Palestine Journal of Mathematics

Vol 13(4)(2024) , 285–292

Skew derivations with homomorphism condition on anticommutators of prime rings

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Communicated by Ayman Badawi

MSC 2020 Classifications: 16W25, 16N20,16N60,16W10, 16R60,16Rxx.

Keywords and phrases: Prime ring, derivation, involution, maximal symmetric ring of quotients, functional identity, rings with polynomial identity.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract Suppose $\Re$ is a prime ring with char($\Re$) $\neq 2$. Further, $\Re$ has center $\mathcal{Z}(\Re)$ and extended centroid $\mathcal{C}$ whereas $\mathcal{Q}$ is the Utumi ring of quotients and Q_r is the Martindale ring of quotients. We ascertain that if a skew derivation of $\Re$ say $\mathcal{K}$ related with an automorphism η of $\Re$, fulfills homorphism condition on anticommuators, then $\mathcal{K}$ is either a zero map or $\Re$ is a commutative ring.

1 Introduction

For the purpose of our study, until the end of the paper, $\Re$ is a prime ring. For handling notation throughout our work, we call $\mathcal{Z}(\Re)$ as center of $\Re$, the Martindale ring of quotients is assigned as Q_r and Utumi ring of quotients as $\mathcal{Q}$ wherein $\mathcal{C} = \mathcal{Z}(\mathcal{Q}_r)$ as the center of $\mathcal{Q}_r$ termed as the extended centroid of $\Re$. These over rings Q_r, $\mathcal{Q}$ and $\mathcal{C}$ has illustrious properties already discussed in [3]. We confront the fact that if primeness prevails in $\Re$ it is prevalent in the Martindale ring of quotients as well. Here, in the present article, our study accomplish the discussion on certain additive maps under the impact of peculiar algebraic identities. A commutator or Lie product of $h, g \in \Re$ is $[h, g] = hg-gh$ and skew-commutator or Jordan product of $h, g \in \Re$ is $h \circ g = hg+gh$ and shall be utilized often in the present paper without recalling specifically each time. Note that $M_h(\mathcal{C})$ depicts a ring comprising of matrices of order $h \times h$ over $\mathcal{C}$ the extended centroid.

Throughout this paper, we shall employ the following operators say skew derivations and its particular form termed as inner skew derivation that behaves as central tool to our investigation. In this sequel, an additive map $\mathcal{K} : \Re \to \Re$ is called as skew derivation with an automorphism η adjoined with it, if $\mathcal{K}(xy) = \eta(x)\mathcal{K}(y) + \mathcal{K}(x)y$, holds for every $x, y \in \Re$. If automorphism η acts identically the skew derivation $\mathcal{K}$ is called as derivation. A skew derivation is inner skew derivation if $\mathcal{K}(x) = \theta x - \eta(x)\theta$, where $\theta \in \mathcal{Q}$.

A plethora of literature has been developed related with these operators associated with specific identities called as functional identity. The prime motives of these works are to render the ring structure as matrix ring over division ring or to reduce ring as commutative. Another objective is to characterize them as centralizer map or scalar map. Some unusual works in this vane include [6], [11], [12] and [20].

2 Motivation

The bird view of functional identities (FI's) will depict an identical relation wherein values of arbitrary functions are augmented with arbitrary elements of a ring. The newcomer can understand this by encountering a FI of the type $X(t)r + Y(r)t = 0$, for every $r, t \in \Re$ where X and Y are arbitrary functions. The motive of introducing functional identity are many. The FI theory are beneficial, if we wish to obtain the ring structure with certainty or the form of arbitrary functions involved therein. Despite an advanced theory, it has shortcomings of its own kind. The challenge here is to put our problem into the framework of standard FI theory. Once we reach this goal, we can find many conclusions with utmost strata of generality. Without doubt, one can find in FI, $X(t)r + Y(r)t = 0$, for every $r, t \in \Re$ where X and Y are arbitrary functions, $X = 0 = Y$ are solutions. But questions revolves in our mind about non-zero solutions to such FI's. Like, for a moment if we restrict our ring as commutative ring, then $X = -Y$ is a solution or if $\Re$ contains a central ideal I then $X(t) = -Y(t) = ct$, where $c \in I$ is still a solution.

In this paper also we study certain FI, take $\mathcal{K}(x) \circ \mathcal{K}(y) = \mathcal{K}(x \circ y)$ where $\mathcal{K}$ is a skew derivation on prime ring $\Re$. Here, we always find $\mathcal{K} = 0$ as a trivial solution but the question still moves in our mind that is it the only possible solution or we have any chance of other forms of $\mathcal{K}$ satisfying such FI? Our results discusses all possibilities of the form of $\mathcal{K}$ making use of differential identity theory.

