(t)\sum_{u=j-i+1}^{l-1}\Sigma_u^{(2)}\right\} &= \delta_i^{(1)}\delta_j^{(1)}E\left\{\sum_{u=j-i+1}^{j-i+r-1}\Sigma_u^{(2)} + \sum_{u=j-i+r}^{l-1}\Sigma_u^{(2)}\right\} \\
&= \delta_i^{(1)}\delta_j^{(1)}E\left\{\sum_{u=j-i+1}^{j-i+r-1}\Sigma_u^{(2)}\right\} + \delta_i^{(1)}\delta_j^{(1)}\underline{\delta}'_{R\setminus\{j-i\}}\left\{\prod_{v=2}^{j-i+1}\Gamma_v\right\} \\
&\quad \times \sum_{u=j-i+r}^{l-1} E\left\{\prod_{v=j-i+2}^{u-r+1}\Gamma_v\right\}E\left\{\left\{\prod_{v=u-r+2}^{u}\Gamma_v\right\}\underline{\sigma}_u^{(2)}(t)\right\}.
\end{aligned}$$

Moreover, the expectation of $\left\{\prod_{v=u-r+2}^{u}\Gamma_v\right\}\underline{\sigma}_u^{(2)}(t)$ is not a function of u for $u \geq j-i+2p-1$. In the next stage, we introduce some symbols to formulate a concise expression. For any $u \geq j-i+2p-1$, $E\left\{\left\{\prod_{v=u-r+2}^{u}\Gamma_v\right\}\underline{\sigma}_u^{(2)}(t)\right\} = \underline{\delta}_{(j-i+r+1)}^{(j-i+2r-1)}E\left\{\sigma_t^2\right\}$ and

$$E\left\{\underline{\gamma}'_{R\setminus\{j-i\}}(t)\left\{\prod_{v=2}^{u}\Gamma_v\right\}\underline{\sigma}_u^{(2)}(t)\right\} = \underline{\delta}_{R\setminus\{j-i\}}^{(2;u)}E\left\{\sigma_t^2\right\}.$$

Furthermore, Condition (2.7) holds iff $\sum_{u=j-i+r}^{l-1} E\left\{\prod_{v=j-i+2}^{u-r+1}\Gamma_v\right\} \underset{l\uparrow\infty}{\longrightarrow} \left(I_{\widetilde{r}}-\Lambda\right)^{-1}$. The rest follows immediately. □

Now, utilizing the previous Lemmas 2.1 – 2.4 and Condition (2.7), we conclude that the mixed moment $E\left\{\gamma_i(t)\gamma_j(t)\sigma_{t-i}\sigma_{t-j}\right\}$ converges to a finite value. This value is a linear function of $E\left\{\sigma_t\right\}$ and $E\left\{\sigma_t^2\right\}$ as $l \uparrow \infty$. We express this result formally in the following proposition

Proposition 2.5. *Under Condition (2.7), we have*

$$E\left\{\gamma_i(t)\gamma_j(t)\sigma_{t-i}\sigma_{t-j}\right\} = \delta_i^{(1)}\left(a_0\delta_j^{(1)}\Xi_1(i,j)E\left\{\sigma_t\right\} + \Xi_2(i,j)E\left\{\sigma_t^2\right\}\right),$$

where, for $j > i+1$,

$$\Xi_1(i,j) = 1 + \underline{\delta}'_{R\setminus\{j-i\}}\left(\sum_{u=1}^{j-i-1}\left\{\prod_{v=2}^{u}\Lambda_v\right\}\underline{e}_1 + \left\{\prod_{u=2}^{j-i}\Lambda_u\right\}\left(\Lambda_{j-i+1}\left(I_{\widetilde{r}}-\Lambda\right)^{-1}\underline{e}_{r-1} + \underline{1}_{r-1}\right)\right),$$

and $\Xi_2(i,j) = \Xi_2^{(1)}(i,j) + \delta_j^{(1)} \sum_{u=2}^{4} \Xi_2^{(u)}(i,j)$ *such that for* $j > i+1$,

