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Magnetic Curves in 3-dimensional C_{12}-Manifolds

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Abstract *Motivated by the recent studies of C_{12}-manifolds, which are non-normal almost contact metric manifolds, in this article we investigate the magnetic curves with respect to contact magnetic fields in C_{12}-manifolds. Moreover, some properties of twin magnetic curves are investigated. Finally, some examples are presented.*

1 Introduction

Magnetic curves have many applications in physics and differential geometry and play an important role in these fields. When a charged particle enters a magnetic field, the particle's Serret-Frenet vectors are influenced by this field, resulting in a force called the Lorentz force. With the effect of this force, the particle follows a new trajectory within this field. This trajectory is called a magnetic curve.

The study of magnetic curves in arbitrary Riemannian manifolds was further developed mostly in the early 1990s, even though related pioneer works were published much earlier. We can refer to Arnold's problems concerning charges in magnetic fields on Riemannian manifolds of arbitrary dimensions, commented on by Ginzburg in [13], and references therein.

Due to the importance of curves and surfaces in physical sciences. Recently, several works have appeared related on this topics, especially in three-dimensional case with different conditions. We mention, for example, [15, 18].

One can easily understand the relation between magnetic fields and almost-contact metric manifolds in the following manner: Denoting by (M, g) an oriented 3-dimensional Riemannian manifold endowed with the volume form dv_g, it is known that the space of all smooth 2-forms is identified with the space of all smooth vector fields via the Hodge star operator. Let now F denote a magnetic field on (M, g, dv_g), V its corresponding divergence free vector field, and $V^\flat$ the dual 1-form of V with respect to the metric g. If V is unitary, then one can show that $(\Phi, V, V^\flat)$ is an almost contact structure on M where Φ is a skew-symmetric tensor field of type $(1, 1)$ called the Lorentz force corresponding to F via g. In other words, a dynamical system (M, g, F), given by an oriented 3-dimensional Riemannian manifold (M, g) together with a magnetic field F having the corresponding divergence-free vector field unitary, may be thought of as an almost contact metric manifold with closed fundamental 2-form [2]. For this reason, recently, several works have focused on magnetic curves corresponding to magnetic fields in normal almost contact metric manifolds. For example, see [11, 12, 14, 16, 17].

The almost contact metric manifolds were completely classified in [9]. An important class of these manifolds are the C_{12}-manifolds which are not normal. Recently, several works have been published on this class of almost contact metric manifolds, for example, see [3, 4, 5, 6, 8].

The main objective of the present paper is the study of trajectories for particles moving under the influence of a contact magnetic field in C_{12}-manifolds.
The article is organized as follows: After collecting in Sections 2 and 3 the necessary general results needed in the sequel, in Section 4 we study contact magnetic trajectories in 3-

dimensional C_{12}-manifolds and give the necessary and sufficient conditions for a regular curve on 3-dimensional C_{12}-manifolds to be a normal magnetic curve. Moreover, we establish an interesting class of magnetic curves called "twin magnetic curves" and give a nice characterization for them. We also study a more special type of normal magnetic curve; we will call them "Bi-magnetic curves", which correspond to two contact magnetic fields at the same time. Finally, we present many examples that justify their study.

2 Three dimensional C_{12}-manifold

2.1 Definitions and properties

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$-tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2(X) = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \end{cases} \tag{2.1}$$

for any vector fields X, Y on M.

In particular, in an almost contact metric manifold we also have

$$\varphi\xi = 0 \qquad and \qquad \eta \circ \varphi = 0.$$

The fundamental 2-form ϕ is defined by

$$\phi(X,Y) = g(X, \varphi Y). \tag{2.2}$$

It is known that the almost contact structure (φ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X,Y) = N_\varphi(X,Y) + 2d\eta(X,Y)\xi = 0, \tag{2.3}$$

for any X, Y on M, where N_φ denotes the Nijenhuis torsion of φ, given by

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.4}$$

In the classification of D. Chinea and C. Gonzalez [9] of almost contact metric manifolds there is a class called C_{12}-manifolds which can be integrable but never normal. In this classification, C_{12}-manifolds are defined by

$$(\nabla_X \phi)(Y,Z) = \eta(X)\eta(Z)(\nabla_\xi \eta)\varphi Y - \eta(X)\eta(Y)(\nabla_\xi \eta)\varphi Z. \tag{2.5}$$

In [6] and [8], The $(2n+1)$-dimensional C_{12}-manifolds are characterized by:

$$(\nabla_X \varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big), \tag{2.6}$$

for any X and Y vector fields on M, where $\omega = -\big(\nabla_\xi \xi\big)^\flat = -\nabla_\xi \eta$ and ψ is the vector field given by

$$\omega(X) = g(X, \psi) = -g(X, \nabla_\xi \xi),$$

for all X vector field on M.

Moreover, in [6] the $(2n+1)$-dimensional C_{12}-manifolds is also characterized by

$$\mathrm{d}\eta = \omega \wedge \eta \qquad \mathrm{d}\phi = 0 \qquad and \qquad N_\varphi = 0. \tag{2.7}$$

Here, we emphasize that the almost C_{12}-manifolds is defined by the following:

Definition 2.1. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. M is called almost C_{12}-manifold if there exists a one-form ω which satisfies

$$\mathrm{d}\eta = \omega \wedge \eta \qquad and \qquad \mathrm{d}\phi = 0.$$

In addition, if $N_\varphi = 0$ we say that M is a C_{12}-manifold.

In dimension 3, one can get the following:

Theorem 2.2. *[6] Let $(M^3, \varphi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. M is C_{12}-manifold if and only if*

$$\nabla_X \xi = -\eta(X)\psi, \tag{2.8}$$

where $\psi = -\nabla_\xi \xi$.

All of the above definitions of the C_{12}-Manifolds (2.5),(2.6) and (2.7) have been shown to be equivalent (see [6, 8]). The following proposition provides another characterization of 3-dimensional C_{12}-Manifolds.

