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SOME REMARKS ON SIGNLESS LAPLACIAN OF STRONG POWER GRAPHS

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Abstract For a group G of order m, Singh and Manilal [2] conceptualized a new graph structure called strong power graph, denoted by $\mathscr{P}_S(G)$, as a graph with G as the vertex set in which a pair of distinct vertices v_i and v_j are adjacent if $v_i^{k_1} = v_j^{k_2}$ in G and k_1, k_2 are positive integers strictly less than m. In this paper, we have formulated the signless characteristic polynomial of $\mathscr{P}_S(G)$ for any finite group G of a given order. We have also characterized the cases where the signless laplacian is integral. We have evaluated the signless laplacian energy for the strong power graph for (i) any cyclic group of prime order and (ii) cyclic group of order m (where m is composite and is of the form $m = 2^r3^s; r, s > 1$). Also we have computed the number of spanning trees of $\mathscr{P}_S(\mathbb{Z}_m)$ when m is composite.

1 Introduction

Associating graphs to algebraic structures and investigating the graph theoretic properties using the algebraic properties of the parent structure has been a fascinating area of research during the past few decades. Consequently, numerous graph structures have evolved from various algebraic structures of which *power graph* is a noteworthy example. It was Kelarev and Quinn [9] who proposed the concept of *directed power graphs* for semigroups. Subsequently, Chakrabarty together with Sen and Ghosh [1] conceived the idea of undirected power graph $\mathscr{P}(S)$ associated with a semigroup S to be a graph with S as the vertex collection wherein a pair of different vertices v_i and v_j form an edge if $v_i = v_j^\alpha$ or $v_j = v_i^\beta$ for some $\alpha, \beta \in \mathbb{N}$. Several studies have been carried out in this regard and can be found in [3]. Over the years, Singh and Manilal [2] introduced the concept of *strong power graph* for finite groups. For more undefined terminologies regarding the spectral characterization of arbitrary graphs, the reader is referred to [10].

Analogous to the graph structure of the power graph, Singh and Manilal introduced the notion of *strong power graph* for any finite group. They defined strong power graph, denoted by $\mathscr{P}_S(G)$ as a graph with vertex set G in which a pair of distinct vertices v_i and v_j are adjacent if $v_i^{k_1} = v_j^{k_2}$ in G and k_1, k_2 are positive integers strictly less than m where k is the order of group G.

2 Preliminaries

Throughout this study, we consider only finite groups and simple graphs. Suppose that Γ is any graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Let v_i denote any arbitrary vertex in $V(\Gamma)$ and d_i denote its corresponding degree. The *degree* matrix corresponding to Γ denoted by $D(\Gamma)$ is the diagonal matrix with degrees of vertices along the main diagonal and the *adjacency* matrix denoted by $A(\Gamma)$ is the binary matrix with entries 1's whenever two vertices are adjacent and 0's elsewhere. Associated with the same graph Γ, there are several other matrix representations such as the *Signless Laplacian* matrix $Q(\Gamma)$, *Laplacian* matrix $L(\Gamma)$, etc. to name a few. Given any graph Γ, there is an empirical relation between all the matrices described above as demonstrated by:

$$L(\Gamma) = D(\Gamma) - A(\Gamma) \qquad \textit{and} \qquad Q(\Gamma) = D(\Gamma) + A(\Gamma) \tag{2.1}$$

For any matrix B, the characteristic polynomial of B is given by $det(\lambda I - B)$. In particular, if $B = Q(\Gamma)$, then we call $det(\lambda I - Q(\Gamma))$ as the *signless laplacian characteristic polynomial* [4] and denote it by $\varphi(Q(\Gamma); \lambda)$ or simply $\varphi(\Gamma; \lambda)$. The eigenvalues obtained from the signless laplacian matrix along with its multiplicities constitute the *Signless Laplacian spectrum* of Γ. The notion of *energy*, conceptualized by Ivan Gutman [5, 6] was initially defined for the adjacency matrix of a graph. Gutman further developed the concept of *signless laplacian energy* [7], denoted by $LE^+(\Gamma)$ given by $\sum_{i=1}^{p} |\lambda_i - \frac{2q}{p}|$ where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the signless laplacian eigenvalues obtained from $Q(\Gamma)$ and p, q denote the respective order and size of Γ.

