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Periodic solutions for a class of seventh-order differential equations

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Abstract The objective of this paper is to investigate sufficient conditions for the existence of periodic solutions of seventh-order differential equation

$$x^{(7)} + (\alpha^2 + \beta^2 + 1)x^{(5)} + (\alpha^2(\beta^2 + 1) + \beta^2)\ddddot{x} + (\alpha\beta)^2\dot{x} = \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, x^{(5)}, x^{(6)}),$$

where $\alpha,\ \beta$ are rational numbers different from $-1, 0, 1$, and $\alpha \neq \pm\beta$ with ε sufficiently small, and F is a nonlinear autonomous function. Moreover we provide some applications.

1 Introduction and statement of the main results

In the qualitative theory of ordinary differential equations, one of the main problems is the study of their limit cycles. A limit cycle of a differential equation is a periodic orbit isolated from the set of all periodic orbits of the differential equation. There are several theories and methods for studying limit cycles; one of the most important perturbative methods is the averaging theory.

In [11], the authors studied the limit cycles of the following third-order differential equation

$$\dddot{x} - \mu\ddot{x} + \dot{x} - \mu x = \varepsilon F(x, \dot{x}, \ddot{x}),$$

where μ and ε are real parameters, ε is a small and the function $F \in \mathcal{C}^2$ is a nonlinear autonomous function.

In [10], the authors studied the limit cycles of the following fourth-order differential equation

$$\ddddot{x} + (1 + p^2)\ddot{x} + p^2 x = \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}),$$

where p is a rational number different from 0, ε is a small real parameter, and F is a nonlinear autonomous function.

In [5], the authors studied the limit cycles of the following fifth-order differential equation

$$x^{(5)} - \lambda\ddddot{x} + (1 + p^2)\dddot{x} - \lambda(1 + p^2)\ddot{x} + p^2\dot{x} - \lambda p^2 x = \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}),$$

where λ and ε are real parameters, p is rational number different from $-1, 0, 1$, ε is a small enough, and $F \in \mathcal{C}^2$ is a nonlinear autonomous function.

In [6], the authors studied the limit cycles of the following fifth-order differential equation

$$x^{(6)} + (p^2 + q^2 + 1)\ddddot{x} + (p^2 + q^2 + p^2q^2)\ddot{x} + p^2q^2 x = \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, x^{(5)}),$$

where p and q are rational numbers different from $1, 0, 1$, and $p \neq q$, ε is a small enough parameter, and $F \in \mathcal{C}^2$ is a nonlinear autonomous function.

The objective of this paper is to apply the averaging theory to studying the periodic solutions for a class of seventh-order autonomous ordinary equation

$$x^{(7)} + (\alpha^2+\beta^2+1)x^{(5)} + (\alpha^2(\beta^2+1)+\beta^2)\dddot{x} + (\alpha\beta)^2\dot{x} = \varepsilon F(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x},x^{(5)},x^{(6)}), \quad (1.1)$$

where $\alpha,\ \beta$ are rational numbers different from $-1,0,1$, and $\alpha \neq \pm\beta$, with ε sufficiently small, and F is a nonlinear autonomous function.

There are many paper that studied seventh-order differential equations in different ways. Thus, our class of equations is not far from the ones studied in [7, 9, 16]. But our main tool for studying the periodic orbits of equation (1.1) is completely different to the tools of the mentioned papers, and consequently the results obtained seem distinct and new. We shall use the averaging theory, more precisely Theorem 2.1. Many of the quoted papers dealing with the periodic orbits of differential equations use Schauder's or Leray–Schauder's fixed point theorem, see for instance [2, 8], the non-local reduction method or variational methods see [1]. For studying the periodic solutions of Lotka–Volterra system see [3, 13].

In general to obtain analytically periodic solutions of a differential system is a very difficult task, usually impossible see. Here with the averaging theory this difficult problem for the differential equation (1.1) is reduced to find the zeros of a non-linear function. We must say that the averaging theory for finding periodic solutions in general does not provide all the periodic solutions of the system. For more information about the averaging theory see section 2 and the references quoted there, and the books [15, 17].