Prestigiacomo [23] gave the characterization of generalized derivation satisfying special yet typical identities discussed above. In other words, the author paved a way to the development of the following theorem.

Theorem 2.1. *Let $\Re$ be a prime ring with char($\Re$) $\neq 2$ along with a non-central Lie ideal $\mathcal{P}$ of $\Re$. Suppose $\mathcal{Q}$ is its Martindale quotient ring and $\mathcal{C}$ is its extended centroid. If $\mathcal{W} : \Re \to \Re$ and $\mathcal{U} : \Re \to \Re$ be nonzero generalized derivations on $\Re$ such that $[\mathcal{W}(x), \mathcal{W}(y)]_k = \mathcal{U}([x, y]_k)$, for all $x, y \in \mathcal{P}$ and fixed positive integer k. Then there exists $\beta \in \mathcal{C}$ such that $\mathcal{U}(x) = \beta^{k+1}x$ and $\mathcal{W}(x) = \beta x$, for any $x \in \Re$, unless $\Re \subseteq M_2(\mathcal{C})$, where $\bar{\mathcal{C}}$ is the algebraic closure of $\mathcal{C}$.*

Ashraf et al. in their paper [[1], Theorem 2.1] established the result as following.

Theorem 2.2. *Let $\mathcal{I}$ be a non-zero ideal of a prime ring $\Re$ and char($\Re$) $\neq 2$. Consider $\mathcal{F}$ to be a generalized derivation associated with a non-zero derivation μ such that $(\mathcal{F}(x) \circ \mathcal{F}(y))^m = \mathcal{F}(x \circ_t y)$ stands true for every $x,\ y \in \mathcal{I}$, where m, t be the fixed positive integers, then $\Re$ is commutative.*

With this result at our disposal and Prestigiacomo's above theorem proved in [23], we have established the following theorem. Note that throughout the proof of the following theorem we assume char($\Re$) $\neq 2$.

Theorem 2.3. *Consider $\Re$ to be a prime ring and associated with $\Re$, $\mathcal{Q}$ is the Utumi ring of quotients and $\mathcal{C}$ is the extended centroid. Suppose $\mathcal{K}$ is a skew derivation and η is the automorphism of $\Re$ associated with $\mathcal{K}$. If $\mathcal{K}(x) \circ \mathcal{K}(y) = \mathcal{K}(x \circ y)$ holds for every $x,\ y \in \Re$. Then $\Re$ is commutative or $\mathcal{K} = 0$.*

The following remark is backbone to the proof of our main result in Theorem 2.4.

Remark 2.4. *The non-commutative standard polynomial $s_4(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ in four non-commuting indeterminates $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ is a polynomial of the form*

$$s_4 = s_4(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \sum_{t \in S_4} sgn(t)\zeta_{t(1)}\zeta_{t(2)}\zeta_{t(3)}\zeta_{t(4)}$$

and sgn(t) is the signature of the permutation t of S_4, where S_4 is the group of permutations. The above expression is an identity known as standard polynomial identity (PI) on a subset say $\mathcal{G}$ of $\Re$ if

$$s_4(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = 0, \ \text{whenever each } \zeta_i \to g_i, \text{ where } g_i \in \mathcal{G}$$

We will make frequent use of such standard polynomial defined as above.

Remark 2.5. *For a prime ring $\Re$, the extended centroid $\mathcal{C}$ of $\Re$ is notably a field also called as the field of quotients of $\mathcal{Z}(\Re)$. Let $Y = \{y_1, y_2, \cdots\}$, be the set consisting of the non-commuting indeterminates say $y_1, y_2, \cdots$ which are countable. Let $\mathcal{C}\{Y\}$ be the free $\mathcal{C}$ algebra of the set Y. Consider $\mathcal{Q}\{Y\} = \mathcal{Q} *_{\mathcal{C}} \mathcal{C}\{Y\}$, the free $\mathcal{C}$-product of $\mathcal{Q}$ and $\mathcal{C}\{Y\}$. The elements of $\mathcal{Q}\{Y\}$ are called the GP (the generalized polynomials). By a nontrivial GP, we mean a non-vanishing element of $\mathcal{Q}\{Y\}$. Every element $w \in \mathcal{Q}\{Y\}$ is of the peculiar form $w = \zeta_o w_1 \zeta_1 w_2 \zeta_2 \cdots w_n \zeta_n$,* *where $\{\zeta_o, \cdots, \zeta_n\} \subseteq \mathcal{Q}$ and $\{w_1, \cdots, w_n\} \subseteq Y$, is called a monomial where $\zeta_o, \cdots, \zeta_n$ are called the coefficients of w. Each $g \in \mathcal{Q}\{Y\}$ constitutes of such monomials as a finite sum. Such representation is easily seen to be not unique. If we have a GPI, $\Gamma(\eta(u_i), \mathcal{K}(v_j))$ equipped with automorphism η and skew derivation $\mathcal{K}$ acting as unary operators on distinct indeterminates u_i, v_i then if η is X-outer automorphism and $\mathcal{K}$ is X-outer skew derivation, then the GPI $\Gamma(\eta(u_i), \mathcal{K}(v_j))$ is reduced to the form of $\Gamma(h_i, g_i)$ where h_i, g_i are distinct indeterminates. For a thorough inquiry see [7], [10].*

