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Kernel function-based primal-dual interior-point methods for linear optimization

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Abstract These days, interior-point methods are one of the most valuable tools for solving linear problems. A significantly improved primal-dual interior point algorithm for linear optimization (LO) is presented based on a novel kernel function. We show a primal-dual interior-point technique for linear optimization based on an eligible class of kernel functions. This research suggests a new efficient kernel function for (LO) problems. The kernel function described here extends the one proposed in [2]. Using a few new technical lemmas, we get the iteration complexity bound that meets the best-known iteration bounds for large-update methods: $O\left(\sqrt{n}\log n \log \frac{n}{\epsilon}\right)$. This iteration complexity bound is the same as the best-known iteration bounds for large-update methods obtained, which coincides with the currently best iteration complexity bounds for large-update methods. We test the effectiveness and validity of our algorithm using calculation tests. Then we compare our numerical results with those obtained by algorithms based on various kernel functions.

1 Introduction

We consider the linear optimization (LO) problem in its standard form.

$$(P) \qquad min\left\{c^T x : Ax = b,\ x \geq 0\right\},$$

with its dual problem

$$(D) \quad max\left\{b^T y : A^T y + s = c,\ s \geq 0\right\},$$

where $A \in \mathbb{R}^{m\times n}$ is a real $m \times n$ matrix with rank m, and $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. The dual problem of (P) is given with $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$.

The majority of the interior-point methods $(IPMs)$ for LO are based on the logarithmic barrier function (El Ghami et al. [11]. Roos et al. [17]. Peng et al. [16] introduced novel IPM versions based on a non-logarithmic kernel function. This kind of function is strongly convex, smooth, and coercive in its domain. They obtained the best-known complexity results for large- and small-update methods. For further research on primal-dual IPMs based on kernel functions, see [17], [1], [10], [3], [11], [9], [14]. Recently, Bai et al. [2] developed a generic primal-dual interior-point method for LO. Instead of the classical primal-dual logarithmic barrier function, they used the barrier function of an eligible kernel function. For some related study see [18], [19], [20]

The majority of the kernel functions that are utilized in IPMs may be categorized as either logarithmic, simple algebraic, exponential, or trigonometric, given the precedence that has been established. The remaining kernel functions are a binary combination of these different kinds. For more research on primal-dual IPMs that are based on a kernel function, various authors, including, Boudjellal et al. [6], Z.Moaberfardi et al. [13], Bounibane and Djeffal [7], Benhadid and Merahi [4] and Sajad Fathi-Hafshejani et al. [12]. Inspired by their previous work, this study defines a new kernel function that uses the kernel functions described in [2] as exceptional cases. Using a variety of technical lemmas and taking $a = 1 + 2\left(\frac{n\theta+2\tau+2\sqrt{2n\tau}}{2(1-\theta)}\right)^{\frac{1}{2}}$, we show that the method has a good iteration complexity bound, namely $O(\sqrt{n}\log n)\log\frac{n}{\epsilon}$, which is the most well-known iteration complexity bound for large-update IPMs . Some preliminary numerical results also confirm the algorithm's efficacy and consistency. Our paper is structured as follows: Section 2 starts by reviewing the basics of IPMs for LO, like the central path. Section 3 presents details concerning the parametric kernel function and barrier function. We show that the kernel function meets the conditions for eligibility. In Section 4, we derive the algorithm's inner iteration bound and total iteration bound. The results of experimental tests are presented in Section 5. Section 6 is the concluding part of the paper. It offers some conclusions and remarks.

2 Preliminaries

2.1 Central path and classical Newton search direction for LO

We consider (P) and (D) to meet the interior point condition (IPC), i.e., there exists an (x_0, s_0, y_0) such that

$$Ax_0 = b, \quad x_0 > 0, \quad A^T y_0 + s_0 = c, \quad s_0 > 0.$$

We suppose $x_0 = s_0 = \mathbf{1}$, where $\mathbf{1}$ denotes the all-one vector, i.e., $\mathbf{1} = (1, 1, ..., 1)^T$. We recommend the most up-to-date works, including those written by Peng et al. [17] and Roos et al. [16] Finding the optimal solution for (P) and (D) corresponds to solving the nonlinear system shown below.

$$\begin{cases} Ax = b, \quad x \geq 0, \\ A^T y + s = c, \quad s \geq 0, \\ xs = 0. \end{cases} \tag{2.1}$$

The key principle of primal-dual IPM is to replace the complementarity condition in (2.1) with the parameterized equation $xs = \mu.\mathbf{1}$, $(\mu > 0)$. This provides the next system.

$$\begin{cases} Ax = b, \quad x \geq 0, \\ A^T y + s = c, \quad s \geq 0, \\ xs = \mu.\mathbf{1}. \end{cases} \tag{2.2}$$

If the IPC is satisfied, the parameterized system (2.2) has a unique solution $(x(\mu), y(\mu), s(\mu))$ for each $\mu > 0$, which is called a μ-center of (P) and (D). The set of μ-centers is said to be the central path of (P) and (D). If $\mu \longrightarrow 0$, then the limit of the central path exists. Since the limit point satisfies the complementarity condition, the limit point yields optimal solutions for (P) and (D). For fixed $\mu > 0$, by applying Newton's method to the parameterized system (2.2), we obtain the search direction $(\Delta x, \Delta y, \Delta s)$ from the following Newton system

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu\mathbf{1} - xs. \end{cases} \tag{2.3}$$

Since A has a full-row rank, the system (2.3) has a unique solution $(\Delta x, \Delta y, \Delta s)$ for any $x > 0$ and $s > 0$.

We can generate a new iteration (x_+, y_+, s_+) by taking a step along the search direction $(\Delta x, \Delta y, \Delta s)$ according to

$$x_+ := x + \alpha\Delta x, \; y_+ := y + \alpha\Delta y, \; s_+ := s + \alpha\Delta s,$$

where the step size α satisfies $0 < \alpha \leq 1$.