Now, the main results for the periodic solutions of equation (1.1) are the following:

Theorem 1.1. *Assume that α,β are rational numbers different from $-1,0,1$ and $\alpha\neq\pm\beta$, in differential equation (1.1). For every simple zero $(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*,S_0^*)$ with $r_0^*>0$ solution of the system*

$$\mathcal{F}_i(r_0,Z_0,U_0,V_0,W_0,S_0)=0,\ i=\overline{1,6}, \quad (1.2)$$

satisfying

$$\det\left(\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4,\mathcal{F}_5,\mathcal{F}_6)}{\partial(r_0,Z_0,U_0,V_0,W_0,S_0)}\Big|_{(r_0,Z_0,U_0,V_0,W_0,S_0)=(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*,S_0^*)}\right)\neq 0, \quad (1.3)$$

where

$$\begin{aligned}
\mathcal{F}_1(r_0,Z_0,U_0,V_0,W_0,S_0) &= \tfrac{1}{2\pi\alpha^2}\int_0^{2\pi\alpha}\cos(\theta)F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\,d\theta,\\
\mathcal{F}_2(r_0,Z_0,U_0,V_0,W_0,S_0) &= -\tfrac{1}{2\pi\alpha^3}\int_0^{2\pi\alpha}\tfrac{\beta U_0\sin(\theta)-r_0\alpha\cos(\frac{\beta}{\alpha}\theta)}{r_0}F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\,d\theta,\\
\mathcal{F}_3(r_0,Z_0,U_0,V_0,W_0,S_0) &= \tfrac{1}{2\pi\alpha^3}\int_0^{2\pi\alpha}\tfrac{\beta Z_0\sin(\theta)-r_0\alpha\sin(\frac{\beta}{\alpha}\theta)}{r_0}F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\,d\theta,\\
\mathcal{F}_4(r_0,Z_0,U_0,V_0,W_0,S_0) &= -\tfrac{1}{2\pi\alpha^3}\int_0^{2\pi\alpha}\tfrac{W_0\sin(\theta)-r_0\alpha\cos(\frac{\theta}{\alpha})}{r_0}F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\,d\theta,\\
\mathcal{F}_5(r_0,Z_0,U_0,V_0,W_0,S_0) &= \tfrac{1}{2\pi\alpha^3}\int_0^{2\pi\alpha}\tfrac{Z_0\sin(\theta)-r_0\alpha\sin(\frac{\theta}{\alpha})}{r_0}F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\,d\theta,\\
\mathcal{F}_6(r_0,Z_0,U_0,V_0,W_0,S_0) &= \tfrac{1}{2\pi\alpha^2}\int_0^{2\pi\alpha}F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\,d\theta,
\end{aligned} \quad (1.4)$$

with

$$\begin{aligned}
A_1 &= -\tfrac{r\cos(\theta)}{(\alpha^2-1)(\alpha^2-\beta^2)\alpha^2}+\tfrac{Z}{(\beta^2-1)(\alpha^2-\beta^2)\beta^2}-\tfrac{V}{(\alpha^2-1)(\beta^2-1)}+\tfrac{S}{\alpha^2\beta^2},\\
A_2 &= \tfrac{r\sin(\theta)}{(\alpha^2-1)(\alpha^2-\beta^2)\alpha}-\tfrac{U}{(\beta^2-1)(\alpha^2-\beta^2)\beta}+\tfrac{W}{(\alpha^2-1)(\beta^2-1)},\\
A_3 &= \tfrac{r\cos(\theta)}{(\alpha^2-1)(\alpha^2-\beta^2)}-\tfrac{Z}{(\beta^2-1)(\alpha^2-\beta^2)}+\tfrac{V}{(\alpha^2-1)(\beta^2-1)},\\
A_4 &= -\tfrac{(\alpha^2\beta^2-\alpha)r\sin(\theta)+(\beta-\alpha^2\beta)U+(\alpha^2-\beta^2)W}{(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)},\\
A_5 &= -\tfrac{\alpha^2 r\cos(\theta)}{(\alpha^2-1)(\alpha^2-\beta^2)}+\tfrac{\beta^2 Z}{(\beta^2-1)(\alpha^2-\beta^2)}-\tfrac{V}{(\alpha^2-1)(\beta^2-1)},\\
A_6 &= \tfrac{(\alpha^3\beta^2-\alpha^3)r\sin(\theta)+(\beta^3-\alpha^2\beta^3)U+(\alpha^2-\beta^2)W}{(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)},\\
A_7 &= \tfrac{\alpha^4 r\cos(\theta)}{(\alpha^2-1)(\alpha^2-\beta^2)}-\tfrac{\beta^4 Z}{(\beta^2-1)(\alpha^2-\beta^2)}+\tfrac{V}{(\alpha^2-1)(\beta^2-1)},
\end{aligned} \quad (1.5)$$