For a lucid explanation of the notion of a non-triviality of a GPI, let us look at the following simple example.

Example: Let Π be the ring of real quaternions. Then, for every $\gamma \in \Pi$, where $\gamma = \gamma_o + \gamma_1 i + \gamma_2 j + \gamma_3 k$ and $\gamma_p \in \mathbb{R}$, for every $p \in \{0, 1, 2, 3\}$, the identity $(\gamma i)^2 - (i\gamma)^2 + (\gamma j)^2 - (j\gamma)^2 + (\gamma k)^2 - (k\gamma)^2 = 0$ called the non-trivial GPI satisfied by Π.

Remark 2.6. *Let $\mathcal{A}$ be a two-sided ideal of $\Re$. Then $\mathcal{A}$, $\Re$ and $\mathcal{Q}$ all the three satisfy the same GPI with coefficients from $\mathcal{Q}$. See [[7], Theorem 2] to corroborate this statement. Also, $\mathcal{A}$, $\Re$ and $\mathcal{Q}$ all the three satisfy the same GPI with coefficients from $\mathcal{Q}$ and a single automorphism.*

Remark 2.7. *Let $\Re$ be a prime ring with extended centroid $\mathcal{C}$, then the following conditions are equivalent:*

(a) *The linear space $\Re\mathcal{C}$ over $\mathcal{C}$ has atmost four dimension;*

(b) *The standard identity $s_4(x_1, \cdots, x_4)$ is satisfied by $\Re$;*

(c) *For a certain field $\mathcal{F}$, $\Re$ embeds in $M_2(\mathcal{F})$ or $\Re$ is commutative;*

(d) *$\Re$ is algebraic of bounded degree two;*

(e) *$\Re$ satisfies the PI $[[x^2, y], [x, y]] = 0$.*

Remark 2.8. *Recall that, in case char($\Re$) $= 0$, an automorphism η of Q is called Frobenius if $\eta(p) = p$ for all $p \in \mathcal{C}$. Moreover, in case char($\Re$) $= q \geq 2$, an automorphism η of Q is called Frobenius if there exists a fixed integer t such that $\eta(z) = z^{q^t}$ for all $z \in \mathcal{C}$. In [[8], Theorem 2] Chuang proves that if $\pi(z_i, \eta(z_i))$ is a generalized polynomial identity for $\Re$, where $\Re$ is a prime ring and $\eta \in Aut(\Re)$ which is not Frobenius, then $\Re$ also satisfies the non-trivial GPI $\pi(z_i, w_i)$, where z_i and w_i are distinct indeterminates.*

Remark 2.9. Let $\Re$ be a domain and $\eta \in \mathrm{Aut}(\Re)$ be an automorphism of $\Re$ which is outer. In [16], Kharchenko proved that if $\phi(x_i, \eta(x_i))$ is a generalized polynomial identity for $\Re$, then $\Re$ also satisfies the non-trivial generalized polynomial identity $\phi(x_i, y_i)$, where x_i, y_i are distinct indeterminates.

3 Proof of Theorem 2.3

We divide the proof of this theorem into two cases.