Now, to facilitate the analysis of the algorithm, we introduce the following notations: for any feasible $x > 0$ and any feasible $s > 0$

$$v := \sqrt{\frac{xs}{\mu}}, \quad d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s}. \tag{2.4}$$

Then the Newton system (2.3) can be rewritten as

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^T\Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{2.5}$$

where

$$\bar{A} := \frac{1}{\mu}AV^{-1}X, \ V := diag(v), \ X := diag(x).$$

Note that d_x and d_s are orthogonal vectors since d_x belongs to the null space and d_s to the row space of the matrix $\bar{A}$;

Hence, we will have

$$d_x = d_s = 0 \Longleftrightarrow v - v^{-1} = 0, \ x = x(\mu), \ s = s(\mu).$$

Note that the right-hand side of the third equation in (2.5) equals the negative gradient of the logarithmic barrier function $\Psi_l(v)$, i.e.,

$$d_x + d_s = -\nabla \, \Psi_l(v),$$

and system (2.5) can be rewritten as follows:

$$\begin{cases} \bar{A}d_x = 0; \\ \bar{A}^T\Delta y + d_s = 0; \\ d_x + d_s = -\nabla \, \Psi_l(v), \end{cases} \tag{2.6}$$

where the barrier function $\Psi_l(v) : \mathbf{R}^n_{++} \to \mathbf{R}_+$ is defined as follows:

$$\begin{aligned} \Psi_l(v) &= \Psi_l(x, s; \mu) = \sum_{i=1}^{n} \psi_l(v_i), \\ \psi_l(v_i) &= \frac{v_i^2 - 1}{2} - \log{(v_i)}. \end{aligned}$$

For a given $\mu > 0$, we use $\Psi_l(v)$ as the proximity function to calculate the distance between the current iterate and the μ-center.

We call $\psi_l(t)$ the kernel function of the logarithmic barrier function $\Psi_l(v)$. As defined in Sect. 3, we replace $\psi_l(t)$ with a new kernel function $\psi_a(t)$ and $\Psi_l(v)$ with a new barrier function $\Psi(v)$ in this paper. As a result, the generic primal-dual interior-point algorithm works as follows.

Generic Primal-Dual Algorithm for LO

Input:
A proximity function $\Psi(v)$;
athreshold parameter τ, $\tau > 0$;
an accuracy parameter ϵ, $\epsilon > 0$;
a fixed barrier update parameter θ, $0 < \theta < 1$;
begin
$x := e$; $s := e$; $\mu := 1$; $v := e$.
While $n\mu \geq \epsilon$ do
begin (outer iteration)
 $\mu := (1 - \theta)\mu$;
while $\Psi(v) > \tau$ do
begin (inner iteration)
 solve the system (2.6) to obtain $(\Delta x, \Delta y, \Delta s)$;

choose a suitable step size α;
$x := x + \alpha\Delta x$, $y := y + \alpha\Delta y$, $s := s + \alpha\Delta s$,
$v = \sqrt{\frac{xs}{\mu}}$
end
end
end

3 Kernel-function properties

First, we study the basic characteristics of the kernel function $\psi(t)$.

3.1 Kernel function characteristics

In the analysis of the algorithm, we also use the norm-based proximity measure $\delta(v) : \mathbf{R}^n_{++} \to \mathbf{R}_+$, defined by

$$\delta(v) = \frac{1}{2} \|\nabla\ \Psi(v)\| = \frac{1}{2} \|dx + ds\| \,. \tag{3.1}$$

Since $\Psi(v)$ is strictly convex and attains its minimum value of zero at $v = e$, we have

$$\Psi(e) = 0 \Leftrightarrow \delta(v) = 0 \Leftrightarrow v = e.$$

Now, the new function $\psi_a(t)$ is defined as follows:

$$\psi_a(t) = \frac{t^2 - 1}{2} - \int_1^t a^{\left(\frac{1}{x}-1\right)} dx \quad a \geq e,\ t > 0. \tag{3.2}$$

In the sequel, we derive the three first derivatives of $\psi_a(t)$ with respect to t as follows:

$$\begin{aligned}
\psi_a'(t) &= t - a^{\left(\frac{1}{t}-1\right)}, && (3.3)\\
\psi_a''(t) &= 1 + \frac{\log a}{t^2} a^{\left(\frac{1}{t}-1\right)}, && (3.4)\\
\psi_a'''(t) &= -\frac{\log a\,(2t + \log a)}{t^4} a^{\left(\frac{1}{t}-1\right)}. && (3.5)
\end{aligned}$$

We can deduce from (3.4) that $\psi_a''(t) > 1$ for $t > 0$, implying that $\psi_a(t)$ is strongly convex over $\mathbb{R}_{++}$. There is also $\psi_a(1) = \psi_a'(1) = 0$. Thus, $\psi_a(t)$ is indeed a kernel function.

Because of the conditions $\psi_a(1) = \psi_a'(1) = 0$, we can completely describe $\psi_a(t)$ by its second derivative:

$$\psi_a(t) = \int_1^t \int_1^\xi \psi_a''(\zeta)\, d\zeta d\xi.$$

In [2], the authors introduced a class of eligible kernel functions by using the following conditions:

Lemma 3.1. *Let the function $\psi_a(t)$ be defined as in (3.2). Then, we have*

$$\begin{aligned}
t\psi_a''(t) + \psi_a'(t) &> 0, \quad t < 1, && (3.6)\\
t\psi_a''(t) - \psi_a'(t) &> 0, \quad t > 1, && (3.7)\\
\psi_a'''(t) &< 0 \quad t > 0, && (3.8)\\
2\left(\psi_a''(t)\right)^2 - \psi_a'(t)\psi_a'''(t) &> 0, \quad t < 1, && (3.9)\\
\psi_a''(t)\psi_a'(\beta t) - \beta\psi_a'(t)\psi_a''(\beta t) &> 0,\ t > 1,\ \beta > 1. && (3.10)
\end{aligned}$$

Proof. For (3.6) and all $t > 0$, we get the following:

$$t\psi_a''(t) + \psi_a'(t) = 2t + \left(\frac{\log a}{t} - 1\right) a^{\left(\frac{1}{t}-1\right)}$$

Using the following inequality

$$\exp^x \geq 1 + x,\ \forall\, x \in \mathbb{R},$$

we obtain for $t \in (0, 1)$

$$t\psi_a''(t) + \psi_a'(t) \geq \left[2t + \left(\frac{\log a}{t} - 1\right)\left(1 + \left(\frac{1-t}{t}\right)\log a\right)\right].$$

$$t\psi_a''(t) + \psi_a'(t) \geq 0 \Leftrightarrow \log a - t \geq 0$$

This last inequality is due to the fact that $a \geq e$, and $t \in (0, 1)$, this shows that the condition (3.6) is satisfied.

For (3.7) , by substituting $\psi_a'(t)$ and $\psi_a''(t)$, we obtain,

$$t\psi_a''(t) - \psi_a'(t) = \left(\frac{\log a + t}{t}\right) a^{\left(\frac{1}{t}-1\right)} > 0,\ t > 0.$$

For (3.8). It is simple to observe $\psi_a'''(t) < 0$ from (3.5).