Proposition 2.3. *Let $(M^3, \varphi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. M is C_{12}-manifold if and only if*

$$\nabla_{\varphi X} \xi = 0. \tag{2.9}$$

Proof. It is sufficient to prove that $\nabla_{\varphi X}\xi = 0$ and $\nabla_X \xi = -\eta(X)\psi$ are equivalent with $\psi = -\nabla_\xi \xi$. Suppose that $\nabla_X \xi = -\eta(X)\psi$, so it is easy to see that $\nabla_{\varphi X}\xi = 0$.

Conversely, suppose that $\nabla_{\varphi X}\xi = 0$ and replacing X by φX using the formula $\varphi^2 X = -X + \eta(X)\xi$, we obtain $\nabla_X \xi = \eta(X)\nabla_\xi \xi$. This completes the proof. □

In [6], the authors studied the 3-dimensional unit C_{12}-manifold i.e. the case where ψ is a unit vector field and ω is a closed 1-form. That is

$$\eta(\xi) = \omega(\psi) = 1, \qquad \eta(\psi) = \omega(\xi) = 0 \qquad \textit{and} \qquad \mathrm{d}\omega = 0.$$

We get immediately that $\{\xi, \psi, \varphi\psi\}$ is an orthonormal frame, and as long as the three vector fields for this basis are global, the manifold is parallelizable, this means that its tangent bundle is trivial (that is, isomorphic to the product, $M \times \mathbb{R}^3$).

In this case, we are interested in the study of magnetic curves, for this we need the following Proposition

Proposition 2.4. *[6] Let $(M^3, \varphi, \xi, \eta, g)$ be a 3-dimensional unit C_{12}-manifold. We have*

$$\begin{array}{lll} \nabla_\xi \xi = -\psi & \nabla_\xi \psi = \xi & \nabla_\xi \varphi\psi = 0 \\ \nabla_\psi \xi = 0 & \nabla_\psi \psi = 0 & \nabla_\psi \varphi\psi = 0 \\ \nabla_{\varphi\psi} \xi = 0 & \nabla_{\varphi\psi} \psi = (-1 + \mathrm{div}\psi)\varphi\psi & \nabla_{\varphi\psi} \varphi\psi = (1 - \mathrm{div}\psi)\psi. \end{array}$$

To clarify these notions, we give the following class of examples:

2.2 Class of examples

We denote the Cartesian coordinates in a 3-dimensional Euclidean space $\mathbb{R}^3$ by (x_1, x_2, x_3) and define a symmetric tensor field g by

$$g = \begin{pmatrix} \tau(x_1,x_2,x_3)^2 & 0 & 0 \\ 0 & \kappa(x_1,x_2,x_3)^2 & 0 \\ 0 & 0 & \mu(x_1,x_2,x_3)^2 \end{pmatrix},$$

where τ, κ and μ are functions on $\mathbb{R}^3$ and $\tau\kappa\mu \neq 0$ everywhere. Further, we define an almost contact structure (φ, ξ, η) on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu}{\kappa} \\ 0 & \frac{\kappa}{\mu} & 0 \end{pmatrix}, \qquad \xi = \frac{1}{\tau}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \eta = \big(\tau, 0, 0\big).$$

Notice that $\mathrm{d}\eta = \tau_2 dx_2 \wedge dx_1 + \tau_3 dx_3 \wedge dx_1 = \omega \wedge \eta$ with $\omega = \frac{\tau_2}{\tau}dx_2 + \frac{\tau_3}{\tau}dx_3$, where $\tau_i = \frac{\partial \tau}{\partial x_i}$ (The index always indicates the derivative). Therefore

$$\psi = \frac{\tau_2}{\tau\kappa^2}\frac{\partial}{\partial x_2} + \frac{\tau_3}{\tau\mu^2}\frac{\partial}{\partial x_3}.$$

Now, we can see that

$$|\psi|^2 = \frac{1}{\tau^2}\Big(\frac{\tau_2^2}{\kappa^2} + \frac{\tau_3^2}{\mu^2}\Big) \qquad and \qquad \mathrm{d}\omega = \Big(\frac{\tau_2}{\tau}\Big)_1 dx_1 \wedge dx_2 + \Big(\frac{\tau_3}{\tau}\Big)_1 dx_1 \wedge dx_3.$$

We give the following orthonormal basis

$$\xi = \frac{1}{\tau}\frac{\partial}{\partial x_1}, \qquad e_1 = \frac{1}{\kappa}\frac{\partial}{\partial x_2}, \qquad e_2 = \frac{1}{\mu}\frac{\partial}{\partial x_3}.$$

So, the components of the Levi-Civita connection corresponding to g are written

$$\begin{array}{ccc} \nabla_\xi \xi = -\frac{\tau_2}{\kappa\tau}e_1 - \frac{\tau_3}{\tau\mu}e_2, & \nabla_\xi e_1 = \frac{\tau_2}{\kappa\tau}\xi, & \nabla_\xi e_2 = \frac{\tau_3}{\tau\mu}\xi, \\ \nabla_{e_1}\xi = \frac{\kappa_1}{\kappa\tau}e_1, & \nabla_{e_1}e_1 = -\frac{\kappa_1}{\kappa\tau}\xi - \frac{\kappa_3}{\kappa\mu}e_2, & \nabla_{e_1}e_2 = \frac{\kappa_3}{\kappa\mu}e_1, \\ \nabla_{e_2}\xi = \frac{\mu_1}{\tau\mu}e_2, & \nabla_{e_2}e_1 = \frac{\mu_2}{\kappa\mu}e_2, & \nabla_{e_2}e_2 = -\frac{\mu_1}{\tau\mu}\xi - \frac{\mu_2}{\kappa\mu}e_1. \end{array}$$