$$\begin{aligned}
\Xi_2^{(1)}(i,j) &= \widetilde{\delta}_{(j-i)}^{(j)} + \underline{\delta}'_{R\setminus\{j-i\}} \sum_{u=1}^{j-i-1} \left\{ \prod_{v=2}^{u} \Lambda_v \right\} \underline{e}_1 \widetilde{\delta}_{(j-i-u)}^{(j)}, \\
\Xi_2^{(2)}(i,j) &= \sum_{u=1}^{j-i-1} \delta_u^{(1)} \Xi_2^{(2)}(i+u,j) + \sum_{u=j-i+1}^{r} \widetilde{\delta}_{(u-j+i)}^{(u)}, \\
\Xi_2^{(3)}(i,j) &= \sum_{u=1}^{r-1} \underline{\delta}_{R\setminus\{j-i\}}^{(2;u+j-i-1)}, \\
\Xi_2^{(4)}(i,j) &= \underline{\delta}'_{R\setminus\{j-i\}} \left\{ \prod_{u=2}^{j-i+1} \Lambda_u \right\} \left(I_{\widetilde{r}} - \Lambda\right)^{-1} \underline{\delta}_{(j-i+r+1)}^{(j-i+2r-1)},
\end{aligned}$$

where $\widetilde{\delta}_{(i)}^{(j)} := E\{\gamma_i(t+i)\gamma_j(t+j)\}$ *for* $i < j$.

Proof. To evaluate $E\left\{\gamma_i(t)\gamma_j(t)\Sigma_0^{(1)}\right\}$ and $E\left\{\gamma_i(t)\gamma_j(t)\Sigma_0^{(2)}\right\}$, we employ similar methods as before, we get

$$\begin{aligned}
E\left\{\gamma_i(t)\gamma_j(t)\Sigma_0^{(1)}\right\} &= a_0\delta_i^{(1)}\delta_j^{(1)} \left(1 + \underline{\delta}'_{R\setminus\{j-i\}} \left(\sum_{u=1}^{j-i-1} \left\{ \prod_{v=2}^{u} \Lambda_v \right\} \underline{e}_1 + \left\{ \prod_{u=2}^{j-i} \Lambda_u \right\} \underline{1}_{r-1} \right)\right) \\
&\quad \times E\{\sigma_t\}.
\end{aligned}$$

For $j > l+1$,

$$\begin{aligned}
& E\left\{\gamma_i(t)\gamma_j(t)\left(\Sigma_0^{(2)} - \underline{\gamma}'_{R\setminus\{j-i\}}(t)\left\{\prod_{j=2}^{j-i}\Gamma_j\right\}\underline{\sigma}_{R_{j-i}\setminus\{(j-i)+r\}}^{(1)}\right)\right\} \\
&= \delta_i^{(1)}\left(\widetilde{\delta}_{(j-i)}^{(j)} + \underline{\delta}'_{R\setminus\{j-i\}} \sum_{u=1}^{j-i-1}\left\{\prod_{v=2}^{u}\Lambda_v\right\}\underline{e}_1\widetilde{\delta}_{(j-i-u)}^{(j)}\right) E\left\{\sigma_t^2\right\}.
\end{aligned}$$

To calculate $E\left\{\gamma_i(t)\gamma_j(t)\underline{\gamma}'_{R\setminus\{j-i\}}(t)\left\{\prod_{j=2}^{j-i}\Gamma_j\right\}\underline{\sigma}_{R_{j-i}\setminus\{(j-i)+r\}}^{(1)}\right\}$, we use recursion for $j = i+1, ..., i+r-1$, we find the following