Given any graph G with m vertices, McLeman and McNicholas [11] defined a graph parameter, namely the *coronal (or A-coronal)*, which is given by $\mathbf{1}_m^T(\lambda I_m - A(G))^{-1}\mathbf{1}_m$. Later, Shu-Yu Cui and Gui-Xian Tian [12] generalized the concept of coronal to any matrix B associated with G. Given a graph G with vertex set cardinality m and any matrix B associated with G, the *B-coronal* is given by $\Gamma_B(\lambda) = \mathbf{1}_m^T(\lambda I_m - B)^{-1}\mathbf{1}_m$ where $\mathbf{1}_m$ stands for the vector of order $m \times 1$ with each of its entries equal to one.

In this study, we have formulated the signless characteristic polynomial of the strong power graph $\mathscr{P}_S(\mathbb{Z}_m)$ in the cases where m is prime and where m is composite. Also, we have determined the signless characteristic polynomial of $\mathscr{P}_S(G)$ when G is non-cyclic. Also, we have analyzed the cases where the signless laplacian is integral (i.e., all the signless eigenvalues are integers). We have also reformulated the number of spanning trees of $\mathscr{P}_S(\mathbb{Z}_m)$ when m is composite. We have also evaluated the signless laplacian energy of the strong power graph in the following cases :

i. when m is prime

ii. when m is composite and is of the form $m = 2^r 3^s$.

Throughout the work, A, L, Q stands for its true meanings (the matrix representations) unless mentioned otherwise.

The following results are vital for the subsequent reading of the paper :

Proposition 2.1. *[12] Consider any r-regular graph G of order m, then the corresponding Q-coronal of G is*

$$\Gamma_{Q(G)}(\lambda) = \frac{m}{\lambda - 2r}$$

Proposition 2.2. *[13] Consider two arbitrary graphs H_1 and H_2 with orders m_1 and m_2 respectively. Then*

$$\varphi_Q(H_1 \vee H_2 \,;\, \lambda) = \varphi_Q(H_1 \,;\, \lambda - m_2) \cdot \varphi_Q(H_2 \,;\, \lambda - m_1) \cdot [1 - \Gamma_{Q(H_1)}(\lambda - m_2) \cdot \Gamma_{Q(H_2)}(\lambda - m_1)]$$

where $\varphi_Q(H \,;\, \lambda)$ represents the signless characteristic polynomial of H.

Proposition 2.3. *The signless characteristic polynomial of K_m is*

$$\varphi_Q(K_m \,;\, \lambda) = [\lambda - 2(m-1)][\lambda - (m-2)]^{m-1}$$

3 Signless Laplacian of Strong Power Graphs

At the outset, we deduce the signless characteristic spectrum of $Q(\mathscr{P}_S(\mathbb{Z}_m))$ when m is prime as demonstrated in the following theorem :

Theorem 3.1. *For every prime m,*

$$\varphi_Q(\mathscr{P}_S(\mathbb{Z}_m)\ ;\ \lambda) = \lambda(\lambda - 2(m-2))(\lambda - (m-3))^{(m-2)}$$

Proof. Let $V(\mathscr{P}_S(\mathbb{Z}_m)) = \{v_0, v_1, v_2, \dots, v_{m-1}\}$. Then by definition, the vertices v_i and v_j are adjacent for all $i, j \neq 0$. On the other hand, the vertices v_i and v_0 do not form an edge for any i. Thus, $\mathscr{P}_S(\mathbb{Z}_m) \cong K_{m-1} \cup K_1$. Indexing the rows and columns in the order of generators and finally the identity element of $\mathbb{Z}_m$, the signless laplacian matrix corresponding to $\mathscr{P}_S(\mathbb{Z}_m)$ is given by

$$Q(\mathscr{P}_S(\mathbb{Z}_m)) = \begin{bmatrix} m-2 & 1 & \dots & 1 & 0 \\ 1 & m-2 & \dots & 1 & 0 \\ \vdots & & & & \\ 1 & 1 & \dots & m-2 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{m \times m}$$