For (3.9) , we have

$$\begin{aligned}
2\left(\psi_a''(t)\right)^2 - \psi_a'(t)\psi_a'''(t) &= 2\left[1 + \frac{\log a}{t^2} a^{\left(\frac{1}{t}-1\right)}\right]^2 \\
&\quad + \left[\frac{(2t + \log a)\log a}{t^4} a^{\left(\frac{1}{t}-1\right)}\right] \times \left[t - a^{\left(\frac{1}{t}-1\right)}\right] \\
&= \left[2 + \frac{t\log a\,(\log a + 6t)\, a^{\left(\frac{1}{t}-1\right)} + \log a\,(\log a - 2t)\, a^{2\left(\frac{1}{t}-1\right)}}{t^4}\right],
\end{aligned}$$

If $a \geq e^2$, the condition (3.9) is unquestionably satisfied for $0 < t < 1$. So it's still clear that:

$$\begin{aligned}
&\left[2\left(\psi_a''(t)\right)^2 - \psi_a'(t)\psi_a'''(t) > 0\right] \\
\Leftrightarrow\ &\left[t\log a\,(\log a + 6t)\, a^{\left(\frac{1}{t}-1\right)} + \log a\,(\log a - 2t)\, a^{2\left(\frac{1}{t}-1\right)} > 0\right]
\end{aligned} \tag{3.11}$$

Let's examine the case $0 < a < e^2$ and $t \in \left(0, \frac{\log a}{2}\right)$ .The relationship (3.11) is obviously satisfied. It's sufficient to prove that (3.9) holds for

$$\begin{cases} t \in \left(\frac{\log a}{2}, 1\right) \\ a \in \left[e, e^2\right[. \end{cases}$$

Then

$$\begin{aligned}
\left[2\left(\psi_a''(t)\right)^2 - \psi_a'(t)\psi_a'''(t) > 0\right] &\Leftrightarrow \left[a^{\left(\frac{1}{t}-1\right)} \leq \frac{(\log a + 6t)\, t}{2t - \log a}\right] \\
&\Leftrightarrow \left[a^{\left(\frac{1}{t}-1\right)} \leq \left(\begin{array}{c} \frac{t\log a}{2t - \log a} \\ + \frac{6t^2}{2t - \log a} \end{array}\right)\right],
\end{aligned}$$

and this is obviously true if

$$a^{\left(\frac{1}{t}-1\right)} < \frac{\log a}{2 - \frac{\log a}{t}} \tag{3.12}$$

Let $u = \frac{1}{t}$. The relation (3.12) can then be expressed as follows:

$$a^{u-1} < \frac{\log a}{2 - u\log a},\ u \in \left(\frac{1}{\log a}, \frac{2}{\log a}\right),$$

which to

$$1 > (\frac{2}{\log a} - u)a^{u-1}. \tag{3.13}$$

For (3.13) , let $h(u) = 1 - (\frac{2}{\log a} - u)a^{u-1}$, then

$$\begin{cases} h'(u) = a^{u-1}(-1 + u\log a) \\ h''(u) = a^{u-1}.(\log a)^2 u > 0 \text{ for } t > 0. \end{cases}$$

If we set $h'(u) = 0$, we obtain $u = \frac{1}{\log a}$. Since $h(u)$ is strictly convex and has a global minimum, $h\left(\frac{1}{\log a}\right) > 0$

We have the result □

As an observation, we give some results about the new kernel function.

Lemma 3.2. *For $\psi_a(t)$ with, $a \geq e$, we get:*

$$\begin{aligned} \frac{1}{2}(t-1)^2 &\leq \psi_a(t) \leq \frac{1}{2}\psi_a'(t)^2, \ t > 0, && (3.14) \\ \psi_a(t) &\leq \frac{1}{2}\psi_a''(1)(t-1)^2, \ t \geq 1, && (3.15) \\ \|v\| &\leq \sqrt{n} + \sqrt{2\,\Psi(v)}. && (3.16) \end{aligned}$$

Proof. For (3.14) , according to the definition of $\psi_a(t)$, we have:

$$\psi_a(t) \geq \frac{1}{2}(t-1)^2,$$

which proves the first inequality. The second inequality is obtained as follows:

$$\begin{aligned} \psi_a(t) &= \int_1^t \int^\xi \psi_a''(\zeta)\,d\zeta d\xi \leq \int_1^t \int^\xi \psi_a''(\xi)\,\psi_a''(\zeta)\,d\zeta d\xi \\ &= \int_1^t \psi_a''(\xi)\,\psi_a'(\xi)\,d\xi \\ &= \int_1^t \psi_a'(\xi)\,d\psi_a'(\xi) = \frac{1}{2}(\psi_a'(t))^2. \end{aligned}$$

For (3.15), since $\psi_a(1) = \psi_a'(1) = 0$, $\psi_a'''(t) < 0$, $\psi_a''(1) = 1 + \log a$, and by using Taylor's expansion we have for some ξ, such that $1 \leq \xi \leq t$.

$$\begin{aligned} \psi_a(t) &= \psi_a(1) + \psi_a'(1)(t-1) + \frac{1}{2}\psi_a''(1)(t-1)^2 + \frac{1}{6}\psi_a'''(\xi)(\xi-1)^3 \\ &= \frac{1}{2}\psi_a''(1)(t-1)^2 + \frac{1}{6}\psi_a'''(\xi)(\xi-1)^3 < \frac{1}{2}\psi_a''(1)(t-1)^2. \end{aligned}$$

Which completes the proof.

For (3.16) , then by using the left hand side of (3.14) and the Cauchy-Schwarz inequality, one can obtain,

$$\begin{aligned} 2\,\Psi(v) &= 2\sum_{i=1}^n \psi_a(v_i) \geq \sum_{i=1}^n (v_i - 1)^2 \\ &= \left[\sum_{i=1}^n v_i^2 - 2\sum_{i=1}^n v_i + n\right] \\ &= \|v\|^2 - 2\mathbf{e}^T v + \|\mathbf{e}\|^2 \geq \left(\|v\|^2 - 2\|v\|\,\|\mathbf{e}\| + n\right) \\ &= (\|v\| - \|\mathbf{e}\|)^2, \end{aligned}$$

that is to say

$$\|v\| \leq \|\mathbf{e}\| + \sqrt{2\,\Psi\,(v)} = \sqrt{n} + \sqrt{2\,\Psi\,(v)},$$

where $\mathbf{e} = \{1, 1, ..., 1\}$ denotes the all-one vector.
This completes the proof. □