the differential equation (1.1) has a periodic solution $x(t,\varepsilon)$ tending to the periodic solution

$$x_0(t) = -\frac{r_0\cos(t)}{(\alpha^2-1)(\alpha^2-\beta^2)\alpha^2}+\frac{Z_0}{(\beta^2-1)(\alpha^2-\beta^2)\beta^2}-\frac{V_0}{(\alpha^2-1)(\beta^2-1)}+\frac{S_0}{\alpha^2\beta^2}$$

of $x^{(7)}+(\alpha^2+\beta^2+1)x^{(5)}+(\alpha^2(\beta^2+1)+\beta^2)\dddot{x}+(\alpha\beta)^2\dot{x}=0$ *when* $\varepsilon\to 0$. *Note that this solution is periodic of period* $2\pi\alpha$.

Theorem 1.1 is proved in section 3. Its proof is based on the averaging theory for computing periodic orbits, see section 2. An application of Theorem 1.1 is given in the following corollary.

Corollary 1.2. *if* $f(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x},x^{(5)},x^{(6)}) = -x^2+1$, *then the differential system* (1.1) *with* $\alpha=2,\ \beta=3,$ *has six periodic solutions* $x_i(t,\varepsilon)$ *with* $i=1..6$ *tending to the periodic solutions*

$$\begin{aligned} x_1(t) &= \tfrac{690214}{488055}cos(t),\\ x_2(t) &= \tfrac{403163}{532200}cos(t)+\tfrac{84958338371891}{212745327291720},\\ x_3(t) &= \tfrac{10365}{15214}cos(t)-\tfrac{10688}{193932},\\ x_4(t) &= \tfrac{10365}{15214}cos(t)-\tfrac{37920534167429}{20199739916160},\\ x_5(t) &= \tfrac{135527}{308340}cos(t)-\tfrac{104112649759}{112115190960},\\ x_6(t) &= \tfrac{135527}{308340}cos(t)-\tfrac{104112649759}{112115190960}, \end{aligned} \tag{1.6}$$

of differential equation $x^{(7)}+14x^{(5)}+49\dddot{x}+36\dot{x}=0$ *when* $\varepsilon\to 0$.

Remark 1.3. In the case α and β are rational numbers different from $1,0,-1$, and $\alpha=\pm\beta$, then we cannot apply Theorem 2.1 for studying the periodic orbits.

2 Basic results on averaging theory

In this section, we present the basic result of the averaging theory that we will use to demonstrate the main results of this paper.

We provide the problem of studying T–periodic solutions of differential systems of the form

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x})+\varepsilon F_1(t,\mathbf{x})+\varepsilon^2F_2(t,\mathbf{x},\varepsilon), \tag{2.1}$$

with ε sufficiently small, the functions $F_0,F_1:\mathbb{R}\times\Omega\to\mathbb{R}^n$ and $F_2:\mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ are of class $\mathcal{C}^2$, $T-$periodic with respect to the first variable, and Ω is an open subset of $\mathbb{R}^n$. We suppose that for the unperturbed system

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}). \tag{2.2}$$

There exists a submanifold consisting of periodic solutions. The averaging theory provides a solution to this problem.

We express by $\mathbf{x}(t,\mathbf{z})$ the solution of system (2.2) such that $\mathbf{x}(0,\mathbf{z})=\mathbf{z}$. The linearized system of the unperturbed system (2.2) along a periodic solution $\mathbf{x}(t,\mathbf{z})$ is

$$\dot{\mathbf{y}} = D_xF_0(t,\mathbf{x}(t,\mathbf{z}))\mathbf{y}. \tag{2.3}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (2.3), and by $\xi:\mathbb{R}^k\times\mathbb{R}^{n-k}\mapsto\mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1,\dots,x_n)=(x_1,\dots,x_k)$.

Consider V as an open and bounded set with $Cl(V)\subset\Omega$, such that for each $\mathbf{z}\in Cl(V)$. In this context $\mathbf{x}(t,\mathbf{z})$ represents the periodic solution of the unperturbed system (2.2) with $\mathbf{x}(0,\mathbf{z})$. The set $Cl(V)$ is isochronous for the system (2.1); i.e., it is a set formed only by periodic orbits, all of them having the same period. The following result provides an answer to the problem of the bifurcation of T–periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z})$ that are contained in $Cl(V)$.