- **$\mathcal{K}$ is inner skew derivation.**

Here, in this situation when $\mathcal{K}$ is a skew inner derivation that is $\mathcal{K}$ will assume the form $\mathcal{K}(x) = \theta x - \eta(x)\theta$, for every $x \in \Re$ due to certain $a \in \mathcal{Q}$ and $\eta \in \mathrm{Aut}(\Re)$. If η is inner then there exists unit $\epsilon \in \mathcal{Q}$ such that $\eta(t) = \epsilon t \epsilon^{-1}$ for every $t \in \Re$. Then by our hypothesis, we have $\mathcal{K}(x) \circ \mathcal{K}(y) = \mathcal{K}(x \circ y)$ stands true for every $x,\ y \in \Re$. Further, we observe that $\Re$ satisfies the GPI

$$\nabla(y, z) = (\theta y - \eta(y)\theta) \circ (\theta z - \eta(z)\theta) - \theta(y \circ z) + \eta(y \circ z)\theta.$$

For the sake of establishment of our main results, we commence our proof with the following prerequisite results.

Proposition 3.1. *Suppose $\Re$ is a prime ring and $\mathcal{K}$ is an inner skew derivation induced by the element $\theta \in \mathcal{Q}$ and associated with inner automorphism η such that following relation, $(\theta y - \eta(y)\theta) \circ (\theta z - \eta(z)\theta) = \theta(y \circ z) - \eta(y \circ z)\theta$, stands true for every $y, z \in \Re$, then either $\mathcal{K} = 0$ or $\Re$ is commutative.*

Proof. Our hypothesis here prompts $\Re$ to satisfy the following GPI

$$\nabla(y, z) = (\theta y - \eta(y)\theta) \circ (\theta z - \eta(z)\theta) - \theta(y \circ z) + \eta(y \circ z)\theta. \tag{3.1}$$

Beidar [[2], Theorem 2] renders the status of $\nabla(y, z)$ to be a GPI in $\mathcal{Q}$. In the context when $\mathcal{C}$ has infinite cardinality, then $\nabla(y, z) = 0$ for every $y, z \in \mathcal{Q} \otimes \overline{\mathcal{C}}$, where we take $\overline{\mathcal{C}}$ as the closure of extended centroid $\mathcal{C}$ in the algebraic sense. It is sincerely observed that both $\mathcal{Q} \otimes \overline{\mathcal{C}}$ and $\mathcal{Q}$ are centrally closed (owing to [[13], Theorems 2.5 and 3.5]). Henceforth, we may replace $\Re$ by $\mathcal{Q}$ or $\mathcal{Q} \otimes \overline{\mathcal{C}}$, as per the situation of $\mathcal{C}$ to be of finite or infinite cardinality. Thus we may suppose that $\Re$ is centrally closed over $\mathcal{C}$ which is either finite or closed in the algebraic sense. If $\epsilon^{-1}\theta \in \mathcal{C}$, then we arrive at the asserted conclusion that $\mathcal{K} = 0$. Henceforth, we proceed with the advantage that $\epsilon^{-1}\theta \notin \mathcal{C}$. Then by the powerful result of Chuang in [7], $\nabla(y, z)$ is a nontrivial GPI for $\Re$. Now, with the aid of Martindales Theorem from [21], $\Re$ is taken to be a primitive ring with nonzero socle $\mathcal{H}$ where $\mathcal{C}$ is the related division ring. In this sequel, a result due to Jacobson [[14], pg 75] yields that $\Re$ can be viewed as a dense ring of linear transformations on some vector space $\mathcal{V}$ over $\mathcal{C}$. Then, recalling the density of $\Re$ on $\mathcal{V}$, we have $\Re \cong M_k(\mathcal{C})$. In pretext to our assumption $dim_{\mathcal{C}}(\mathcal{V}) \geq 1$.

• **Automorphism η is identity.**

That is $\eta(z) = z$ for every $z \in \Re$. Owing to Ashraf et al [[1], Theorem 2.1], we get the required result.

• **Automorphism η is inner but non-identity.**

Hence we have $\eta(x) = \epsilon x \epsilon^{-1}$. Let us first deal with the case of $\dim_{\mathcal{C}}(\mathcal{V}) \geq 2$. For any $u \in \mathcal{V}$, we first show that the vectors $0 \neq u$ and $\epsilon^{-1}\theta u$ are linearly $\mathcal{C}$-dependent. In this view, we suppose that for certain non-zero u, the set $\{u, \epsilon^{-1}\theta u\}$ is linearly $\mathcal{C}$-independent and show that a contradiction is triggered. The suitable exploitation of the density of $\Re$ guarantees the existence $y, z \in \Re$ and when $\epsilon^{-1}u \notin span\{u, \epsilon^{-1}\theta u\}$, then $\{\epsilon^{-1}u, u, \epsilon^{-1}\theta u\}$ is $\mathcal{D}$-linearly Independent. Then consider

$$yu = 0, \ zu = 0 \text{ and } y\epsilon^{-1}\theta u = 0, \ z\epsilon^{-1}\theta u = \epsilon^{-1}u \text{ and } y\epsilon^{-1}u = u, \ z\epsilon^{-1}u = \epsilon^{-1}u.$$