Using Theorem **??**, *one can check that* $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ *is a* 3*-parameter family of* C_{12}*-manifolds if and only if*

$$\nabla_{e_i}\xi = -\eta(e_i)\psi = -\eta(e_i)\Big(\frac{\tau_2}{\kappa\tau}e_1 + \frac{\tau_3}{\mu\tau}e_2\Big),$$

where $i \in \{0, 1, 2\}$ *with* $e_0 = \xi$*, i.e.*

$$\nabla_\xi \xi = -\frac{\tau_2}{\kappa\tau}e_1 - \frac{\tau_3}{\tau\mu}e_2, \qquad \nabla_{e_1}\xi = \nabla_{e_2}\xi = 0.$$

From the above components of the Levi-Civita connection, we get

$$\kappa_1 = \mu_1 = 0.$$

3 Magnetic Curve

3.1 Slant curves

Let (M, g) be a 3-dimensional Riemannian manifold with Levi-Civita connection ∇. γ is said to be a Frenet curve if there exists an orthonormal frame $\{E_1 = \dot\gamma, E_2, E_3\}$ along γ such that

$$\nabla_{E_1}E_1 = \kappa E_2, \qquad \nabla_{E_1}E_2 = -\kappa E_1 + \tau E_3 \qquad \nabla_{E_1}E_3 = -\tau E_2. \tag{3.1}$$

The curvature κ is defined by the formula

$$\kappa = |\nabla_{\dot\gamma}\dot\gamma|. \tag{3.2}$$

The second unit vector field E_2 is thus obtained by

$$\nabla_{\dot\gamma}\dot\gamma = \kappa E_2. \tag{3.3}$$

Next, the torsion τ and the third unit vector field E_3 are defined by the formulas

$$\tau = |\nabla_{\dot\gamma}E_2 + \kappa E_1| \qquad and \qquad \nabla_{\dot\gamma}E_2 + \kappa E_1 = \tau E_3. \tag{3.4}$$

The concept of slant curve in almost contact metric geometry was introduced in [10] with the constant angle θ between the tangent and the Reeb vector field. The particular case of $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ is very important since we recover the Legendre curves of [3].

Definition 3.1. $\gamma : I \to M$ be a Frenet curve on M. The structural angle of γ is the function $\theta : I \to [0, 2\pi[$ given by

$$\cos\theta = g(\dot\gamma, \xi) = \eta(\dot\gamma). \tag{3.5}$$

where $\dot\gamma = \frac{d\gamma}{ds}$ with s is the arc length parameter. The curve γ is a slant curve if θ is a constant function. particularly if $\eta(\dot\gamma) = 0$ the curve γ is called Legendre curve.

3.2 Magnetic curves

Magnetic curves represent, in Physics, the trajectories of charged particles moving on a Riemannian manifold under the action of magnetic fields. A magnetic field F on a Riemannian manifold (M, g) is a closed 2-form F and the Lorentz force associated to F is an endomorphism field Φ such that, for any X and Y vector fields on M

$$F(X, Y) = g(\Phi X, Y). \tag{3.6}$$

The magnetic trajectories of F are curves γ in M that satisfy the Lorentz equation (called also the Newton equation)

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \varphi\dot{\gamma}, \tag{3.7}$$

which generalizes the equation of geodesics under arc length parametrization, namely, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Here ∇ denotes the Levi-Civita connection associated to the metric g. A magnetic field F is said to be uniform if $\nabla F = 0$.

It is well-known that the magnetic trajectories have constant speed. When the magnetic curve $\gamma(s)$ is parametrized by the arc length, it is called normal magnetic curve.

4 Magnetic Curves in Three-Dimensional C_{12}-manifold

4.1 Magnetic Curves in Three-Dimensional C_{12}-manifold

In the following we investigate contact magnetic curves on 3-dimensional C_{12}-manifold. Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional C_{12}-manifold and ϕ the fundamental 2-form defined by (2.2). Since $\mathrm{d}\phi = 0$, we can define a magnetic field on M by

$$F_q(X, Y) = q\phi(X, Y), \tag{4.1}$$

for all X and Y vector fields on M and q is a real constant. We call F_q the contact magnetic field with strength q. It should be pointed out that there is no danger of confusion in using the word contact for the magnetic field, recall that our setting is a C_{12}-manifold which is not contact and not normal.

Notice that if $q = 0$, then the contact magnetic field vanishes identically and the magnetic curves are the geodesics of M. In the sequel we assume $q \neq 0$.

The Lorentz force Φ_q associated to the contact magnetic field F_q may be easily determined combining (2.2),(4.1) and (4.1), that is

$$\Phi_q = -q\varphi, \tag{4.2}$$

where φ is the field of endomorphisms of the almost contact metric structure. Let γ be a normal magnetic trajectory in a three dimensional C_{12}-manifold M with respect to the Lorentz force $q\varphi$. Namely, $\gamma = \gamma_1\xi + \gamma_2\psi + \gamma_3\varphi\psi$ is parametrized by the arclength and it satisfies

$$\nabla_{\dot{\gamma}}\dot{\gamma} = q\varphi\dot{\gamma}, \tag{4.3}$$

where q is a real constant. The solutions of Eq. (4.3) are called normal magnetic curves or trajectories for F_q. The first fundamental result is the following one.