Theorem 2.1. *Assume that there exists an open and bounded set* V *with* $Cl(V)\subset\Omega$ *such that, for each* $\mathbf{z}\in Cl(V)$, *the solution* $\mathbf{x}(t,\mathbf{z})$ *is* T*–periodic, considering a function* $\mathcal{F}:Cl(V)\to\mathbb{R}^n$ *as defined by*

$$\mathcal{F}(\mathbf{z}) = \frac{1}{T}\int_0^T M_{\mathbf{z}}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z}))\,dt. \tag{2.4}$$

If there exists $a\in V$ *such that* $\mathcal{F}(a)=0$, *and* $\det((d\mathcal{F}/d\mathbf{z})(a))\neq 0$, *then there exists a* $T-$*periodic solution* $\varphi(t,\varepsilon)$ *to system* (2.1) *such that* $\varphi(0,\varepsilon)\to a$ *as* $\varepsilon\to 0$.

This theorem dates back to Malkin [12] and Roseau [14], for a shorter proof see Buică [4].

3 Proof of the results

Proof of Theorem 1.1. Introducing the variable $y = \dot{x},\ z = \ddot{x},\ u = \dddot{x},\ v = \ddddot{x},\ w = x^{(5)},\ s = x^{(6)}$ we can be written the seventh-order differential equation (1.1) as a first-order differential system defined in an open subset Ω of $\mathbb{R}^7$. Thus we have the differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = v, \\ \dot{v} = w, \\ \dot{w} = s, \\ \dot{s} = -(\alpha^2+\beta^2+1)w - (\alpha^2(\beta^2+1)+\beta^2)u - (\alpha\beta)^2 y + \varepsilon F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, x^{(5)}, x^{(6)}). \end{cases} \tag{3.1}$$

For $\varepsilon = 0$ the unperturbed system of (3.1) has a unique singular point, the origin. The eigenvalues of the linear part of this system are $\pm i,\ \pm i\alpha,\ \pm i\beta$ and 0. By the linear invertible transformation

$$(X, Y, Z, U, V, W, S)^T = B(x, y, z, u, v, w, s)^T,$$

where B given by

$$\begin{pmatrix} 0 & 0 & \beta^2 & 0 & \beta^2+1 & 0 & 1 \\ 0 & \alpha\beta^2 & 0 & \alpha(\beta^2+1) & 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 & 0 & \alpha^2+1 & 0 & 1 \\ 0 & \beta\alpha^2 & 0 & \beta(\alpha^2+1) & 0 & \beta & 0 \\ 0 & 0 & (\alpha\beta)^2 & 0 & \alpha^2+\beta^2 & 0 & 1 \\ 0 & (\alpha\beta)^2 & 0 & \alpha^2+\beta^2 & 0 & 1 & 0 \\ (\alpha\beta)^2 & 0 & (\alpha\beta)^2+\alpha^2+\beta^2 & 0 & \alpha^2+\beta^2+1 & 0 & 1 \end{pmatrix},$$

the differential system (3.1) becomes

$$\begin{cases} \dot{X} = -\alpha Y + \varepsilon G(X, Y, Z, U, V, W, S), \\ \dot{Y} = \alpha X, \\ \dot{Z} = \beta Z + \varepsilon G(X, Y, Z, U, V, W, S), \\ \dot{U} = -\beta U, \\ \dot{V} = -W + \varepsilon G(X, Y, Z, U, V, W, S), \\ \dot{W} = V, \\ \dot{S} = \varepsilon G(X, Y, Z, U, V, W, S), \end{cases} \tag{3.2}$$

where $G(X, Y, Z, U, V, W, S) = F(A_1, A_2, A_3, A_4, A_5, A_6, A_7)$.