For the smooth handling of action of u on our GPI, we take

$$\nabla_1(y, z) = (\theta y - \eta(y)\theta)(\theta z - \eta(z)\theta), \ \nabla_2(y, z) = (\theta z - \eta(z)\theta)(\theta y - \eta(y)\theta), \ \nabla_3(y, z) = \theta(y \circ z) - \eta(y \circ z)\theta.$$

Finally we have expressed the generalized polynomial (3.1) in the following simple form

$$\nabla = \nabla_1 + \nabla_2 - \nabla_3 \tag{3.2}$$

Right multiplying by u in relation (3.2), we get $(\nabla_1 + \nabla_2)u - \nabla_3 u = 0$. That is, $\epsilon u = 0$, a contradiction to the fact that u is non-zero.

on considering $\epsilon^{-1}u \in span\{u, \epsilon^{-1}\theta u\}$ we have $\epsilon^{-1}u = \zeta u + \lambda\epsilon^{-1}\theta u$ for certain $\zeta, \lambda \in \mathcal{D}$. Owing to the density of $\Re$ we have the following relation as below

$$yu = 0, \ zu = 0, \ y\epsilon^{-1}\theta u = \lambda\epsilon^{-1}u, \ z\epsilon^{-1}\theta u = \lambda\epsilon^{-1}u. \text{ We see } y\epsilon^{-1}u = \lambda^2\epsilon^{-1}u, \ z\epsilon^{-1}u = \lambda^2\epsilon^{-1}u.$$

Right multiplying by u in relation (3.2), we get $(\nabla_1 + \nabla_2)u - \nabla_3 u = 0$. That is, $2(\lambda^2 + \lambda^3)u = 0$ since $u \neq 0$, we have $\lambda^2 + \lambda^3 = 0$. Thus, $\lambda = 0$ or λ =-1.

◇ **When $\lambda = 0$.** Suppose $yu = 0, \ zu = 0$ and $y\epsilon^{-1}\theta u = 2^{-1}\epsilon^{-1}u, \ z\epsilon^{-1}\theta u = 2\epsilon^{-1}u$. One can observe that $y\epsilon^{-1}u = 0, \ z\epsilon^{-1}u = 0$. Right multiplying by u in relation (3.2), we get $(\nabla_1 + \nabla_2)u - \nabla_3 u = 0$. That is, $2u = 0$ since $u \neq 0$, we have a contradiction. Therefore,

◇ **When λ =-1.** Recurrent use of Density Theorem allows us to pick y, z as $yu = 0, \ zu = 0$ and $y\epsilon^{-1}\theta u = 2\epsilon^{-1}u, \ z\epsilon^{-1}\theta u = \epsilon^{-1}u$. One can observe that $y\epsilon^{-1}u = 2\lambda\epsilon^{-1}u, \ z\epsilon^{-1}u =$

$\lambda\epsilon^{-1}u$. Right multiplying by u in relation (3.2), we get $2u = 0$ since $u \neq 0$, we have a contradiction.

Hence the vectors u and $\epsilon^{-1}\theta u$ are linearly $\mathcal{C}$-dependent for every $u \in \mathcal{V}$. It is easy consequence that $\epsilon^{-1}\theta u = cu$ where c is a fixed element from $\mathcal{C}$ irrespective of the choice of u from [[4], Lemma 7.1]. Further, assume that for $r \in \Re$ and $u \in \mathcal{V}$, we have $0 = \mathcal{K}(z)\mathcal{V} = (\theta z - \epsilon z\epsilon^{-1}\theta)\mathcal{V}$ as $\mathcal{V}$ is faithfully, we have $\mathcal{K}(z) = 0$. This tempts a contradiction. □

We now deduce some results pertaining to any automorphism associated with a skew derivation.

Proposition 3.2. *Suppose $\Re$ is a prime ring and $\mathcal{K}$ be an inner skew derivation induced by element $\theta \in \mathcal{Q}$ and associated with automorphism η. If $\mathcal{K}(y) \circ \mathcal{K}(z) = \mathcal{K}(y \circ z)$ stands true for every $y, z \in \Re$ and fixed positive integers $m > 1$ then either $\mathcal{K} = 0$ or $\Re$ is commutative.*

Proof. Our hypothesis here prompts $\Re$ to satisfy the following GPI

$$\nabla(y, z) = (\theta y - \eta(y)\theta) \circ (\theta z - \eta(z)\theta) - \theta(y \circ z) + \eta(y \circ z)\theta. \tag{3.3}$$

• **Suppose η is inner.**

Due to Proposition (3.1) our assertion follows.