Proposition 4.1. *Every normal contact magnetic curve on a 3-dimensional unit C_{12}-manifold satisfies*

$$\begin{cases} \ddot{\gamma}_1 = -\dot{\gamma}_1\dot{\gamma}_2, \\ \ddot{\gamma}_2 = -q\dot{\gamma}_3 + \dot{\gamma}_1^2 - \dot{\gamma}_3^2(1 - \mathrm{div}\psi), \\ \ddot{\gamma}_3 = q\dot{\gamma}_2 + \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi). \end{cases} \tag{4.4}$$

Proof. If we denote $\gamma(s) = \gamma_1(s)\xi + \gamma_2(s)\psi + \gamma_3(s)\varphi\psi$ then

$$\begin{aligned} \dot{\gamma}_1 = \eta(\dot{\gamma}) \Rightarrow \ddot{\gamma}_1 &= \frac{d}{ds}g(\dot{\gamma}, \xi) \\ &= g(\nabla_{\dot{\gamma}}\dot{\gamma}, \xi) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\xi) \\ &= g(q\varphi\dot{\gamma}, \xi) - \eta(\dot{\gamma})g(\gamma, \psi) \\ &= -\dot{\gamma}_1\dot{\gamma}_2. \end{aligned}$$

Knowing that $\dot{\gamma}_2 = g(\dot{\gamma}, \psi)$ and using Proposition 2.4, we have

$$\begin{aligned}\ddot{\gamma}_2 &= \frac{d}{ds} g(\dot{\gamma}, \psi) \\ &= g(\nabla_{\dot{\gamma}}\dot{\gamma}, \psi) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\psi) \\ &= g(q\varphi\dot{\gamma}, \psi) + \dot{\gamma}_1 g(\dot{\gamma}, \nabla_\xi \psi) + \dot{\gamma}_2 g(\dot{\gamma}, \nabla_\psi \psi) + \dot{\gamma}_3 g(\dot{\gamma}, \nabla_{\varphi\psi}\psi) \\ &= -q\dot{\gamma}_3 + \dot{\gamma}_1^2 - \dot{\gamma}_3^2(1 - \mathrm{div}\psi).\end{aligned}$$

Also, we have

$$\begin{aligned}\ddot{\gamma}_3 &= \frac{d}{ds} g(\dot{\gamma}, \varphi\psi) \\ &= g(\nabla_{\dot{\gamma}}\dot{\gamma}, \varphi\psi) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\varphi\psi) \\ &= q\dot{\gamma}_2 + \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi).\end{aligned}$$

This completes the proof. □

Conversely, we have the following proposition:

Proposition 4.2. *Any regular curve γ in a 3-dimensional unit C_{12}-manifold satisfies system (4.16) such that*

$$\frac{1}{\dot{\gamma}_2}\big(\ddot{\gamma}_3 - \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi)\big) = const \quad with \quad \dot{\gamma}_2 \neq 0$$

or

$$\frac{1}{\dot{\gamma}_3}\big(\dot{\gamma}_1^2 - \ddot{\gamma}_2 - \dot{\gamma}_3^2(1 - \mathrm{div}\psi)\big) = const \quad with \quad \dot{\gamma}_3 \neq 0,$$

is a normal contact magnetic curve.

Proof. Let $\gamma : I \to M$ be a curve on M. Then we have

$$\begin{aligned}\nabla_{\dot{\gamma}}\dot{\gamma} &= \nabla_{\dot{\gamma}}(\dot{\gamma}_1\xi + \dot{\gamma}_2\psi + \dot{\gamma}_3\varphi\psi) \\ &= \ddot{\gamma}_1\xi + \dot{\gamma}_1\nabla_{\dot{\gamma}}\xi + \ddot{\gamma}_2\psi + \dot{\gamma}_2\nabla_{\dot{\gamma}}\psi + \ddot{\gamma}_3\varphi\psi + \dot{\gamma}_3\nabla_{\dot{\gamma}}\varphi\psi \\ &= \ddot{\gamma}_1\xi - \dot{\gamma}_1\eta(\dot{\gamma})\psi + \ddot{\gamma}_2\psi + \dot{\gamma}_2\big(\dot{\gamma}_1\nabla_\xi\psi + \dot{\gamma}_2\nabla_\psi\psi + \dot{\gamma}_3\nabla_{\varphi\psi}\psi\big) \\ &\quad + \dot{\gamma}_3\big(\dot{\gamma}_1\nabla_\xi\varphi\psi + \dot{\gamma}_2\nabla_\psi\varphi\psi + \dot{\gamma}_3\nabla_{\varphi\psi}\varphi\psi\big),\end{aligned}$$

with the help of Proposition 2.4, one can get

$$\begin{aligned}\nabla_{\dot{\gamma}}\dot{\gamma} = (\ddot{\gamma}_1 + \dot{\gamma}_1\dot{\gamma}_2)\xi + \big(\ddot{\gamma}_2 - \dot{\gamma}_1^2 + \dot{\gamma}_3^2(1 - \mathrm{div}\psi)\big)\psi \\ + \big(\ddot{\gamma}_3 - \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi)\big)\varphi\psi.\end{aligned} \tag{4.5}$$

Let's use the first hypothesis in system (4.16) i.e. $\ddot{\gamma}_1 + \dot{\gamma}_1\dot{\gamma}_2 = 0$, we obtain

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \big(\ddot{\gamma}_2 - \dot{\gamma}_1^2 + \dot{\gamma}_3^2(1 - \mathrm{div}\psi)\big)\psi + \big(\ddot{\gamma}_3 - \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi)\big)\varphi\psi. \tag{4.6}$$

knowing that $|\dot{\gamma}| = 1$ i.e. $\dot{\gamma}_1\ddot{\gamma}_1 + \dot{\gamma}_2\ddot{\gamma}_2 + \dot{\gamma}_3\ddot{\gamma}_3 = 0$ then, one can note that there are two cases for discussion:
1) If $\dot{\gamma}_2 \neq 0$ then, from (4.6) we get

$$\begin{aligned}\dot{\gamma}_2\nabla_{\dot{\gamma}}\dot{\gamma} &= -\dot{\gamma}_3\big(\ddot{\gamma}_3 - \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi)\big)\psi + \dot{\gamma}_2\big(\ddot{\gamma}_3 - \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi)\big)\varphi\psi \\ &= \big(\ddot{\gamma}_3 - \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi)\big)(-\dot{\gamma}_3\psi + \dot{\gamma}_2\varphi\psi) \\ &= \big(\ddot{\gamma}_3 - \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi)\big)\varphi\dot{\gamma}.\end{aligned}$$