Note that the linear part of the differential system (3.2) at the origin is in its real normal form of Jordan. We switch now from the cartesian variables (X, Y, Z, U, V) to the cylindrical variables (r, θ, Z, U, V) of $\mathbb{R}^6$, with $X = r\cos(\theta),\ Y = r\sin(\theta)$, and we find

$$\begin{cases} \dot{r} = \varepsilon\cos(\theta)G(r, \theta, Z, U, V, W, S), \\ \dot{\theta} = \alpha - \dfrac{\varepsilon}{r} sin(\theta)G(r, \theta, Z, U, V, W, S), \\ \dot{Z} = -\beta U + \varepsilon G(r, \theta, Z, U, V, W, S), \\ \dot{U} = \beta Z, \\ \dot{V} = -W + \varepsilon G(r, \theta, Z, U, V, W, S), \\ \dot{W} = V, \\ \dot{S} = \varepsilon G(r, \theta, Z, U, V, W, S). \end{cases} \tag{3.3}$$

Dividing by $\dot{\theta}$ the system (3.3) becomes

$$\begin{cases} \dfrac{dr}{d\theta} = \dfrac{\varepsilon}{\alpha}\cos(\theta)G + O(\varepsilon^2), \\ \dfrac{dZ}{d\theta} = -\dfrac{\beta U}{\alpha} - \varepsilon\dfrac{\beta U\sin(\theta) - \alpha r}{\alpha^2 r}G + O(\varepsilon^2), \\ \dfrac{dU}{d\theta} = \dfrac{\beta Z}{\alpha} + \varepsilon\dfrac{\beta Z\sin(\theta)}{\alpha^2 r}G + O(\varepsilon^2), \\ \dfrac{dV}{d\theta} = -\dfrac{W}{\alpha} - \varepsilon\dfrac{W\sin(\theta) - \alpha r}{\alpha^2 r}G + O(\varepsilon^2), \\ \dfrac{dW}{d\theta} = \dfrac{V}{\alpha} + \varepsilon\dfrac{V\sin(\theta)}{\alpha^2 r}G + O(\varepsilon^2), \\ \dfrac{dS}{d\theta} = \dfrac{\varepsilon}{\alpha}G + O(\varepsilon^2), \end{cases} \tag{3.4}$$

where $G = G(r,\theta,Z,U,V,W,S)$.

We shall apply Theorem 2.1 to the our differential system. We note that system (3.4) can be written as system (2.1) taking

$$\mathbf{x} = \begin{pmatrix} r \\ Z \\ U \\ V \\ W \\ S \end{pmatrix}, \ F_0(\theta,\mathbf{x}) = \begin{pmatrix} 0 \\ -\dfrac{\beta}{\alpha}U \\ \dfrac{\beta}{\alpha}Z \\ -\dfrac{1}{\alpha}W \\ -\dfrac{1}{\alpha}V \\ 0 \end{pmatrix}, \ F_1(\theta,\mathbf{x}) = \begin{pmatrix} \dfrac{\cos(\theta)}{\alpha}G \\ \dfrac{\beta U\sin(\theta) - \alpha r}{\alpha^2 r}G \\ \dfrac{\beta Z\sin(\theta)}{\alpha^2 r}G \\ \dfrac{W\sin(\theta) - \alpha r}{\alpha^2 r}G \\ \dfrac{V\sin(\theta)}{\alpha^2 r}G \\ 0 \end{pmatrix},$$

system (3.4) with $\varepsilon = 0$ has the $2\pi\alpha$-periodic solutions given by

$$\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \\ W(\theta) \\ S(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ Z_0\cos(\frac{\beta}{\alpha}\theta) - U_0\sin(\frac{\beta}{\alpha}\theta) \\ U_0\cos(\frac{\beta}{\alpha}\theta) + Z_0\sin(\frac{\beta}{\alpha}\theta) \\ V_0\cos(\frac{\theta}{\alpha}) - W_0\sin(\frac{\theta}{\alpha}) \\ W_0\cos(\frac{\theta}{\alpha}) + V_0\sin(\frac{\theta}{\alpha}) \\ S_0 \end{pmatrix},$$

for $(r_0, Z_0, U_0, V_0, W_0, S_0) \in \mathbb{R}$ with $r_0 > 0$. To look for the periodic solutions of our equation (3.4) we must calculate the zeros $\mathbf{z} = (r_0, Z_0, U_0, V_0, W_0, S_0)$ of the system $\mathcal{F}(\mathbf{z}) = 0$ where $\mathcal{F}(\mathbf{z})$ given in (2.4). The fundamental matrix $M(\theta)$ of the system (3.4) with $\varepsilon = 0$ along any periodic solution is