• **Suppose η is not inner.**

By remark 2.6, $\mathcal{Q}$ satisfies (3.3). Thus by [8], the ring $\mathcal{Q}$ satisfies a non trivial GPI. By Martindales Theorem [21], we have, $\mathcal{Q}$ is primitive ring having non-zero socle and related with division ring $\mathcal{D}$ which is finite dimensional over $\mathcal{C}$. Hence, $\mathcal{Q}$ is isomorphic to a dense subring of linear transformation of a vector space $\mathcal{V}$ over $\mathcal{D}$ and constitute of non-zero finite rank linear transformations.

Let us prioritize the discussion of $\dim_{\mathcal{D}}(\mathcal{V}) \geq 2$.

(i) **Recall that whenever η is non Frobenius.**

By remark 2.8, the relation (3.3) is reduced as

$$\nabla(y, z) = (\theta y - w\theta) \circ (\theta z - h\theta) - \theta(y \circ z) + (w \circ h)\theta. \tag{3.4}$$

For any $v \in \mathcal{V}$, we first show that the vectors $0 \neq v$ and bv are linearly $\mathcal{C}$-dependent. In this view, we suppose that for certain non-zero v, the set $\{v, \theta v\}$ is linearly $\mathcal{C}$-independent and show that a contradiction is triggered. The suitable exploitation of the density of $\Re$, guarantees the existence $y, \gamma, z, h \in \Re$, so that the following relation stands true

$$yv = 0,\ \gamma v = 0,\ zv = 0,\ hv = 0 \text{ and}$$

$$y\theta v = -v,\ \gamma\theta v = -v,\ z\theta v = -v,\ h\theta v = -v \text{ and}$$

Right multiplying by v in relation (3.4) we get $2v = 0$, a contradiction to the fact that v is non-zero. Hence the vectors v and θv are linearly $\mathcal{C}$-dependent for every $v \in \mathcal{V}$. It is easy consequence that $\beta v = \theta v$ where β is fixed element from $\mathcal{C}$ irrespective of the choice of v from [[4], Lemma 7.1]. Further, assume that for $u \in \Re$ and $v \in \mathcal{V}$, we have $[\theta, u]v = \theta(uv) - u(\theta v) = \beta uv - u(\beta v) = 0$. Hence $[u, \theta]\mathcal{V} = 0$, as $[u, \theta]$ is a linear transformation that acts faithfully on the vector space $\mathcal{V}$. Therefore, $[u, \theta] = 0$, for every $u \in \Re$. Thus $\theta \in \mathcal{C}$, a contradiction as relation (3.4) is reduced into a PI and on putting $w = h = 0$ in relation (3.4), we have

$$(\theta^2 - \theta)(y \circ z) = 0$$

and following well versed technique we encounter a contradiction. Hence, $\Re$ is commutative or $\mathcal{K} = 0$.

(ii) **suppose η is Frobenius.**

A quick look confirms that char($\Re$) $\neq 0$. So, take char($\Re$) $= q > 0$. such that $\eta(\gamma) = \gamma^{q^t}$, for every $\gamma \in \mathcal{C}$ and fixed positive integer t. Hence, for certain $\zeta \in \mathcal{C}$, $\zeta^{q^t} \neq \zeta$ or $\zeta^{q^t-1} \neq 1$. Moreover, by [[14], pg 79], there exists a semi-linear automorphism $N \in End(\mathcal{V})$ such that $\eta(z) = NzN^{-1}$ for every $z \in \mathcal{Q}_r$. Hence, $\mathcal{Q}_r$ satisfies

$$\nabla(y,z) = (\theta y - NyN^{-1}\theta) \circ (\theta z - NzN^{-1}\theta) - \theta(y \circ z) + N(y \circ z)N^{-1}\theta. \tag{3.5}$$

Consider the case of $\dim_{\mathcal{D}}(\mathcal{V}) \geq 2$. For any $u \in \mathcal{V}$, we first show that the vectors $0 \neq u$ and $N^{-1}\theta u$ are linearly $\mathcal{D}$-dependent. In this view, we suppose that for certain non-zero u, the set $\{u, N^{-1}\theta u\}$ is linearly $\mathcal{D}$-independent and show that a contradiction is triggered.