$$\begin{aligned}
\text{if } j &= i+1,\ E\left\{\gamma_i(t)\gamma_j(t)\underline{\gamma}'_{R\setminus\{1\}}(t)\underline{\sigma}_{R_1\setminus\{r+1\}}^{(1)}\right\} = \delta_i^{(1)}\delta_j^{(1)}\left(\sum_{j=2}^{r}\widetilde{\delta}_{(j-1)}^{(j)}\right)E\left\{\sigma_t^2\right\}, \\
\text{if } j &= i+2,\ E\left\{\gamma_i(t)\gamma_j(t)\underline{\gamma}'_{R\setminus\{2\}}(t)\Gamma_2\underline{\sigma}_{R_2\setminus\{r+2\}}^{(1)}\right\} = \delta_i^{(1)}\delta_j^{(1)} \\
&\times\left(\delta_1^{(1)}\sum_{j=2}^{r}\widetilde{\delta}_{(j-1)}^{(j)} + \sum_{j=3}^{r}\widetilde{\delta}_{(j-2)}^{(j)}\right)E\left\{\sigma_t^2\right\}.
\end{aligned}$$

Taking $\Xi_2^{(2)}(j-1,j) = \sum_{j=2}^{r}\widetilde{\delta}_{(j-1)}^{(j)}$, we then obtain $\Xi_2^{(2)}(j-2,j) = \delta_1^{(1)}\Xi_2^{(2)}(j-1,j) + \sum_{j=3}^{r}\widetilde{\delta}_{(j-2)}^{(j)}$. Continuing this process, we arrive at the desired result. In the latter part, utilizing the above results, the proof is completed. □

Below are given the most important result in this section

Theorem 2.6. *Under the Condition* (2.7)*, a necessary and sufficient condition for the existence of the fourth unconditional moment of* (X_t) *is given by*

$$\lambda_{(2)} := \delta_2 + 2\sum_{i<j}\delta_i^{(1)}\Xi_2(i,j) < 1 \tag{2.10}$$

while a detailed expression for the fourth moment

$$E\left\{X_t^4\right\} = a_0^2\kappa \frac{(1-\delta_1)^{-1}\Upsilon}{1-\lambda_{(2)}},$$

where $\Upsilon := 1+\delta_1+2\sum_{i<j}\delta_i^{(1)}\delta_j^{(1)}\Xi_1(i,j)$ *and the kurtosis coefficient of the process is calculated as*

$$\kappa_4 := \frac{E\left\{X_t^4\right\}}{\left(E\left\{X_t^2\right\}\right)^2} = \kappa\frac{(1-\delta_1)\Upsilon}{1-\lambda_{(2)}}.$$

Proof. The proof is straight forward. Therefore, we omit the details. □

Example 2.7. In the following table, we present the necessary and sufficient conditions for the existence of $E\left\{X_t^4\right\}$ and an expression for some specific cases

	$BL-GARCH(2,2,2)$	$BL-GARCH(1,1,1)$
$\lambda_{(1)} < 1$	$\delta_2^{(1)} < 1$	$\delta_1^{(1)} < 1$
$\lambda_{(2)} < 1$	$\delta_2 + 2\delta_1^{(1)}\widetilde{\delta}_1^{(2)}\left(1-\delta_2^{(1)}\right)^{-1} < 1$	$\delta_1^{(2)} < 1$
$E\left\{X_t^4\right\}$	$a_0^2\kappa\frac{1+\delta_1+2\delta_1^{(1)}\delta_2^{(1)}\left(1-\delta_2^{(1)}\right)^{-1}}{(1-\delta_1)\left(1-\lambda_{(2)}\right)}$	$a_0^2\kappa\frac{1+\delta_1}{(1-\delta_1)\left(1-\delta_1^{(2)}\right)}$
Kurtosis	$\kappa\frac{1-\delta_1^2+2\delta_1^{(1)}\delta_2^{(1)}\left(1-\delta_2^{(1)}\right)^{-1}(1-\delta_1)}{1-\lambda_{(2)}}$	$\kappa\frac{1-\delta_1^2}{1-\delta_1^{(2)}}$

Table 1 : Condition (2.10) pertains to the existence of $E\left\{X_t^4\right\}$ and provides an expression as well as the kurtosis coefficient.