Lemma 3.3. *Let* $\beta \geq 1$. *Then*

$$\psi_a\,(\beta t) \leq \psi_a\,(t) + \frac{1}{2}\left(\beta^2 - 1\right) t^2.$$

Proof. Defining

$$\psi_b\,(t) = -\int_1^t a^{\left(\frac{1}{t}-1\right)} < 0,$$

we have $\psi_b'\,(t) = -a^{\left(\frac{1}{t}-1\right)} < 0$ i.e., $\psi_b\,(t)$ is thus a decreasing function when $t > 0$. Thus $\psi_b\,(\beta t) \leq \psi_b\,(t)$ for $\beta \geq 1$. So

$$\psi_a\,(\beta t) - \psi_a\,(t) = \frac{1}{2}\left(\beta^2 - 1\right) t^2 + \psi_b\,(\beta t) - \psi_b\,(t) \leq \frac{1}{2}\left(\beta^2 - 1\right) t^2.$$

That implies the lemma. □

4 Algorithm analysis for LO

In this section, we analyze the complexity of the *LO* interior-point algorithm for large updates. The algorithm is analyzed using the norm-based proximity measure.

Lemma 4.1. *Let* $\varrho : [0, +\infty) \longrightarrow [1, +\infty)$ *be the inverse function of* $\psi_a\,(t)$ *for* $t \geq 1$ *and* $\rho : [0, +\infty) \longrightarrow (0, 1]$ *the inverse function of* $\frac{-1}{2}\psi_a'\,(t)$ *for* $t \in (0, 1]$, *we have:*

$$\sqrt{2s+1} \quad \leq \quad \varrho\,(s) \leq \sqrt{2s} + 1, \quad s \geq 0, \tag{4.1}$$

$$\rho\,(z) \quad \geq \quad \frac{1}{1 + \frac{\log(1+2z)}{\log a}}, \quad z \geq 0. \tag{4.2}$$

Proof. For (4.1), let $s = \psi_a\,(t)$ for $t \geq 1$. Then $\varrho\,(s) = t$, $t \geq 1$, using (3.14) of lemma 3.2, we have $s = \psi_a\,(t) \geq \frac{1}{2}\,(t-1)^2$, so $t = \varrho\,(s) \leq \sqrt{2s} + 1$. By the definition of $\psi\,(t)$ we have

$$\begin{aligned} s &= \psi_a\,(t) = \psi_b\,(t) + \frac{t^2-1}{2} \leq \frac{t^2-1}{2} \\ &\Leftrightarrow 2s \leq t^2 - 1 \\ &\Leftrightarrow t = \varrho\,(s) \geq \sqrt{1+2s}. \end{aligned}$$

Thus

$$t = \varrho\,(u) \geq \sqrt{1+2s}.$$

For (4.2). To find the inverse function of the restriction of $\frac{-1}{2}\psi_a'\,(t)$ in the interval $(0, 1]$, we need to solve the equation $\frac{-1}{2}\psi_a'\,(t) = z$ for $t \in (0, 1]$. To do so, we have

$$2z = -\psi_a'\,(t) \Leftrightarrow -\left(t - a^{\left(\frac{1}{t}-1\right)}\right) = 2z.$$

This implies that

$$\begin{aligned} a^{\left(\frac{1}{t}-1\right)} &= t + 2z \leq 1 + 2z \\ &\Leftrightarrow \frac{1}{t} \leq 1 + \frac{\log\,(1+2z)}{\log a} \end{aligned}$$

Hence we have

$$t = \rho\,(z) \geq \frac{1}{1 + \frac{\log(1+2z)}{\log a}},$$

This completes the proof. □

Theorem 4.2 (Lemma 2.4 in [2].). *Assume that* ϱ *is defined as in Lemma* 4.1. *Then*

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right), \; v \in \mathbf{R}^n_{++}, \; \beta \geq 1.$$

Lemma 4.3. *Let* $0 \leq \theta < 1$ *and* $v_+ = \frac{v}{\sqrt{1-\theta}}$. *If* $\Psi(v) \leq \tau$ *then we have:*

$$\Psi(v_+) \leq \Psi(v) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left[n + 2\,\Psi(v) + 2\sqrt{2n\,\Psi(v)}\right], \tag{4.3}$$

Proof. For (4.3), using Lemma 3.3 with $\beta = \frac{1}{\sqrt{1-\theta}}$, and Lemma 3.2 (3.16), we obtain

$$\begin{aligned}
\Psi(v_+) &= \Psi(\beta v) = \sum_{i=1}^{n}\psi_a(\beta v_i) \leq \sum_{i=1}^{n}\left[\psi_a(v_i) + \frac{1}{2}\left(\beta^2 - 1\right)v_i^2\right] \\
&= \Psi(v) + \frac{1}{2}\left(\beta^2 - 1\right)\sum\nolimits_{i=1}^{n} v_i^2 \\
&= \Psi(v) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\|v\|^2 \\
&\leq \Psi(v) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(\sqrt{n} + \sqrt{2\,\Psi(v)}\right)^2 \\
&= \Psi(v) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(n + 2\,\Psi(v) + 2\sqrt{2n\,\Psi(v)}\right).
\end{aligned}$$

Since $\Psi(v) \leq \tau$, we have

$$\Psi(v_+) \leq \tau + \frac{\theta}{2(1-\theta)}\left(n + 2\tau + 2\sqrt{2n\tau}\right).$$

This completes the proof. □

Denote

$$\bar{\Psi}_0 = \frac{2\tau + n\theta + 2\theta\sqrt{2n\tau}}{2(1-\theta)}, \tag{4.4}$$

We'll utilize $\bar{\Psi}_0$ for the upper bounds of $\Psi(v)$ for large-update methods throughout the algorithm.

Remark 4.4. For the large-update method, by taking $\tau = \mathcal{O}(n)$, $\theta = \Theta(1)$, $\bar{\Psi}_0 = \mathcal{O}(n)$.