Then $\varphi(Q(\mathscr{P}_S(\mathbb{Z}_m)); \lambda)$ is determined by

$$\det(\lambda I_m - Q) = \begin{vmatrix} \lambda - (m-2) & -1 & \dots & -1 & 0 \\ -1 & \lambda - (m-2) & \dots & -1 & 0 \\ \vdots & & & & \\ -1 & -1 & \dots & \lambda - (m-2) & 0 \\ 0 & 0 & \dots & 0 & \lambda \end{vmatrix}_{m \times m}$$

$$= \lambda \begin{vmatrix} \lambda - (m-2) & -1 & \dots & -1 \\ -1 & \lambda - (m-2) & \dots & -1 \\ \vdots & & & \\ -1 & -1 & \dots & \lambda - (m-2) \end{vmatrix}_{m-1 \times m-1}$$

Applying the following elementary column and row operations :

(i) $C_1 \to C_1 + C_2 + \dots C_{m-1}$

(ii) $R_i \to R_i - R_1\ ;\ i = 2, 3, \dots, m-1$

we arrive at the desired result. □

Our immediate aim is to determine the signless laplacian spectrum of $Q(\mathscr{P}_S(\mathbb{Z}_m))$ when m is composite and is given by the following theorem :

Theorem 3.2. *For every composite m,*

$$\varphi_Q(\mathscr{P}_S(\mathbb{Z}_m); \lambda) = (\lambda - (m-2))^{m-\phi(m)-1}(\lambda - (m-3))^{\phi(m)-1}.f(\lambda)$$

where $f(\lambda) = \lambda^2 - 3m\lambda + \lambda\phi(m) + 5\lambda + 2m^2 - 2m\phi(m) - 8m + 6\phi(m) + 6$

Proof. Let $V(\mathscr{P}_S(\mathbb{Z}_m)) = \{v_0, v_1, v_2, \dots, v_{m-1}\}$. Let v_i, v_j, v_k and $v_0 \in V(\mathscr{P}_S(\mathbb{Z}_m))$ that corresponds to generators i and j, non-generator k and identity 0 in $\mathbb{Z}_m$. Then the following are true :

(i) v_i and v_j are adjacent.

(ii) Both v_i and v_j are adjacent to v_k (respectively).

(iii) v_k and v_0 are adjacent

(iv) Both v_i and v_j are not adjacent to v_0.

Since this is true for every generator and non-generator, $m - \phi(m) - 1$ vertices (non-generators of $\mathbb{Z}_m$) form an edge, one with each vertex of $\mathscr{P}_S(\mathbb{Z}_m)$. On the other hand, $\phi(m)$ vertices (generators of $\mathbb{Z}_m$) form an edge, one with each of the vertices in $\mathscr{P}_S(\mathbb{Z}_m)$ except v_0.

$$\text{Thus, } \mathscr{P}_S(\mathbb{Z}_m) \cong [K_{\phi(m)} \cup K_1] \vee K_{m-\phi(m)-1}$$

Let $H_1 = K_{\phi(m)} \cup K_1$ and $H_2 = K_{m-\phi(m)-1}$ be two graphs with $m_1 = \phi(m) + 1$ and $m_2 = m - \phi(m) - 1$ vertices. Then by Proposition 2.3,

$$\psi(Q(H_2); \lambda - m_1) = (\lambda - 2m + \phi(m) + 3)(\lambda - m + 2)^{m-\phi(m)-2}$$

Also, $\psi(Q(H_1); \lambda - m_2) = (\lambda - m + \phi(m) + 1)(\lambda - m - \phi(m) + 3)(\lambda - m + 3)^{\phi(m)-1}$

Since H_2 is regular with regularity $r = m - \phi(m) - 2$, by Proposition 2.1,

$$\Gamma_{Q(H_2)}(\lambda - m_1) = \frac{m - \phi(m) - 1}{\lambda - 2m + \phi(m) + 3}$$

Now to find $\Gamma_{Q(H_1)}(\lambda - m_2)$, we observe that $(\lambda - m_2)I_{m_1} - Q(H_1)=$

$$\left[\begin{array}{ccccc|c} \lambda-m+2 & -1 & -1 & \dots & -1 & -1 \\ -1 & \lambda-m+2 & -1 & \dots & -1 & -1 \\ -1 & -1 & \lambda-m+2 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & \lambda-m+2 & -1 \\ \hline 0 & 0 & 0 & \dots & 0 & \lambda-m+\phi(m)+1 \end{array}\right]_{(\phi(m)+1)\times(\phi(m)+1)} \tag{3.1}$$