Remark 2.8. From Table 1, it is noteworthy that the outcomes derived from the BL-GARCH(1,1,1) model align with those obtained by Storti and Vitale [20]. Additionally, the results for the Standard GARCH(2,2) model, which can be derived from the BL-GARCH(2,2,2) models by setting $c_1 = c_2 = 0$, coincide with findings from He and Teräsvirta [3].

3 The autocorrelation function for the square process

In this section, we derive the autocorrelation function for the squared observations, touching on some models included in the main model, such as $ARCH(p)$ by Milhϕj [16], $GARCH(p,q)$ by He and Teräsvirta [3] and the first-order $BL-GARCH$ model by Storti and Vitale [20]. Regardless, the autocorrelation function of $\left(X_t^2\right)$ at lag h is given by

$$\rho_{-h} = \rho_h = \rho\left(X_t^2, X_{t-h}^2\right) = \frac{Cov\left(X_t^2, X_{t-h}^2\right)}{Var\left(X_t^2\right)} \text{ for any } h > 0.$$

The next step is to find a detailed expression for the mixed moment $E\left\{X_t^2X_{t-h}^2\right\}$ as a function of $E\left\{\sigma_t\right\}$ and $E\left\{\sigma_t^2\right\}$. In the following lemma, we address this problem by searching for a suitable expression for the mixed moment $E\left\{X_t^2X_{t-h}^2\right\}$, $1 \leq h \leq r$

Lemma 3.1. *For* $1 \leq h \leq r$*, the previous result mentioned in Proposition 2.5 gives*

$$E\left\{X_t^2X_{t-h}^2\right\} = a_0^2\Xi_1(0,h)E\left\{\sigma_t\right\} + \Xi_2(0,h)E\left\{\sigma_t^2\right\}. \quad (3.1)$$

Here, $\Xi_2^{(1)}(0,h)$ *has changed to become* $\Xi_2^{(1)}(0,h) = \widehat{\delta}_h + \underline{\delta}'_{R\backslash\{h\}}\sum_{u=1}^{h-1}\left\{\prod_{v=2}^{u}\Lambda_v\right\}\underline{e}_1\widehat{\delta}_{h-u}$ *where* $\widehat{\delta}_u := E\left\{e_t^2\gamma_u(t+u)\right\}.$

Proof. The proof is directly verified with the use of Proposition 2.5. □

Now, we seek to obtain the same Equation (3.1) but with different coefficients for both $E\left\{\sigma_t\right\}$ and $E\left\{\sigma_t^2\right\}$, for $h>r$. For this purpose, we have the following results

Lemma 3.2. *For $h>r$, the expectation of the $\left\{\prod_{u=0}^{r-1}\widetilde{\Gamma}_{h+u-r+1}\right\}\underline{\widetilde{\sigma}}_t^{(2)}$ does not depend on h and we have*

$$E\left\{e_{t-h}^2\left\{\prod_{u=0}^{r-1}\widetilde{\Gamma}_{h+u-r+1}\right\}\left(0,1,\underline{\gamma}'_{(R-1)}\left(t-h\right)\right)'\right\}=\sum_{u=1}^{r}\left(\Xi_2^{(1)}\left(0,u\right)+\Xi_2^{(2)}\left(0,u\right)\right)\underline{\widetilde{e}}_{r-u+2}.$$

Proof. It is noted that the product $\prod_{u=0}^{r-1}\widetilde{\Gamma}_{h+u-r+1}$ is not stochastically independent of $\underline{\widetilde{\sigma}}_t^{(2)}$. After some calculations, we find that:

$$\prod_{u=0}^{r-1}\widetilde{\Gamma}_{h+u-r+1}=\Pi_h=\begin{pmatrix}1 & 0 & \cdots & 0\\ \psi_{21}\left(h\right) & \psi_{22}\left(h\right) & \cdots & \psi_{2,r+1}\left(h\right)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_{r+1,1}\left(h\right) & \psi_{r+1,2}\left(h\right) & \cdots & \psi_{r+1,r+1}\left(h\right)\end{pmatrix}\text{ and }\Pi_h=\Pi_{h-1}\widetilde{\Gamma}_h,$$