4.1 An Estimation of the Step Size

Lemma 4.5. *Let* $\delta(v)$ *be defined as in* (3.1).

$$\delta(v) \geq \sqrt{\frac{\Psi(v)}{2}}. \tag{4.5}$$

Proof. Using (3.14)

$$\begin{aligned}
\Psi(v) &= \sum_{i=1}^{n}\psi_a(v_i) \leq \sum_{i=1}^{n}\frac{1}{2}\left[\psi_a'(v_i)\right]^2 \\
&= \frac{1}{2}\|\nabla\,\Psi\|^2 = 2\delta(v)^2.
\end{aligned}$$

So that $\delta(v) \geq \sqrt{\frac{\Psi(v)}{2}}$. This finishes the proof. □

Remark 4.6. We always assume that $\tau \geq 1$. During this work, we use Lemma 4.5 and the assumption $\Psi(v) \geq \tau \geq 1$ we have

$$\delta(v) \geq \sqrt{\frac{1}{2}}.$$

In preparation for the next discussion, the difference in proximities between a new iterate and a current iterate for fixed μ is defined, for $\alpha > 0$.

$$f(\alpha) = \Psi(v_+) - \Psi(v).$$

Now, we give many lemmas that will be used to determine an appropriate lower bound for the step size α

Lemma 4.7 (Lemma 4.3 in [2]). *Let $\delta(v)$ be defined as in (3.1), then the largest possible value of the step size of α is given by*

$$\bar{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta).$$

Lemma 4.8 (Lemma 4.4 in [2]). *One has*

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

Define

$$\tilde{\alpha} = \frac{1}{\psi''(\rho(2\delta))}. \tag{4.6}$$

Then, $\tilde{\alpha} \leq \bar{\alpha}$ in the next step, we made this the default value for step size in Algorithm 1.

Lemma 4.9. *Let ρ and $\bar{\alpha}$ defined in Lemma 4.7 if $\Psi(t) \geq \tau \geq 1$, then we have*

$$\bar{\alpha} \geq \frac{1}{8\delta \log a \left[1 + \frac{\log(1+4\delta)}{\log a}\right]^2}.$$

Proof. As a result, by substituting $t = \rho(2\delta)$, which is equivalent to $4\delta = -\psi_a'(t)$, we get

$$\frac{1}{t} \leq 1 + \frac{\log(1+4\delta)}{\log a}.$$

Furthermore, using the, defintion of $\psi_a''(t)$ for $z = 2\delta$ and (4.2), we conclude that

$$\begin{aligned} \bar{\alpha} &\geq \frac{1}{\psi_a''(\rho(2\delta))} \\ &= \frac{1}{1 + \frac{\log a}{(\rho(2\delta))^2} a^{\left(\frac{1}{\rho(2\delta)} - 1\right)}} \\ &\geq \frac{1}{1 + \log a\,(4\delta + 1)\left[\frac{\log(4\delta+1)}{\log a} + 1\right]^2}. \end{aligned}$$

Using Remark 2, one has

$$\begin{aligned} \bar{\alpha} &\geq \frac{1}{\sqrt{2}\delta \log a + \left(1 + \frac{\log(1+4\delta)}{\log a}\right)^2 \left(4\delta + \sqrt{2}\delta\right) \log a} \\ &\geq \frac{1}{2\delta \log a + \left(1 + \frac{\log(1+4\delta)}{\log a}\right)^2 (4\delta + 2\delta) \log a}. \end{aligned}$$

This implies that

$$\bar{\alpha} \geq \frac{1}{8\delta \left(1 + \frac{\log(1+4\delta)}{\log a}\right)^2 \log a}.$$

This completes the proof. □

Denoting

$$\tilde{\alpha} = \frac{1}{8\delta\left[1+\frac{\log(1+4\delta)}{\log a}\right]^2\log a}, \tag{4.7}$$

Lemma 4.10. *If the step size $\tilde{\alpha}$ in as (4.6) Then we have*

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))}. \tag{4.8}$$

Lemma 4.11. *Let $\tilde{\alpha}$ be as defined in (4.7) and $\Psi(v) \geq 1$. Then we have the following upper bound for $f(\tilde{\alpha})$:*

$$f(\tilde{\alpha}) \leq -\frac{\sqrt{\Psi}}{16\log a\left[1+\frac{\log\left(1+2\sqrt{\Psi_0}\right)}{\log a}\right]^2}. \tag{4.9}$$

Proof. According to Lemma 10, with $\alpha = \tilde{\alpha}$ and (4.7), we have

$$\begin{aligned} f(\tilde{\alpha}) &\leq -\tilde{\alpha}\delta^2 \\ &= -\frac{\delta^2}{8\delta\log a\left[1+\frac{\log(1+4\delta)}{\log a}\right]^2} \leq -\frac{\sqrt{\Psi}}{16\log a\left[1+\frac{\log\left(1+2\sqrt{\Psi_0}\right)}{\log a}\right]^2}. \end{aligned}$$

This proves the theorem. □

Lemma 4.12 (Proposition 1.3.2 in [15]). *Suppose that a sequence $\{t^k > 0,\ k = 0,1,2,...,K\}$ is satisfying the following inequality:*

$$t_{k+1} \leq t_k - \eta t_k^{1-\gamma}, \quad k = 0,1,2,...,K-1,$$

where $\eta > 0$ and $\gamma \in (0,1]$. Then $K \leq \left\lceil \frac{t_0^\gamma}{\eta\gamma} \right\rceil$.

(4.8) shows the diminution of every inner iteration. In [15] we may obtain the proper values of η and $\gamma \in (0,1]$.

$$\eta = \frac{1}{16\log a\left(1+\frac{\log\left(1+2\sqrt{\Psi_0}\right)}{\log a}\right)^2}, \quad \gamma = \frac{1}{2}.$$

Theorem 4.13. *Let $\bar{\Psi}_0$ be defined as in (4.4) and let L be the total number of inner iterations in the outer iteration for large-update methods. We have*

$$L \leq 32\log a\left(1+\frac{\log\left(1+2\sqrt{\bar{\Psi}_0}\right)}{\log a}\right)^2 \bar{\Psi}_0^{\frac{1}{2}},$$

Proof. By Lemma 4.12 and Theorem 4.2, we have

$$L \leq \frac{\bar{\Psi}_0^\gamma}{\eta\gamma} = 32\log a\left(1+\frac{\log\left(1+2\sqrt{\bar{\Psi}_0}\right)}{\log a}\right)^2 \bar{\Psi}_0^{\frac{1}{2}}.$$

This completes the proof. □

The number of outer iterations is bounded above by $\frac{\log\frac{n}{\epsilon}}{\theta}$ (see [17] Lemma II.17, page116). By multiplying the number of outer iterations by the number of inner iterations, we get an upper bound for the total number of iterations, which is

$$\left\lceil 32\log a\left(\frac{\log a+\log\left(1+2\sqrt{\bar{\Psi}_0}\right)}{\log a}\right)^2 \bar{\Psi}_0^{\frac{1}{2}}\frac{1}{\theta}\log\frac{n}{\epsilon}\right\rceil, \text{ for large -update methods.}$$

For large-update methods, set $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$
By choosing

$$a = 1 + 2\left(\frac{n\theta + 2\tau + 2\sqrt{2n\tau}}{2(1-\theta)}\right)^{\frac{1}{2}}$$

The iteration bound reduces to $\mathcal{O}\left(\sqrt{n}(\log n)\log n\right)$, which matches the currently best-known iteration bound for large-update IPMs.