and when $N^{-1}u \notin span\{u, N^{-1}\theta u\}$, then $\{N^{-1}u, u, N^{-1}\theta u\}$ is $\mathcal{D}$-linearly Independent. Then consider $yu = 0,\ zu = 0,\ yN^{-1}\theta u = 0,\ zN^{-1}\theta u = N^{-1}u$ and $yN^{-1}u = u,\ zN^{-1}u = N^{-1}u$. Suppose for smooth handling

$$\nabla_1(y,z) = (\theta y - \eta(y)\theta)(\theta z - \eta(z)\theta) \text{ and } \nabla_2(y,z) = (\theta z - \eta(z)\theta)(\theta y - \eta(y)\theta).$$

Further, we also consider

$$\nabla_3(y,z) = \theta(y \circ z) - \eta(y \circ z)\theta.$$

Finally we have expressed the generalized polynomial (3.1) discussed above in the following simple form,

$$\nabla = \nabla_1 + \nabla_2 - \nabla_3 \tag{3.6}$$

Right multiplying by u in relation (3.6), we get $(\nabla_1 + \nabla_2)u - \nabla_3 u = 0$. That is, $Nu = 0$, a contradiction to the fact that u is non-zero.

When $N^{-1}u \in span\{u, N^{-1}\theta u\}$ then $N^{-1}u = \zeta u + \varrho N^{-1}\theta u$ for certain $\zeta, \varrho \in \mathcal{D}$. Owing to the density of $\Re$ we have the following relation as below;

$$yu = 0,\ zu = 0 \text{ and } yN^{-1}\theta u = \varrho N^{-1}u,\ zN^{-1}\theta u = \varrho N^{-1}u.$$

One can observe that $yN^{-1}u = \varrho^2 N^{-1}u,\ zN^{-1}u = \varrho^2 N^{-1}u$. Right multiplying by u in relation (3.6), we get $2(\varrho^2 + \varrho^3)u = 0$ as $u \neq 0$, $\varrho^2 + \varrho^3 = 0$. Thus, $\varrho = 0$ or $\varrho = -1$.

$\diamondsuit$ **When $\varrho = 0$.**

Suppose $yu = 0,\ zu = 0$ and $yN^{-1}\theta u = 2^{-1}N^{-1}u,\ zN^{-1}\theta u = 2N^{-1}u$. One can observe that $yN^{-1}u = 0,\ zN^{-1}u = 0$. Right multiplying by u in relation (3.6), we get $(\nabla_1 + \nabla_2)u - \nabla_3 u = 0$. That is, $2u = 0$ since $u \neq 0$, we have a contradiction.

$\diamondsuit$ **When $\varrho = -1$.**

Again applying Density Theorem allows us to pick y, z as $yu = 0,\ zu = 0$ and $yN^{-1}\theta u = 2N^{-1}u,\ zN^{-1}\theta u = N^{-1}u$. One can observe that $yN^{-1}u = 2\varrho N^{-1}u,\ zN^{-1}u = \varrho N^{-1}u$. Right multiplying by u in relation 3.6, we get $2u = 0$ since $u \neq 0$, we have oa contradiction.

Hence the vectors u and $N^{-1}\theta u$ are linearly $\mathcal{C}$-dependent for every $u \in \mathcal{V}$. It is easy consequence that $N^{-1}\theta u = cu$ where c is a fixed element from $\mathcal{C}$ irrespective of the choice of u from [[4], Lemma 7.1]. Further, assume that for $r \in \Re$ and $u \in \mathcal{V}$, we have $0 = \mathcal{K}(z)\mathcal{V} = (\theta z - NzN^{-1}\theta)\mathcal{V}$ as $\mathcal{V}$ is faithful, we have $\mathcal{K}(z) = 0$.

We are left with the only possible case of $dim_{\mathcal{D}}(\mathcal{V}) = 1$. For the situation when $\mathcal{C}$ is finite $\mathcal{D}$ is also finite and hence a field by Wedderburn's Theorem. Thus, $\mathcal{Q} \cong \mathcal{D}$ that is $\mathcal{Q}$ is commutative or $\Re$ is commutative.