where $\psi_{k,m}\left(s\right)$ is determined recursively with respect to integers $s=h-r+1,...,h$ such that

$$\psi_{km}\left(s\right)=\begin{cases}\psi_{k2}\left(s-1\right)\gamma_{m-1}\left(t-s+2m-3\right)+\psi_{k,m+1}\left(s-1\right) & \text{for } m=2,...,r\\ \psi_{k2}\left(s-1\right)\gamma_r\left(t-s+1\right) & \text{for } m=r+1\end{cases},$$

with the initial values $\psi_{k,m}\left(h-r+1\right)=\gamma_{km}$ for any k,m, where γ_{km} is the $(k,m)-th$ element of matrix $\widetilde{\Gamma}_{h-r+1}$, we find

$$E\left\{e_{t-h}^2\left\{\prod_{u=0}^{r-1}\widetilde{\Gamma}_{h+u-r+1}\right\}\left(0,1,\underline{\gamma}'_{(R-1)}\left(t-h\right)\right)'\right\}=\sum_{k,m=2}^{r+1}E\left\{e_{t-h}^2\psi_{k,m}\left(h\right)\gamma_{m-2}\left(t-h\right)\right\}\underline{\widetilde{e}}_k,$$

by convention $\gamma_0\left(t-h\right)=1$. □

Lemma 3.3. *For $h>r$, we obtain $E\left\{X_t^2X_{t-h}^2\right\}$ as in Equation (3.1) with $\Xi_2\left(0,h\right)$ defined by*

$$\Xi_2\left(0,h\right)=\underline{\widetilde{\delta}}'_R\widetilde{\Lambda}^{h-r-1}\sum_{u=1}^{r}\Xi_2\left(0,u\right)\underline{\widetilde{e}}_{r-u+2},$$

where $\underline{\widetilde{\delta}}_R:=E\left\{\underline{\widetilde{\gamma}}_R\left(t\right)\right\}=\left(a_0,\underline{\delta}'_R\right)'$ and $E\left\{\widetilde{\Gamma}_j\right\}=\widetilde{\Lambda}$ for $j=2,...,h$.

Proof. Enough to note that $\Xi_2\left(0,h\right)$ is the coefficient of $E\left\{\sigma_t^2\right\}$ in Equation (3.1). For any $h>r$ and by Lemma 3.2 we have

$$\begin{aligned}E\left\{e_{t-h}^2\Sigma_0^{(2)}\right\} &= \underline{\widetilde{\delta}}_R\widetilde{\Lambda}^{h-r-1}\sum_{u=1}^{r}\left(\Xi_2^{(1)}\left(0,u\right)+\Xi_2^{(2)}\left(0,u\right)\right)\underline{\widetilde{e}}_{r-u+2}E\left\{\sigma_t^2\right\}\\ &= \left(\Xi_2^{(1)}\left(0,h\right)+\Xi_2^{(2)}\left(0,h\right)\right)E\left\{\sigma_t^2\right\}.\end{aligned}$$