Then, γ is a normal contact magnetic curve with

$$q = \frac{1}{\dot{\gamma}_2}\big(\ddot{\gamma}_3 - \dot{\gamma}_2\dot{\gamma}_3(1 - \mathrm{div}\psi)\big).$$

2) If $\dot{\gamma}_3 \neq 0$ then, from (4.6) we obtain

$$\begin{aligned}
\dot{\gamma}_3 \nabla_{\dot{\gamma}} \dot{\gamma} &= \dot{\gamma}_3\big(\ddot{\gamma}_2 - \dot{\gamma}_1^2 + \dot{\gamma}_3^2(1-\mathrm{div}\psi)\big)\psi + \dot{\gamma}_3\big(\ddot{\gamma}_3 - \dot{\gamma}_2\dot{\gamma}_3(1-\mathrm{div}\psi)\big)\varphi\psi \\
&= \dot{\gamma}_3\big(\ddot{\gamma}_2 - \dot{\gamma}_1^2 + \dot{\gamma}_3^2(1-\mathrm{div}\psi)\big)\psi\big(-\dot{\gamma}_1\ddot{\gamma}_1 - \dot{\gamma}_2\ddot{\gamma}_2 - \dot{\gamma}_2\dot{\gamma}_3^2(1-\mathrm{div}\psi)\big)\varphi\psi \\
&= \dot{\gamma}_3\big(\ddot{\gamma}_2 - \dot{\gamma}_1^2 + \dot{\gamma}_3^2(1-\mathrm{div}\psi)\big)\psi + \big(\dot{\gamma}_1^2\dot{\gamma}_2 - \dot{\gamma}_2\ddot{\gamma}_2 - \dot{\gamma}_2\dot{\gamma}_3^2(1-\mathrm{div}\psi)\big)\varphi\psi \\
&= \dot{\gamma}_3\big(\ddot{\gamma}_2 - \dot{\gamma}_1^2 + \dot{\gamma}_3^2(1-\mathrm{div}\psi)\big)\psi + \dot{\gamma}_2\big(\dot{\gamma}_1^2 - \ddot{\gamma}_2 - \dot{\gamma}_3^2(1-\mathrm{div}\psi)\big)\varphi\psi \\
&= \big(\dot{\gamma}_1^2 - \ddot{\gamma}_2 - \dot{\gamma}_3^2(1-\mathrm{div}\psi)\big)(-\dot{\gamma}_3\psi + \dot{\gamma}_2\varphi\psi) \\
&= \big(\dot{\gamma}_1^2 - \ddot{\gamma}_2 - \dot{\gamma}_3^2(1-\mathrm{div}\psi)\big)\varphi\dot{\gamma}.
\end{aligned}$$

Therefore, γ is a normal contact magnetic curve with

$$q = \frac{1}{\dot{\gamma}_3}\big(\dot{\gamma}_1^2 - \ddot{\gamma}_2 - \dot{\gamma}_3^2(1-\mathrm{div}\psi)\big).$$

□

4.2 Twin Magnetic curves on 3-dimensional unit C_{12}-manifold

A very interesting remark arises in the case of 3*-dimensional unit* C_{12}*-manifold. It's the existence of another interesting closed 2-form on* M*. Knowing that* $\mathrm{d}\eta = \omega \wedge \eta$ *let us set* $\tilde{\phi} = 2\omega \wedge \eta$ *i.e. for all* X *and* Y *vector field on* M *we have*

$$\tilde{\phi}(X,Y) = g\big(\omega(X)\xi - \eta(X)\psi, Y\big). \tag{4.7}$$

Since the 2-form $\tilde{\phi}$ *is closed, we can define a magnetic field on* M *by*

$$\tilde{F}_q(X,Y) = \tilde{q}\tilde{\phi}(X,Y) \tag{4.8}$$

The Lorentz force $\tilde{\Phi}_{\tilde{q}}$ *associated to the magnetic field* $\tilde{F}_{\tilde{q}}$ *may be easily determined by*

$$\tilde{\Phi}_{\tilde{q}} = \tilde{q}\tilde{\varphi}, \tag{4.9}$$

where $\tilde{\varphi}X = \omega(X)\xi - \eta(X)\psi$*.*

A regular curve γ *is said to be a twin normal magnetic curve in a three dimensional unit* C_{12}*-manifold* M *with respect to the Lorentz force* $q\tilde{\varphi}$ *if it is parametrized by the arclength and it satisfies*

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \tilde{q}\tilde{\varphi}\dot{\gamma}, \tag{4.10}$$

where $\tilde{q}$ *is a real constant. The first fundamental result in this direction is the following one.*

Theorem 4.3. *Let* $(M^3, \varphi, \xi, \eta, g)$ *be a* 3*-dimensional unit* C_{12}*-manifold and* $\gamma = \gamma_1\xi + \gamma_2\psi + \gamma_3\varphi\psi$ *be a regular curve on* M*. Then,* γ *is a twin normal magnetic curve if and only if*

$$\begin{cases} \ddot{\gamma}_1 = \tilde{q}\dot{\gamma}_2 - \dot{\gamma}_1\dot{\gamma}_2, \\ \ddot{\gamma}_2 = -\tilde{q}\dot{\gamma}_1 + \dot{\gamma}_1^2 - \dot{\gamma}_3^2(1-\mathrm{div}\psi), \\ \ddot{\gamma}_3 = \dot{\gamma}_2\dot{\gamma}_3(1-\mathrm{div}\psi). \end{cases} \tag{4.11}$$