5 Numerical results

In this section, the principal objective is to compare the number of iterations and the time produced by the algorithm for certain kernel functions to validate the efficacy of our proposed kernel function, where the experiments are conducted using **Dev-Cpp 5.11 TDM-GCC 4.9.2**. Install and run on a computer. We take the accuracy parameter $\epsilon = 10^{-6}$, a threshold parameter $\tau = \sqrt{n}$, barrier update

$\theta \in \{0.1, 0.5, 0.7, 0.9, 0.95, 0.99\}$, various values for the barrier parameter a and the practical value for step size $\alpha_{pra} = \rho \min(\alpha_x, \alpha_s)$ with $\alpha \in (0,1)$ and

$$\alpha_x = \min\begin{cases} \frac{-x_i}{\Delta x_i} & \text{if } \Delta x_i < 0 \\ 1 & \text{else} \end{cases}, \quad \alpha_s = \min\begin{cases} \frac{-s_i}{\Delta s_i} & \text{if } \Delta s_i < 0 \\ 1 & \text{else} \end{cases}$$

We assume that **It** and **T** represent the number of iterations and the time produced by our algorithm, respectively. Recall that a pair of primal-dual linear optimization (LO) problems is defined as

$$(P) \qquad \min\left\{c^T x : Ax = b,\ x \geq 0\right\},$$

with its dual problem

$$(D) \quad \max\left\{b^T y : A^T y + s = c,\ s \geq 0\right\},$$

The results of our numerical comparisons between the $\psi_a(t)$ function described in (3.2) and other existing kernel functions in the literature are mentioned above with different sizes

$$\begin{aligned}
\textbf{1-}\ \psi_1(t) &= t^2 - 1 - \log(t) + \frac{t^{-p} - 1}{p},\ p = \frac{\log n}{2} - 1\ [7] \\
\textbf{2-}\ \psi_2(t) &= t^2 - t - \frac{t^{-p+1} - 1}{p+1} p > 1\ \ [6] \\
\textbf{3-}\ \psi_3(t) &= t^2 - 1 - \frac{t^{-2p+1} - 1}{-2p+1} - \frac{t^{-p+1} - 1}{-p+1}, \quad p > 1\ \ [4] \\
\textbf{4-}\ \psi_{cl}(t) &= \frac{t^2 - 1}{2} - \log t\ \ [11] \\
\textbf{5-}\ \psi_4(t) &= p\frac{t^2 - 1}{2} + \frac{4}{\pi}\left(e^{p\left(\tan\left(\frac{\pi}{2+2t}\right)\right)-1} - 1\right), \quad p \geq 1\ \ [13] \\
\textbf{6-}\ \psi_5(t) &= (p+1)t^2 - \frac{1}{t^p} - (p+2)t, \quad p > 4\ \ [8] \\
\textbf{7-}\ \psi_a(t) &= \frac{t^2 - 1}{2} - \int_1^t a^{\left(\frac{1}{x} - 1\right)} dx\ \ a \geq e,\ \text{New kernel function.}
\end{aligned}$$

Where $\psi_{cl}(t)$ is the classical logarithmic kernel function.

Now, we present some problems.

For $m = 2$ and $n = 4$, the matrix A and the two vectors b and c are defined as follows

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0.5 \end{pmatrix}^T, \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$$

and the initial feasible iterate is given by $\quad x^0 = \begin{pmatrix} 0.5 & 1 & 1 & 1 \end{pmatrix}^T,$

$y^0 = \begin{pmatrix} 0.25 & 0.25 \end{pmatrix}^T, \quad s^0 = \begin{pmatrix} 0.5 & 1.75 & 3 & 4.75 \end{pmatrix}^T$

For $m = 3$ and $n = 5$, the matrix A and the two vectors b and c are defined as follows

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 8 & 7 & 3 \end{pmatrix}^T, \quad c = \begin{pmatrix} 4 & 5 & 2 & 2 & 2 \end{pmatrix}^T$$

and the initial feasible iterate is given by $\quad x^0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T,$

$y^0 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \quad s^0 = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 \end{pmatrix}^T$ The results are summarized in the following table.

The kernel functions		$\theta = 0.1$		$\theta = 0.5$		$\theta = 0.7$		$\theta = 0.9$		$\theta = 0.95$		$\theta = 0.99$	
		It	**T**	**It**	**T**	**It**	**T**	**It**	**T**	**It**	**T**	**It**	**T**
PROBLEM 1	$\psi_1(t)$	192	1.17	51	1.10	41	1.07	20	0.79	17	0.71	16	0.83
	$\psi_a(t)$	**172**	**0.90**	**41**	**0.80**	**29**	**0.76**	**17**	**0.51**	**15**	**0.48**	**15**	**0.32**
	$\psi_2(t)$	191	1.35	41	0.84	30	0.82	17	0.76	16	0.67	16	0.65
	$\psi_3(t)$	229	1.30	63	1.15	55	1.02	23	0.86	19	0.83	18	0.79
	$\psi_{cl}(t)$	176	1.04	45	0.97	35	0.84	18	0.66	18	0.63	17	0.58
	$\psi_4(t)$	186	1.23	45	0.89	30	0.86	20	0.82	18	0.76	16	0.67
	$\psi_5(t)$	217	1.84	60	0.97	51	0.95	22	0.76	21	0.75	19	0.64
PROBLEM 2	$\psi_1(t)$	204	1.33	42	1.03	38	0.94	24	0.91	22	0.87	16	0.78
	$\psi_a(t)$	**182**	**0.95**	**42**	**0.86**	**31**	**0.71**	**21**	**0.70**	**20**	**0.64**	**15**	**0.55**
	$\psi_2(t)$	208	1.55	45	0.95	29	0.80	21	0.78	19	0.75	17	0.63
	$\psi_3(t)$	274	1.54	76	1.42	54	1.26	31	1.02	20	0.83	19	0.76
	$\psi_{cl}(t)$	185	1.12	42	0.97	31	0.91	22	0.81	20	0.75	16	0.72
	$\psi_4(t)$	214	1.17	51	1.10	48	1.00	29	0.97	19	0.80	18	0.74
	$\psi_5(t)$	243	1.76	67	1.58	56	1.26	25	0.91	20	0.84	19	0.78

Table 1. NUMERICAL RESULTS OF OUR TWO PROBLEMS.