$$M(\theta) = M_{\mathbf{z}}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\frac{\beta}{\alpha}\theta) & -\sin(\frac{\beta}{\alpha}\theta) & 0 & 0 & 0 \\ 0 & \sin(\frac{\beta}{\alpha}\theta) & \cos(\frac{\beta}{\alpha}\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\frac{\theta}{\alpha}) & -\sin(\frac{\theta}{\alpha}) & 0 \\ 0 & 0 & 0 & \sin(\frac{\theta}{\alpha}) & \cos(\frac{\theta}{\alpha}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We calculate the $\mathcal{F}(\mathbf{z})$ is given by (2.4) we got that the system $\mathcal{F}(\mathbf{z}) = 0$ can be written as

$$\begin{pmatrix} \mathcal{F}_1(r,Z,U,V,W,S) \\ \mathcal{F}_2(r,Z,U,V,W,S) \\ \mathcal{F}_3(r,Z,U,V,W,S) \\ \mathcal{F}_4(r,Z,U,V,W,S) \\ \mathcal{F}_5(r,Z,U,V,W,S) \\ \mathcal{F}_6(r,Z,U,V,W,S) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.5}$$

we obtain

$$\begin{aligned}
\mathcal{F}_1(r_0,Z_0,U_0,V_0,W_0,S_0) &= \frac{1}{2\pi\alpha^2}\int_0^{2\pi\alpha} \cos(\theta) F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\, d\theta,\\
\mathcal{F}_2(r_0,Z_0,U_0,V_0,W_0,S_0) &= -\frac{1}{2\pi\alpha^3}\int_0^{2\pi\alpha} \frac{\beta U_0\sin(\theta) - r_0\alpha\cos(\frac{\beta}{\alpha}\theta)}{r_0} F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\, d\theta,\\
\mathcal{F}_3(r_0,Z_0,U_0,V_0,W_0,S_0) &= \frac{1}{2\pi\alpha^3}\int_0^{2\pi\alpha} \frac{\beta Z_0\sin(\theta) - r_0\alpha\cos(\frac{\beta}{\alpha}\theta)}{r_0} F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\, d\theta,\\
\mathcal{F}_4(r_0,Z_0,U_0,V_0,W_0,S_0) &= -\frac{1}{2\pi\alpha^3}\int_0^{2\pi\alpha} \frac{W_0\sin(\theta) - r_0\alpha\cos(\frac{\beta}{\alpha}\theta)}{r_0} F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\, d\theta,\\
\mathcal{F}_5(r_0,Z_0,U_0,V_0,W_0,S_0) &= \frac{1}{2\pi\alpha^3}\int_0^{2\pi\alpha} \frac{Z_0\sin(\theta) - r_0\alpha\cos(\frac{\beta}{\alpha}\theta)}{r_0} F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\, d\theta,\\
\mathcal{F}_6(r_0,Z_0,U_0,V_0,W_0,S_0) &= \frac{1}{2\pi\alpha^2}\int_0^{2\pi\alpha} F(A_1,A_2,A_3,A_4,A_5,A_6,A_7)\, d\theta,
\end{aligned}$$

where $A_1, A_2, A_3, A_4, A_5, A_6$ and A_7 as in the statement of Theorem 1.

The zeros $(r_0^*, Z_0^*, U_0^*, V_0^*, W_0^*, S_0^*)$ of system (3.5) with respect to the variables r_0, Z_0, U_0, V_0, W_0 and S_0 provide periodic orbits of system (3.4), with $\varepsilon \neq 0$ sufficiently small and, i.e. if the condition (1.3) is satisfied. Going back though the change of variable, for every simple zero of system (3.5) we obtain a $2\pi\alpha$-periodic solution $x(t,\varepsilon)$ of the differential equation (1.1) for $\varepsilon \neq 0$ sufficiently small such that $x(t,\varepsilon)$ tends to the periodic solution

$$x_0(t) = -\frac{r_0\cos(t)}{(\alpha^2-1)(\alpha^2-\beta^2)\alpha^2} + \frac{Z_0}{(\beta^2-1)(\alpha^2-\beta^2)\beta^2} - \frac{V_0}{(\alpha^2-1)(\beta^2-1)} + \frac{S_0}{\alpha^2\beta^2},$$

of $x^{(7)} + (\alpha^2+\beta^2+1)x^{(5)} + (\alpha^2(\beta^2+1)+\beta^2)\dddot{x} + (\alpha\beta)^2\dot{x} = 0$ when $\varepsilon \to 0$. Note that this solution is periodic of period $2\pi\alpha$.