$= X(\text{say})$

From (3.1), it follows that

$$\det X = \det B \cdot \det C = (\lambda - m + \phi(m) + 1) \cdot \det B \tag{3.2}$$

$$\text{where } B = \begin{bmatrix} \lambda-m+2 & -1 & -1 & \dots & -1 \\ -1 & \lambda-m+2 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & \lambda-m+2 \end{bmatrix}_{\phi(m)\times\phi(m)}$$

$$\text{and } C = \begin{bmatrix} \lambda - m + \phi(m) + 1 \end{bmatrix}_{1\times 1}$$

Applying the elementary column and row operations :

(i) $C_1 \to C_1 + C_2 + \ldots + C_{\phi(m)}$

(ii) $R_i \to R_i - R_1$; $i = 2, 3, \ldots \phi(m)$ on B , we get

$$\det B = (\lambda - m - \phi(m) + 3)(\lambda - m + 3)^{\phi(m)-1}$$

Thus equation 3.2 $\implies$ $\det X = (\lambda - m + \phi(m) + 1)(\lambda - m - \phi(m) + 3)(\lambda - m + 3)^{\phi(m)-1}$

Now to find the adjoint of X, we find the principal minors $X_{1,1}, X_{1,2}, \ldots, X_{\phi(m)+1,\phi(m)+1}$ where $X_{i,j}$ denotes the principal minor obtained after the deletion of i^{th} row and j^{th} column. By applying similar elementary row and column operations we get,

$$X_{i,i} = (\lambda - m + \phi(m) + 1)(\lambda - m - \phi(m) + 4)(\lambda - m + 3)^{\phi(m)-2} \quad \forall\, i = 1, 2, \ldots \phi(m)$$

$$X_{i,j} = (\lambda - m + \phi(m) + 1)(\lambda - m + 3)^{\phi(m)-2} \;\forall\, i, j = 1, 2, \ldots, \phi(m) \text{ such that } i \neq j$$

$$X_{\phi(m)+1,\phi(m)+1} = (\lambda - m - \phi(m) + 3)(\lambda - m + 3)^{\phi(m)-1}$$

$$X_{i,\phi(m)+1} = X_{\phi(m)+1,j} = 0 \qquad \forall\, i, j = 1, 2, \ldots, \phi(m)$$

Then X^{-1} is given by

$$\begin{bmatrix} \frac{\lambda-m-\phi(m)+4}{(\lambda-m-\phi(m)+3)(\lambda-m+3)} & \frac{1}{(\lambda-m-\phi(m)+3)(\lambda-m+3)} & \cdots & \frac{1}{(\lambda-m-\phi(m)+3)(\lambda-m+3)} & 0 \\ \frac{1}{(\lambda-m-\phi(m)+3)(\lambda-m+3)} & \frac{\lambda-m-\phi(m)+4}{(\lambda-m-\phi(m)+3)(\lambda-m+3)} & \cdots & \frac{1}{(\lambda-m-\phi(m)+3)(\lambda-m+3)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ \frac{1}{(\lambda-m-\phi(m)+3)(\lambda-m+3)} & \frac{1}{(\lambda-m-\phi(m)+3)(\lambda-m+3)} & \cdots & \frac{\lambda-m-\phi(m)+4}{(\lambda-m-\phi(m)+3)(\lambda-m+3)} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\lambda-m+\phi(m)+1} \end{bmatrix}_{(\phi(m)+1)\times(\phi(m)+1)}$$

$$\text{Then, } \Gamma_{Q(H_1)}(x-m_2) = \mathbf{1}_{n_1}^T X^{-1}\mathbf{1}_{n_1}$$
$$= \frac{\lambda\phi(m)+\lambda-m\phi(m)+[\phi(m)]^2-m+3}{(\lambda-m-\phi(m)+3)(\lambda-m+\phi(m)+1)}$$

Thus, signless characteristic polynomial of $\mathscr{P}_S(\mathbb{Z}_m)$

$$= \varphi(Q(H_1)\ ;\ \lambda-m_2)\cdot\varphi(Q(H_2)\ ;\ \lambda-m_1)\cdot[1-\Gamma_{Q(H_1)}(\lambda-m_2)\cdot\Gamma_{Q(H_2)}(\lambda-m_1)]$$
$$= (\lambda-m+2)^{m-\phi(m)-1}(\lambda-m+3)^{\phi(m)-1}\times[\lambda^2-3m\lambda+\lambda\phi(m)+5\lambda+2m^2-2m\phi(m)-8m+6\phi(m)+6]$$