We consider the Bouafiaâ's problem [5] with only minor changes. Thus, we have $n = 2m$ and the matrix A of the problem are given by

$A_{ij} = \begin{cases} 1 & \text{if } j = i \text{ or } j = i + m \\ 0 & \text{else} \end{cases}$.

At this stage, we distingue two cases

Case 1. $b = \begin{pmatrix} 2 & \dots & 2 \end{pmatrix}^T$, $c_i = -1$ if $i = 1, \dots, m$, $c_i = 0$ if $i = m + 1, \dots, n$, $s_i^0 = 1$ if $i = 1, \dots, m$, $s_i^0 = 2$ if $i = m + 1, \dots, n$, $y^0 = \begin{pmatrix} -2 & \dots & -2 \end{pmatrix}^T$, $x_i^0 = 1$ if $i = 1, \dots, n$ and the optimal solution is $y^* = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^T$, $s_i^* = 0$, $x_i^* = 2$ if $i = 1, \dots, m$ and $s_i^* = 1$, $x_i^* = 0$ if $i = m + 1, \dots, n$

Case 2. $b = \begin{pmatrix} 2 & \dots & 2 \end{pmatrix}^T$, $c_i = -1$ if $i = 1, \dots, m$, $c_i = 0$ if $i = m + 1, \dots, n$, $s^0 = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^T$, $y^0 = \begin{pmatrix} -2 & \dots & -2 \end{pmatrix}^T$, $x_i^0 = 1.5$ if $i = 1, \dots, m$, $x_i^0 = 0.5$ if $i = m + 1, \dots, n$ and the optimal solution is $s^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \end{pmatrix}^T$, $y^* = \begin{pmatrix} -1 & \dots & -1 \end{pmatrix}^T$ and $x_i^* = 1.4793$ if $i = 1, \dots, m$, $x_i^* = 0.5207$ if $i = m + 1, \dots, n$

The kernel functions			$\theta = 0.1$		$\theta = 0.5$		$\theta = 0.7$		$\theta = 0.9$		$\theta = 0.95$		$\theta = 0.99$	
			It	**T**	**It**	**T**	**It**	**T**	**It**	**T**	**It**	**T**	**It**	**T**
CASE 1	$\psi_1(t)$	$m = 5$	207	0.89	56	0.74	35	0.67	17	0.50	15	0.43	14	0.36
		$m = 10$	213	0.97	68	0.81	40	0.76	19	0.69	18	0.51	17	0.45
		$m = 15$	224	1.23	48	1.08	46	0.96	21	0.89	20	0.71	18	0.56
		$m = 20$	236	1.46	59	1.25	52	1.19	25	0.97	23	0.85	20	0.68
		$m = 25$	243	1.93	103	1.67	67	1.10	30	1.01	27	0.84	25	0.73
	$\psi_a(t)$	$m = 5$	**170**	**0.89**	**36**	**0.86**	**27**	**0.49**	**16**	**0.43**	**13**	**0.40**	**12**	**0.38**
		$m = 10$	**188**	**1.09**	**39**	**0.98**	**30**	**0.90**	**18**	**0.65**	**15**	**0.56**	**14**	**0.43**
		$m = 15$	**196**	**1.16**	**45**	**1.04**	**38**	**1.07**	**20**	**0.67**	**19**	**0.62**	**16**	**0.53**
		$m = 20$	**209**	**1.34**	**58**	**1.22**	**42**	**1.16**	**24**	**0.84**	**22**	**0.73**	**20**	**0.66**
		$m = 25$	**225**	**1.72**	**97**	**1.51**	**56**	**1.08**	**28**	**0.96**	**25**	**0.78**	**23**	**0.69**
	$\psi_2(t)$	$m = 5$	179	0.96	39	0.89	30	0.58	19	0.51	17	0.47	15	0.43
		$m = 10$	190	1.12	45	1.03	34	0.96	20	0.67	18	0.60	17	0.51
		$m = 15$	204	1.24	56	1.15	40	0.98	22	0.86	20	0.76	18	0.60
		$m = 20$	216	1.48	62	1.31	45	1.20	22	1.05	21	0.84	20	0.78
		$m = 25$	238	1.85	1.4	1.69	65	1.12	26	1.02	25	0.87	24	0.82
	$\psi_3(t)$	$m = 5$	171	1.02	38	0.98	28	0.57	17	0.46	16	0.43	14	0.40
		$m = 10$	194	1.17	41	1.12	34	0.96	19	0.70	18	0.60	15	0.45
		$m = 15$	203	1.20	56	1.12	40	1.08	21	0.68	20	0.65	18	0.57
		$m = 20$	214	1.41	61	1.26	45	1.19	25	0.81	23	0.77	20	0.66
		$m = 25$	225	2.77	99	1.58	62	1.31	29	1.00	26	0.89	24	0.68
	$\psi_{cl}(t)$	$m = 5$	182	1.09	42	0.95	30	0.62	21	0.54	19	0.46	15	0.40
		$m = 10$	194	1.14	45	1.03	36	0.94	24	0.71	20	0.56	16	0.44
		$m = 15$	202	1.18	50	1.05	41	0.99	26	0.94	23	0.87	17	0.71
		$m = 20$	214	1.37	64	1.25	56	1.20	27	0.96	25	0.84	21	0.86
		$m = 25$	245	1.76	103	1.58	67	1.36	29	0.99	27	1.91	24	0.94
	$\psi_4(t)$	$m = 5$	177	0.90	38	0.87	29	0.76	17	0.50	15	0.43	14	0.40
		$m = 10$	190	1.11	40	1.02	32	0.95	19	0.72	16	0.58	15	0.46
		$m = 15$	203	1.19	47	1.08	40	1.04	21	0.75	20	0.65	17	0.61
		$m = 20$	216	1.38	61	1.25	45	1.18	25	0.88	23	0.75	20	0.68
		$m = 25$	227	2.71	99	1.58	57	1.10	29	0.98	26	0.80	24	0.72
	$\psi_5(t)$	$m = 5$	210	1.13	57	1.06	34	0.94	22	0.67	19	0.56	15	0.46
		$m = 10$	219	1.25	63	1.11	40	0.99	24	0.87	21	0.67	17	0.52
		$m = 15$	230	1.40	71	1.15	47	1.10	23	0.97	21	0.80	21	0.79
		$m = 20$	244	1.50	78	1.39	56	1.26	26	1.19	26	0.94	23	0.81
		$m = 25$	252	2.01	109	1.76	70	1.23	32	1.25	29	0.99	27	0.87