Proof. According to the notation used, we have

$$\begin{aligned} \tilde{\varphi}\dot{\gamma} &= \dot{\gamma}_1\tilde{\varphi}\xi + \dot{\gamma}_2\tilde{\varphi}\psi + \dot{\gamma}_3\tilde{\varphi}\varphi\psi \\ &= \dot{\gamma}_2\xi - \dot{\gamma}_1\psi, \end{aligned} \tag{4.12}$$

then, with the help of Propositions 2.4, we get

$$\begin{aligned}
\ddot{\gamma}_1 &= \frac{d}{ds}g(\dot{\gamma},\xi) \\
&= g(\nabla_{\dot{\gamma}}\dot{\gamma},\xi) + g(\dot{\gamma},\nabla_{\dot{\gamma}}\xi) \\
&= g(\tilde{q}\tilde{\varphi}\dot{\gamma},\xi) + \dot{\gamma}_1 g(\dot{\gamma},\nabla_\xi\xi) + \dot{\gamma}_2 g(\dot{\gamma},\nabla_\psi\xi) + \dot{\gamma}_3 g(\dot{\gamma},\nabla_{\varphi\psi}\xi) \\
&= \tilde{q}\dot{\gamma}_2 - \dot{\gamma}_1\dot{\gamma}_2.
\end{aligned}$$

And

$$\begin{aligned}\ddot{\gamma}_2 &= \frac{d}{ds} g(\dot{\gamma}, \psi) \\ &= g(\nabla_{\dot{\gamma}}\dot{\gamma}, \psi) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\psi) \\ &= g(\tilde{q}\tilde{\varphi}\dot{\gamma}, \psi) + \dot{\gamma}_1 g(\dot{\gamma}, \nabla_{\xi}\psi) + \dot{\gamma}_2 g(\dot{\gamma}, \nabla_{\psi}\psi) + \dot{\gamma}_3 g(\dot{\gamma}, \nabla_{\varphi\psi}\psi) \\ &= -\tilde{q}\dot{\gamma}_1 + \dot{\gamma}_1^2 - \dot{\gamma}_3^2(1 - \text{div}\psi).\end{aligned}$$

Also, we have

$$\begin{aligned}\ddot{\gamma}_3 &= \frac{d}{ds} g(\dot{\gamma}, \varphi\psi) \\ &= g(\nabla_{\dot{\gamma}}\dot{\gamma}, \varphi\psi) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\varphi\psi) \\ &= g(\tilde{q}\tilde{\varphi}\dot{\gamma}, \varphi\psi) + \dot{\gamma}_1 g(\dot{\gamma}, \nabla_{\xi}\varphi\psi) + \dot{\gamma}_2 g(\dot{\gamma}, \nabla_{\psi}\varphi\psi) + \dot{\gamma}_3 g(\dot{\gamma}, \nabla_{\varphi\psi}\varphi\psi) \\ &= \dot{\gamma}_2\dot{\gamma}_3(1 - \text{div}\psi).\end{aligned}$$

Conversely, just replace equations (4.11) in the formula (4.5), directly we find

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \tilde{q}\tilde{\varphi}\dot{\gamma},$$

which completes the proof. □

While this is an area of possible future research especially for the general case (i.e $|\psi| \neq 1$), we focus in the rest of this paper, on the case where $\text{div}\psi = 1$. *Consequently, from (4.11) we get*

$$\begin{cases} \ddot{\gamma}_1 = (\tilde{q} - \dot{\gamma}_1)\dot{\gamma}_2, \\ \ddot{\gamma}_2 = (\dot{\gamma}_1 - \tilde{q})\dot{\gamma}_1, \\ \ddot{\gamma}_3 = 0. \end{cases} \tag{4.13}$$

We put $\dot{\gamma}_3 = c_0 \in \mathbb{R}$ and discuss the following two cases:
If $\dot{\gamma}_1 = \tilde{q}$ we get $\dot{\gamma}_2 = b_0 \in \mathbb{R}$ then

$$\dot{\gamma} = \tilde{q}\xi + b_0\psi + c_0\varphi\psi,$$

i.e.

$$\gamma = (\tilde{q}s + a_1)\xi + (b_0 s + b_1)\psi + (c_0 s + c_1)\varphi\psi.$$

If $\dot{\gamma}_1 \neq \tilde{q}$, we deriving the first equation in (4.13), we get

$$\dddot{\gamma}_1 = -\ddot{\gamma}_1\dot{\gamma}_2 + (\tilde{q} - \dot{\gamma}_1)\ddot{\gamma}_2.$$

Using the first and the second equations from (4.13), we obtain

$$f\ddot{f} - \dot{f}^2 + f^3(f + \tilde{q}) = 0,$$

where $f = \dot{\gamma}_1 - \tilde{q}$. With the help of computer algebra software, we get two solutions:

$$f_1 = \frac{4e^{\frac{s+\mu}{\lambda}}}{4 + \lambda^2\left(e^{\frac{s+\mu}{\lambda}} + 2\tilde{q}\right)^2}, \qquad f_2 = \frac{4e^{\frac{s+\mu}{\lambda}}}{4e^{\frac{s+\mu}{\lambda}} + \lambda^2\left(2\tilde{q}e^{\frac{s+\mu}{\lambda}} + 1\right)^2} \tag{4.14}$$

where λ and μ are constants and $\lambda \neq 0$. Then

$$\dot{\gamma}_1 = \tilde{q} + f, \qquad \text{and} \qquad \dot{\gamma}_2 = -\frac{\dot{f}}{f},$$

i.e.

$$\gamma_1 = \tilde{q}s + \int f ds \qquad \text{and} \qquad \gamma_2 = -\ln f,$$

where

$$\int f_1 ds = 2\text{arctan}\left(\frac{\lambda}{2}\left(2\tilde{q} + \mathrm{e}^{\frac{s+\mu}{\lambda}}\right)\right),$$

and

$$\int f_2 ds = 2\text{arctan}\left(\lambda\tilde{q} + \frac{2}{\lambda}(1+\lambda^2\tilde{q})^2\mathrm{e}^{\frac{s+\mu}{\lambda}}\right).$$