Moreover, For any $h>r$,

$$\begin{aligned}\lim_{l\uparrow\infty}\sum_{i=h+1}^{l-1}E\left\{e_{t-h}^2\Sigma_i^{(2)}\right\} &= \underline{\widetilde{\delta}}_R\sum_{i=1}^{r-1}\underline{\delta}_{(2)}^{(h+i)}E\left\{\sigma_t^2\right\}+\underline{\widetilde{\delta}}_R\widetilde{\Lambda}^{h-1}\widetilde{\Lambda}_{h+1}\left(I_{\widetilde{r}}-\Lambda\right)^{-1}\underline{\delta}_{(h+r)}^{(h+2r-2)}E\left\{\sigma_t^2\right\}\\ &= \underline{\widetilde{\delta}}_R\sum_{i=1}^{r-1}\widetilde{\Lambda}^{h-p+i}\underline{\delta}_{(3+i)}^{(h+i+1)}E\left\{\sigma_t^2\right\}+\underline{\widetilde{\delta}}_R\widetilde{\Lambda}^{h-1}\widetilde{\Lambda}_{h+1}\left(I_{\widetilde{r}}-\Lambda\right)^{-1}\\ &\quad\times\underline{\delta}_{(2r+1)}^{(3r-1)}E\left\{\sigma_t^2\right\}\\ &= \left(\Xi_2^{(3)}\left(0,h\right)+\Xi_2^{(4)}\left(0,h\right)\right)E\left\{\sigma_t^2\right\}.\end{aligned}$$

□

Now, the next result provides the mixed moment for any $h \geq 1$

Proposition 3.4. *Under Condition (2.7), the validity of Equation (3.1) is established for any $h \geq 1$. Specifically, for $h \geq 1$, we have $\Xi_1(0,h) := \widetilde{\underline{\delta}}_R \widetilde{\Lambda}^{h-1}\left(\underline{a}_0 + \widetilde{\Lambda}_{h+1}(I_{\widetilde{r}} - \Lambda)^{-1}\underline{e}_{r-1}\right)$ with $a_0\underline{a}_0 := \left(1, 0, a_0\underline{1}_{r-1}\right)'$, and $\Xi_2(0,h)$ is defined by Lemma 3.1 and Lemma 3.3.*

Proof. Using Lemma 2.1 is sufficient to note that

$$\begin{aligned} E\left\{e^2_{t-n}\Sigma^{(1)}_0\right\} &= a_0\widetilde{\underline{\delta}}_R\widetilde{\Lambda}^{h-1}\underline{a}_0 E\{\sigma_t\}; \\ \lim_{l\uparrow\infty}\sum_{u=h+1}^{l-1} E\left\{e^2_{t-n}\Sigma^{(1)}_u\right\} &= a_0\widetilde{\underline{\delta}}_R\widetilde{\Lambda}^{h-1}\widetilde{\Lambda}_{h+1}(I_{\widetilde{r}} - \Lambda)^{-1}\underline{e}_{r-1}E\{\sigma_t\}. \end{aligned}$$

□

Below is the most important result in this section

Theorem 3.5. *Consider the $BL-GARCH$ models (1.1) and (1.2). Assume that $\kappa < \infty$ and that Condition (2.10) holds. Then the autocorrelation function of $\left(X_t^2\right)$ is, for any $h \geq 1$*

$$\rho_h = \frac{\Upsilon\Xi_2(0,h) + \left(1-\lambda_{(2)}\right)\left(\Xi_1(0,h) - (1-\delta_1)^{-1}\right)}{\kappa\Upsilon - \left(1-\lambda_{(2)}\right)(1-\delta_1)^{-1}}.$$

4 Conclusion

In conclusion, the analysis of the BL-GARCH model in this scientific inquiry has yielded valuable insights into the dynamic behavior and statistical properties of financial processes. The conditional stability condition (2.7) emerged as a critical determinant of the model's stability, emphasizing the role of joint moments and their persistence in shaping the long-term dynamics. The examination of mixed moments provided a detailed formulation for various random variables and their correlations, shedding light on the distributions of σ_t and σ_t^2. Furthermore, the autocorrelation function for X_t^2 was precisely formulated, unveiling the intricate relationship between values at different time points. The existence conditions for higher-order moments underscored specific requirements for the continued existence of high-order moments, delineating the necessary conditions for such occurrences.

In summary, the BL-GARCH model offers a robust framework for analyzing the dynamic behavior of financial processes. The conditions and results obtained contribute significantly to understanding the complex interactions among different variables, providing valuable insights for modeling and forecasting financial time series. The findings presented here enhance our understanding of the underlying mechanisms governing financial processes and offer avenues for further research in the field.

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