CASE 2	$\psi_1(t)$	$m=5$	227	0.97	68	0.81	39	0.78	19	0.55	17	0.48	15	0.40
		$m=10$	239	1.37	83	1.13	42	0.86	23	0.64	21	0.55	20	0.54
		$m=15$	254	1.53	88	1.48	50	1.03	24	0.95	23	0.74	20	0.69
		$m=20$	269	1.69	93	1.53	62	1.45	27	1.07	26	1.05	24	0.85
		$m=25$	273	2.39	130	1.91	74	1.60	31	1.24	29	1.19	28	0.96
	$\psi_a(t)$	$m=5$	**191**	**1.09**	**40**	**1.06**	**32**	**0.58**	**18**	**0.44**	**14**	**0.41**	**14**	**0.39**
		$m=10$	**208**	**1.39**	**45**	**1.28**	**34**	**0.86**	**20**	**0.69**	**18**	**0.52**	**17**	**0.45**
		$m=15$	**221**	**1.46**	**57**	**1.40**	**48**	**1.07**	**22**	**0.77**	**21**	**0.72**	**17**	**0.61**
		$m=20$	**239**	**1.64**	**72**	**1.52**	**58**	**1.34**	**25**	**1.04**	**24**	**0.83**	**22**	**0.75**
		$m=25$	**245**	**2.04**	**127**	**1.71**	**67**	**1.58**	**30**	**1.16**	**28**	**0.94**	**27**	**0.89**
	$\psi_2(t)$	$m=5$	211	1.05	54	0.95	32	0.86	21	0.68	18	0.52	16	0.48
		$m=10$	233	1.40	56	1.17	43	0.92	24	0.71	23	0.63	19	0.58
		$m=15$	246	1.65	61	1.51	51	1.09	26	1.01	27	0.84	20	0.73
		$m=20$	265	1.84	83	1.60	62	1.52	28	1.12	30	1.09	24	0.87
		$m=25$	281	2.43	129	1.94	80	1.79	32	1.29	35	1.21	29	0.99
	$\psi_3(t)$	$m=5$	191	1.25	44	1.08	36	0.79	21	0.66	19	0.53	17	0.41
		$m=10$	224	1.57	51	1.29	47	0.94	26	0.72	25	0.64	18	0.55
		$m=15$	239	1.71	66	1.42	54	1.26	31	0.84	26	0.82	20	0.75
		$m=20$	244	1.99	73	1.65	61	1.41	35	1.08	28	0.89	25	0.81
		$m=25$	252	2.14	133	1.86	74	1.68	37	1.24	29	0.99	28	0.93
	$\psi_{cl}(t)$	$m=5$	200	1.12	52	1.09	34	0.61	24	0.47	22	0.44	18	0.39
		$m=10$	214	1.40	54	1.34	39	0.96	27	0.77	23	0.56	20	0.46
		$m=15$	225	1.58	63	1.48	51	1.13	29	0.83	25	0.79	21	0.70
		$m=20$	240	1.73	77	1.55	62	1.41	32	1.07	28	0.88	26	0.84
		$m=25$	252	2.08	130	1.82	75	1.63	34	1.22	30	1.02	29	0.96
	$\psi_4(t)$	$m=5$	195	1.11	52	1.09	35	0.63	25	0.50	24	0.48	17	0.44
		$m=10$	211	1.43	55	1.32	39	1.30	29	0.91	27	0.83	20	0.69
		$m=15$	231	1.64	58	1.48	46	1.43	30	1.11	28	0.97	21	0.71
		$m=20$	256	1.82	86	1.65	52	1.58	33	1.13	30	0.85	24	0.80
		$m=25$	279	2.17	130	1.83	70	1.65	35	1.27	32	0.98	29	0.92
	$\psi_5(t)$	$m=5$	244	1.30	72	1.26	75	0.84	25	0.76	21	0.61	16	0.49
		$m=10$	255	1.56	77	1.39	62	0.98	28	0.82	25	0.70	20	0.58
		$m=15$	278	1.92	98	1.52	77	1.26	32	0.94	28	0.95	22	0.73
		$m=20$	315	2.17	115	1.78	85	1.50	49	1.16	31	1.10	26	0.94
		$m=25$	323	2.67	177	2.23	127	1.82	61	1.46	35	1.25	30	0.99

Table 2. NUMERICAL RESULTS OF THE TWO CASES OF PROBLEM 3.

We consider the following problem with $n = 2m$ and the matrix A of the problem is given by

$$A_{ij} = \begin{cases} -2 & \text{if } j = i \text{ or } j = i+m \\ 0 & \text{else} \end{cases}.$$

$b = \begin{pmatrix} 4 & \dots & 4 \end{pmatrix}^T$, $c_i = 3$ if $i = 1, \dots, m$, $c_i = 2$ if $i = m+1, \dots, n$, we choose the strictly feasible initial point as $s_i^0 = 8$ if $i = 1, \dots, m$, $s_i^0 = 5$ if $i = m+1, \dots, n$, $y^0 = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^T$, $x_i^0 = 1$ if $i = 1, \dots, n$ we finish our algorithm with the following solution $y^* = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}^T$, $s_i^* = 0$, $x_i^* = 4$ if $i = 1, \dots, m$ and $s_i^* = 0$, $x_i^* = 3$ if $i = m+1, \dots, n$

Comments

The results of our numerical studies demonstrate the efficacy of our novel efficient kernel function. We see that when the problem dimension grows more significant, the difference in the number of inner iterations and computation time between our novel kernel function, that of Bounibane and Djeffal [7], Boudjellal et al. [6], Benhadid and Merahi [4], El Ghami et al.[11], Z. Moaberfardithatand al. [13] Djeffal and Laouar[8] and that of Bouafia et al. [5] become significant. These numerical results support and reinforce our theoretical findings.

6 Concluding Remarks

Motivated by recent works of Bai et al.[2], we present a novel kernel function that generalizes the kernel function in [2] and establishes a primal-dual IPM for LO problems, which reduces the algorithm's iteration complexity. We have shown that it yields the best possible iteration bounds for large-update methods for a function with a double barrier term. The algorithm produces the iteration bounds $\mathcal{O}\left(\sqrt{n}\log n \log \frac{n}{\epsilon}\right)$, which is currently the best-known iteration bound for such methods for large methods. In addition, we have provided some numerical results to demonstrate the validity of our approach by comparing methods based on distinct kernel functions. According to our numerical results, the new kernel function performed better than the others. Future research could concentrate on the examination of semidefinite optimization.

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