□

Next, we analyse the particular cases where $Q(\mathscr{P}_S(\mathbb{Z}_m))$ is Signless Laplacian integral and one such characterization is given by :

Theorem 3.3. *If $m=2^r3^s$ where r,s are positive integers, then the signless laplacian spectrum of $\mathscr{P}_S(\mathbb{Z}_m)$ is demonstrated as :*

$$\begin{pmatrix} m-2 & m-3 & m-\phi(m)-2 & 2m-3 \\ m-\phi(m)-1 & \phi(m)-1 & 1 & 1 \end{pmatrix}$$

Proof. If $m=2^r3^s$, then $\phi(m)=2^r3^{s-1}=\frac{m}{3}$. Then from Theorem 3.2,

$$\varphi(Q(\mathscr{P}_S(\mathbb{Z}_m)\ ;\ \lambda)) = (\lambda-m+2)^{m-\phi(m)-1}(\lambda-m+3)^{\phi(m)-1}\cdot f(\lambda)$$

where $f(\lambda)=\lambda^2-3m\lambda+\lambda\phi(m)+5\lambda+2m^2-2m\phi(m)-8m+6\phi(m)+6$
Hence the two roots of $f(\lambda)$ are $2m-3$ and $\frac{2m}{3}-2=m-\phi(m)-2$. □

Our next objective is to completely characterize the cases when $Q(\mathscr{P}_S(\mathbb{Z}_m))$ is the Signless Laplacian integral. Before discussing the key result, we revisit a classical result in Number Theory for a comprehensive understanding of the result, which is given by :

Proposition 3.4. *[14] Given any positive integer m, then the Euler-Totient function ϕ and the Möbius function μ are related by the following :*

$$\frac{m}{\phi(m)} = \sum_{d|m}\frac{\mu^2(d)}{\phi(d)}$$

The necessary and sufficient condition for $Q(\mathscr{P}_S(\mathbb{Z}_m))$ to be Signless Laplacian integral is proposed in the following theorem :

Theorem 3.5. *$Q(\mathscr{P}_S(\mathbb{Z}_m))$ is Signless Laplacian integral if and only if m is prime or $m=2^r3^s$, where r,s are positive integers.*

Proof. As a consequence of Theorem 3.1 and Theorem 3.3, the first part holds. Conversely assuming that the signless laplacian spectrum of $\mathscr{P}_S(\mathbb{Z}_m)$ are all integers, we need to prove that m is prime or $m=2^r3^s$, where r,s are positive integers. With reference to Theorem 3.3, it is sufficient to demonstrate that $\phi(m)=m-1$ or $m=3\phi(m)$ is not valid for any m (other than m is prime and $m=2^r3^s$) .
Without loss of generality, we assume and analyze the following cases :

(i) Suppose $m = p^k$ (p stands for any prime)
Case (i) When p is an even prime (i.e, $m = 2^k$),
By Proposition 3.4,

$$\frac{m}{\phi(m)} = \sum_{i=0}^{k} \frac{\mu^2(2^i)}{\phi(2^i)} = 2$$

Case (ii) When $m = p^k;\ p \neq 2$,
By Proposition 3.4,

$$\frac{m}{\phi(m)} = \sum_{i=0}^{k} \frac{\mu^2(p^i)}{\phi(p^i)} = 1 + \frac{1}{p-1} < 2$$

(ii) Suppose $m = q_1^{n_1} q_2^{n_2}$ (where q_1, q_2 are primes such that $q_1, q_2 \neq 2, 3$)
By Proposition 3.4,

$$\frac{m}{\phi(m)} = \frac{\mu^2(1)}{\phi(1)} + \frac{\mu^2(q_1)}{\phi(q_1)} + \frac{\mu^2(q_2)}{\phi(q_2)} + \frac{\mu^2(q_1 q_2)}{\phi(q_1 q_2)} + \sum_{i,j \neq 1} \frac{\mu^2(q_1^i q_2^j)}{\phi(q_1^i q_2^j)}$$
$$= 1 + \frac{1}{q_1 - 1} + \frac{1}{q_2 - 1} + \frac{1}{(q_1 - 1)(q_2 - 1)} < 3$$