Based on these facts, we give the following theorem:

Theorem 4.4. *Let $(M^3, \varphi, \xi, \eta, g)$ be a 3-dimensional unit C_{12}-manifold with $\mathrm{div}\psi = -\mathrm{div}\nabla_\xi\xi = 1$ and consider the contact magnetic field $\tilde{F}_{\tilde{q}}$ for $\tilde{q} \neq 0$ on M. γ is a twin normal magnetic curve associated to $\tilde{F}_{\tilde{q}}$ in M if and only if*

$$\gamma = (\tilde{q}s + a_1)\xi + (b_0 s + b_1)\psi + (c_0 s + c_1)\varphi\psi.$$

i.e. the curve is slant, that is the angle $\theta \in [0, \pi]$ between $T = \dot{\gamma}$ and ξ is constant. Or,

$$\gamma = \left(\tilde{q}s + \int f ds\right)\xi - (\ln f)\psi + (c_0 s + c_1)\varphi\psi.$$

where f is a well-known function given above.

4.3 Bi-Magnetic curves

Since there are two closed 2-forms in 3-dimensional unit C_{12}-manifold, which implies the existence of two magnetic fields on the same manifold, one can ask if it is possible to define a magnetic curves associated to F_q and $\tilde{F}_{\tilde{q}}$ in M at the same time?

In other words, Magnetic curves represent, in Physics, the trajectories of charged particles moving on a Riemannian manifold under the action of magnetic fields. Thus, these charged particles can be exposed to more than one magnetic field. Therefore, the above question is legitimate. Let's give here at least one consistent mathematical answer and related to the space covered in this study.

Let $(M^3, \varphi, \xi, \eta, g)$ be a 3-dimensional unit C_{12}-manifold with $\mathrm{div}\psi = 1$ and consider the contact magnetic fields F_q and $\tilde{F}_{\tilde{q}}$ on M. Let γ be a normal magnetic curve associated to F_q and $\tilde{F}_{\tilde{q}}$ in M at the same time we will call it "normal contact Bi-magnetic curve". Based on the above notation, one can get

$$\nabla_{\dot{\gamma}}\dot{\gamma} = q\varphi\dot{\gamma} + \tilde{q}\tilde{\varphi}\dot{\gamma}. \tag{4.15}$$

The fundamental result here is the following one.

Proposition 4.5. *Every normal contact Bi-magnetic curve on a 3-dimensional unit C_{12}-manifold satisfies*

$$\begin{cases} \ddot{\gamma}_1 = \dot{\gamma}_2(\tilde{q} - \dot{\gamma}_1), \\ \ddot{\gamma}_2 = -\dot{\gamma}_1(\tilde{q} - \dot{\gamma}_1) - \dot{\gamma}_3\big(q + (1 - \mathrm{div}\psi)\dot{\gamma}_3\big), \\ \ddot{\gamma}_3 = \dot{\gamma}_2\big(q + (1 - \mathrm{div}\psi)\dot{\gamma}_3\big). \end{cases} \tag{4.16}$$

Proof. One can adapt Theorem 4.3. □

In this interesting situation, we suffice to note that for $\mathrm{div}\psi = 2$, $\dot{\gamma}_1 = \tilde{q}$ and $\dot{\gamma}_3 = q$ it results

$$\gamma = (\tilde{q}s + a_1)\xi + (a_2 s + b_2)\psi + (qs + b_3)\varphi\psi,$$

where a_i and b_i are arbitrary constants. This confirms the existence of the normal contact Bi-magnetic curves on a 3-dimensional unit C_{12}-manifold.

4.4 Curvature and torsion of twin Magnetic curves

On an arbitrary oriented Riemannian 3-manifold one can canonically define a cross product $\times$ *of two vector fields* X *and* Y *on* M *as follows:*

$$g(X \times Y, Z) = dv_g(X, Y, Z), \tag{4.17}$$

for any vector fields Z *on* M*. where* dv_g *denotes the volume form defined by* g*. When* M *is an almost contact metric 3-manifold, the cross product is given by the formula* X *and* Y *on* M *as follows [7]:*

$$X \times Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y. \tag{4.18}$$

Note that for the fundamental basis $\{\xi, \psi, \varphi\psi\}$ *of a* 3*-dimensional unit* C_{12}*-manifold* $(M, \varphi, \xi, \eta, g)$*, we have*

$$\dot{\gamma} \times \xi = -\varphi\dot{\gamma}, \qquad \dot{\gamma} \times \varphi\psi = \tilde{\varphi}\dot{\gamma} \tag{4.19}$$

Take the Frenet frame field $\{T, N, B\}$ *along* γ *. By definition* $T = \dot{\gamma}$*. Hence, the magnetic equation is written as*

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \tilde{q}\tilde{\varphi}\dot{\gamma} = \kappa N. \tag{4.20}$$

Let's define the twin contact angle as the angle $\tilde{\theta}(s) \in [0, \pi]$ *made by* γ *with the trajectories of* $\varphi\psi$*, that is we have*

$$\cos\tilde{\theta} = g(\dot{\gamma}, \varphi\psi) = c_0.$$

From (4.20), one can get

$$\begin{aligned} \kappa^2 &= \tilde{q}^2 g(\tilde{\varphi}\dot{\gamma}, \tilde{\varphi}\dot{\gamma}) = \tilde{q}^2\big(\dot{\gamma}_1^2 + \dot{\gamma}_2^2\big) = \tilde{q}^2(1 - c_0^2) \\ &= \tilde{q}^2 \sin^2\tilde{\theta}. \end{aligned}$$