In case $\mathcal{C}$ is infinite and since $dim_{\mathcal{D}}(\mathcal{V}) = 1$ we have $\mathcal{Q} \cong \mathcal{D}$, a domain. After employing remark 2.9 we arrive at the following GPI

$$\nabla(y,z) = \{(\theta y - \gamma\theta)o(\theta z - h\theta)\} - \theta(yo_m z) + (\gamma o_m h)\theta. \tag{3.7}$$

Place $\gamma = z = 0$ in above GPI (3.7), we attain the following

$$\nabla(y, z) = \{(\theta y \circ h\theta)\} = 0. \tag{3.8}$$

As the above identity is true for $\mathcal{Q}$ which has no non-trivial nilpotents we have $\theta y \circ h\theta = 0$ for every $y, h \in \mathcal{Q}$. Use $y = h = 1 \in \mathcal{Q}$, we have $2\theta^2 = 0$ or $\theta = 0$. This establishes that $\mathcal{K} = 0$. □

- **General case of skew derivation.**

In this segment of the proof, we begin by considering that $\mathcal{K}$ is a skew derivation associated with automorphism η. In an attempt to develop our main theorem, we assume for certain $\eta \in Aut(\Re)$ and related skew derivation μ of $\Re$, under the pivotal assumption of the article $\mathcal{K}(x) \circ \mathcal{K}(y) = \mathcal{K}(x \circ y)$ stands true for every $x, y \in \Re$. Under the impact of Kharchenko Theory (See [15]), we break our situation as follows.

• When μ is an inner skew derivation.

Then skew derivation is defined as $\mathcal{K}(w) = \theta w - \eta(w)\theta$ for every $w \in \Re$, $\theta \in \mathcal{Q}$ and associated automorphism η. Hence, by recalling Proposition (3.2), we are done.

• When μ is an outer skew derivation.

Due to our pivotal assumption, we have $\mathcal{K}(x) \circ \mathcal{K}(y) = \mathcal{K}(x \circ y)$ stands true for every $x, y \in \Re$ and can be rewritten as

$$\mathcal{K}(x) \circ \mathcal{K}(y) = \mathcal{K}(x) \circ y + \eta(x) \circ \mathcal{K}(y).$$

Thereafter, as $\mathcal{K}$ is outer, from the effect of Kharchenko result, the above relation gives the following

$$uov = u \circ y + \eta(x) \circ v$$

In particular, we put $y = x = 0$ in above relation, we have $u \circ v = 0$. Hence, $\Re$ is commutative.

After a good attempt on establishing the previous theorem readers may feel motivated to solve the following open problems.

Problem 1 Consider $\mathcal{P}$ to be the non-central Lie ideal of prime ring $\Re$ with char($\Re$) $\neq 2$. Let Utumi ring of quotients and the extended centroid of $\Re$ be $\mathcal{Q}$ and $\mathcal{C}$ respectively. Suppose $\Re$ admits a generalized skew derivation $\mathcal{K}$ associated with automorphism h and skew derivation $\mathcal{N}$ of $\Re$. Then

(i) if $(\mathcal{K}(x)o\mathcal{K}(y))^m = \mathcal{K}(xo_t y)$ holds for every $x,\ y \in \mathcal{P}$ with m, t be the fixed positive integers;

(ii) if $\mathcal{K}(x)o_m\mathcal{K}(y) = (\mathcal{K}(xoy))^n$ holds for every $x,\ y \in \mathcal{P}$ with m, n be the fixed positive integers;

(iii) if $(\mathcal{K}(x)o\mathcal{K}(y))^m = (\mathcal{K}(xoy))^n$ holds for every $x,\ y \in \mathcal{P}$ with m, n be the fixed positive integers.

Then what can be interpreted about the structure of ring or the form of $\mathcal{K}$.

Problem 2 Suppose $\mathcal{P}$ to be the non-central Lie ideal of a prime ring $\Re$ with char($\Re$) $\neq 2$. Consider associated with $\Re$, $\mathcal{Q}$ is the Utumi ring of quotients and $\mathcal{C}$ is the extended centroid. Let $\mathcal{K}$ be a X-generalized skew derivation with automorphism h and X-skew derivation $\mathcal{N}$ associated with $\mathcal{K}$. Then

(i) if $(\mathcal{K}(x) \circ \mathcal{K}(y))^m = \mathcal{K}(x \circ_t y)$ holds for every $x,\ y \in \mathcal{P}$ with m, t be the fixed positive integers;

(ii) if $\mathcal{K}(x) \circ_m \mathcal{K}(y) = (\mathcal{K}(x \circ y))^n$ holds for every $x,\ y \in \mathcal{P}$ with m, n be the fixed positive integers;

(iii) if $(\mathcal{K}(x) \circ \mathcal{K}(y))^m = (\mathcal{K}(x \circ y))^n$ holds for every $x,\ y \in \mathcal{P}$ with m, n be the fixed positive integers.

Then what can be interpreted about the structure of ring or the form of $\mathcal{K}$.

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Received: 2023-05-18
Accepted: 2024-02-06