This completes the proof of Theorem 1.1. □

Proof of corollary 1.2 . We have the equation

$$x^{(7)} + 14x^{(5)} + 49\dddot{x} + 36\dot{x} = \varepsilon(-x^2+1), \tag{3.6}$$

which corresponds to the case $\alpha = 2$, $\beta = 3$ and $F(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x},x^{(5)},x^{(6)}) = -x^2+1$. The functions $\mathcal{F}_k(r_0,Z_0,U_0,V_0,W_0,S_0)$ for $k = \overline{1,6}$ of Theorem 1.1 are

$$\begin{aligned}
\mathcal{F}_1(r_0,Z_0,U_0,V_0,W_0,S_0) &= -\tfrac{V_0}{4608} + \tfrac{W_0}{4608} - \tfrac{r_0S_0}{4320} - \tfrac{U_0W_0}{34560} - \tfrac{V_0Z_0}{34560},\\
\mathcal{F}_2(r_0,Z_0,U_0,V_0,W_0,S_0) &= \tfrac{8S_0Z_0r_0 - 9U_0^2V_0 - 135U_0V_0W_0 + 9U_0W_0Z_0 + 36V_0r_0^2}{207360r_0},\\
\mathcal{F}_3(r_0,Z_0,U_0,V_0,W_0,S_0) &= \tfrac{8S_0U_0r_0 + 9U_0V_0Z_0 + 135V_0W_0Z_0 - 9W_0Z_0^2 + 36W_0r_0^2}{207360r_0},\\
\mathcal{F}_4(r_0,Z_0,U_0,V_0,W_0,S_0) &= \tfrac{200S_0V_0r_0 - 5U_0V_0W_0 - 75V_0W_0^2 + 60V_0r_0^2 + 5W_0^2Z_0 + 4Z_0r_0^2}{345600r_0},\\
\mathcal{F}_5(r_0,Z_0,U_0,V_0,W_0,S_0) &= \tfrac{200S_0W_0r_0 + 5U_0V_0^2 + 4U_0r_0^2 + 75V_0^2W_0 - 5V_0W_0Z_0 - 60W_0r_0^2}{345600r_0},\\
\mathcal{F}_6(r_0,Z_0,U_0,V_0,W_0,S_0) &= -\tfrac{r_0^2}{14400} - \tfrac{Z_0^2}{518400} - \tfrac{U_0^2}{518400} - \tfrac{V_0^2}{2304} - \tfrac{W_0^2}{2304} - \tfrac{S_0^2}{2592} + \tfrac{1}{2}.
\end{aligned}$$

The system $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = \mathcal{F}_6 = 0$ has six real solutions with $r_0 > 0$ given by

$$\begin{aligned}
&(\tfrac{2760856}{32537}, 0, 0, 0, 0, 0),\ (\tfrac{403163}{8870}, -\tfrac{275198}{4131}, -\tfrac{92749}{61623}, \tfrac{104667}{27463}, \tfrac{101674}{4061}, \tfrac{69964}{5209}),\\
&(\tfrac{310950}{7607}, 0, -\tfrac{300149}{1646}, 0, \tfrac{248943}{12656}, \tfrac{106889}{5387}),\ (\tfrac{310950}{7607}, \tfrac{300149}{1646}, 0, \tfrac{248943}{12656}, 0, -\tfrac{106889}{5387}),\\
&(\tfrac{135527}{5139}, -\tfrac{74279}{9549}, \tfrac{74279}{9549}, \tfrac{743779}{32614}, \tfrac{743779}{32614}, 0),\ (\tfrac{135527}{5139}, -\tfrac{74279}{9549}, -\tfrac{74279}{9549}, \tfrac{743779}{32614}, -\tfrac{743779}{32614}, 0).
\end{aligned}$$

Since the Jacobian (1.3) for these solutions $(r_0^*, Z_0^*, U_0^*, V_0^*, W_0^*, S_0^*)$

$$(-4.845167479)10^{-14},\ (-7.096581955)10^{-13},$$
$$(4.037375667)10^{-13},\ (4.037375667)10^{-13},$$
$$(1.227210749)10^{-11},\ (1.227210749)10^{-11},$$

by Theorem 1.1 equation (3.6) has six periodic solutions tending to periodic solutions (1.6) given in the statement of the corollary 1.2. □

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