Thus, $\kappa = |\tilde{q}| \sin\tilde{\theta}$ *with* $0 \le \tilde{\theta} \le \pi$*. Assume that* γ *is a non-geodesic normal magnetic curve, then from (4.20) we have*

$$N = \frac{\tilde{q}}{\kappa}\tilde{\varphi}\dot{\gamma}. \tag{4.21}$$

Next, the binormal vector field B *is obtained from the formula*

$$\begin{aligned} B = \dot{\gamma} \times N &= \frac{\tilde{q}}{\kappa}\dot{\gamma} \times \tilde{\varphi}\dot{\gamma} \\ &= \frac{\tilde{q}}{\kappa}\dot{\gamma} \times \big(\dot{\gamma}_2\xi - \dot{\gamma}_1\psi\big) \\ &= \frac{\tilde{q}}{\kappa}\Big(\cos\tilde{\theta}\big(\dot{\gamma}_1\xi + \dot{\gamma}_2\psi\big) - \sin^2\tilde{\theta}\varphi\psi\Big) \\ &= \frac{\tilde{q}}{\kappa}\Big(\cos\tilde{\theta}\,\dot{\gamma} - \varphi\psi\Big). \end{aligned} \tag{4.22}$$

The covariant derivative of the binormal may be computed as

$$\begin{aligned} \nabla_{\dot{\gamma}}B &= \frac{\tilde{q}}{\kappa}\Big(\cos\tilde{\theta}\,\nabla_{\dot{\gamma}}\dot{\gamma} - \nabla_{\dot{\gamma}}\varphi\psi\Big) \\ &= \frac{\tilde{q}^2}{\kappa}\cos\tilde{\theta}\,\tilde{\varphi}\dot{\gamma}. \end{aligned} \tag{4.23}$$

Comparing this with

$$\nabla_{\dot{\gamma}}B = -\tau N = -\frac{\tau\tilde{q}}{\kappa}\tilde{\varphi}\dot{\gamma},$$

we obtain the expression of the torsion of γ *, that is*

$$\tau = -\tilde{q}\cos\tilde{\theta}.$$

Notice that

$$\kappa^2 + \tau^2 = \tilde{q}^2.$$

Theorem 4.6. *Let (M^3,φ,ξ,η,g) be a 3-dimensional unit C_{12}-manifold with* $\mathrm{div}\psi=-\mathrm{div}\nabla_\xi\xi=1$ *and let γ be a twin normal magnetic curve corresponding to the contact magnetic field $\tilde{F}_{\tilde{q}}$ for $\tilde{q}\neq 0$ on M. Then γ is a helix with*

$$\kappa=|\tilde{q}|\sin\tilde{\theta}\quad and\quad \tau=-\tilde{q}\cos\tilde{\theta}.$$

with $0\leq\tilde{\theta}\leq\pi$

Example 4.7. From the above class of examples, let's taking $\tau=\kappa=\frac{1}{y}$ with $y>0$ and $\mu=1$. That is, The metric g becomes

$$g=\frac{1}{y^2}(dx^2+dy^2)+dz^2.$$

With respect to the metric g an orthonormal basis is

$$e_1=y\frac{\partial}{\partial x},\qquad e_2=y\frac{\partial}{\partial y},\qquad e_3=\frac{\partial}{\partial z},$$

and it's dual frame (θ^i) is given by

$$\theta^1=\frac{1}{y}dx,\qquad \theta^2=\frac{1}{y}dy,\qquad \theta^3=dz.$$

The non-zero components of the Levi-Civita connection ∇ with respect to g are

$$\nabla_{e_1}e_1=e_2\qquad and\qquad \nabla_{e_1}e_2=-e_1.$$

Easily, we can verify that $\nabla_{e_i}e_1=\theta^1(e_i)e_2$ for all $i\in\{1,2,3\}$. So, taking

$$\xi=e_1,\quad \eta=\theta^1,\quad \varphi=\theta^3\otimes e_2-\theta^2\otimes e_3,$$

one can see that (M,φ,ξ,η,g) is a 3-dimensional C_{12}-manifold with $\psi=-e_2$ and $\mathrm{div}\psi=1$.

Let us choose a curve : $\gamma:I\to M$ by $\gamma(s)=\alpha se_1+e_2-(\alpha^2-1)se_3$ with $\alpha\in\mathbb{R}-\{-1,0,1\}$. We see that

$$\dot{\gamma}=\alpha e_1-(\alpha^2-1)e_3.$$

Now,

$$\nabla_{\dot{\gamma}}\dot{\gamma}=\nabla_{(\alpha e_1-(\alpha^2-1)e_3)}\alpha e_1-(\alpha^2-1)e_3)=\alpha^2e_2.$$

Again

$$\varphi\dot{\gamma}=(\alpha^2-1)e_2. \tag{4.24}$$

So,

$$\nabla_{\dot{\gamma}}\dot{\gamma}=\frac{\alpha^2}{\alpha^2-1}\varphi\dot{\gamma}.$$

Hence, the curve is a normal contact magnetic curve.

On the other hand, we have

$$\tilde{\varphi}\dot{\gamma}\ =\ \alpha e_2,$$

i.e

$$\nabla_{\dot{\gamma}}\dot{\gamma}=\alpha\tilde{\varphi}\dot{\gamma}.$$

and this means that γ is a twin magnetic curve too.
For γ to be a normal contact Bi-magnetic curve, it is necessary that

$$\alpha\tilde{\varphi}\dot{\gamma}=\frac{\alpha^2}{\alpha^2-1}\varphi\dot{\gamma},$$

witch implies

$$\alpha^2-\alpha-1=0,$$

and this equation has two solutions $\alpha_1=\frac{1+\sqrt{5}}{2}$ (the Golden ratio) and $\alpha_1=\frac{1-\sqrt{5}}{2}$.

4.5 Conclusion remarks and open question

The question remains open about the stability of these results and their validity for $\text{div}\psi \neq 1$ *or for dimensions greater than three. In other words, what happens for the cases where* $\text{div}\psi \neq 1$ *or* $dimM > 3$*?.*

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