(iii) Suppose $m = 2^r 3^s p^t$ (p stands for any prime such that $p \neq 2, 3$)
By Proposition 3.4,

$$\frac{m}{\phi(m)} = \frac{\mu^2(1)}{\phi(1)} + \frac{\mu^2(2)}{\phi(2)} + \frac{\mu^2(3)}{\phi(3)} + \frac{\mu^2(p)}{\phi(p)} + \frac{\mu^2(2.3)}{\phi(2.3)} + \frac{\mu^2(2.p)}{\phi(2.p)} + \frac{\mu^2(3.p)}{\phi(3.p)} + \sum_{i,j,k \neq 1} \frac{\mu^2(2^i 3^j p^k)}{\phi(2^i 3^j p^k)}$$
$$= 3 + \frac{2}{p-1} + \frac{1}{2(p-1)}$$

(iv) When $m = 2^{n_1} p^{n_2}$ (where $p \neq 3$) or $3^{n_1} q^{n_2}$ (where $q \neq 2$)
Case (i) When $m = 2^{n_1} p^{n_2}$ (where $p \neq 3$)
By Proposition 3.4,

$$\frac{m}{\phi(m)} = \frac{\mu^2(1)}{\phi(1)} + \frac{\mu^2(2)}{\phi(2)} + \frac{\mu^2(p)}{\phi(p)} + \frac{\mu^2(2.p)}{\phi(2.p)} + \sum_{i,j \neq 1} \frac{\mu^2(2^i p^j)}{\phi(2^i p^j)} = 2 + \frac{2}{p-1}$$

Case (ii) When $m = 3^{n_1} q^{n_2}$ (where $q \neq 2$)
By Proposition 3.4,

$$\frac{m}{\phi(m)} = \frac{\mu^2(1)}{\phi(1)} + \frac{\mu^2(3)}{\phi(3)} + \frac{\mu^2(q)}{\phi(q)} + \frac{\mu^2(3.q)}{\phi(3.q)} + \sum_{i,j \neq 1} \frac{\mu^2(3^i q^j)}{\phi(3^i q^j)} = \frac{3q}{2(q-1)}$$

Hence in any of the above cases, $\phi(m) = m - 1$ or $m = 3\phi(m)$ does not hold. Extending the idea to the cases where m comprises of more than two prime factors and applying the same, we arrive at the desired result. Hence the proof. □

Next, we compute the Signless Laplacian energy of $\mathscr{P}_S(\mathbb{Z}_m)$) when m is prime :

Theorem 3.6. *Assume that G is any cyclic group of prime order, the signless laplacian energy of the strong power graph of G is evaluated as :*

$$LE^+(\mathscr{P}_S(G)) = 2(m-1) - \frac{4}{m}$$

Proof. From Theorem 3.1, we see that $\mathscr{P}_S(\mathbb{Z}_m)) \cong K_{m-1} \cup K_1$. Thus, the number of edges in $\mathscr{P}_S(\mathbb{Z}_m) = q = \frac{(m-1)(m-2)}{2}$.
From Theorem 3.1, it follows that $0, 2(m-2)$ and $m-3$ are the signless laplacian eigen values of $\mathscr{P}_S(\mathbb{Z}_m)$ with respective multiplicities $1, 1$ and $m-2$. Thus, $LE^+(\mathscr{P}_S(G)) = \sum_{i=1}^{p} |\lambda_i - \frac{2q}{p}| = 2(m-1) - \frac{4}{m}$ □

By Theorem 3.5, $\mathscr{P}_S(\mathbb{Z}_m)$ is Signless Laplacian integral if and only if $m = 2^r 3^s$ (when m is composite). Hence, we discuss the following theorem :

Theorem 3.7. *If G is any cyclic group of order m with $m = 2^r 3^s$; $r, s > 1$, the signless laplacian energy of strong power graph of G is evaluated as :*

$$LE^+(\mathscr{P}_S(G)) = 2(m-2) - \frac{4\phi(m)}{m}$$

Proof. From Theorem 3.3, we see that $\mathscr{P}_S(\mathbb{Z}_m) \cong [K_{\phi(m)} \cup K_1] \vee K_{m-\phi(m)-1}$. Thus, the size of $\mathscr{P}_S(\mathbb{Z}_m) = q = \frac{(m-1)(m-2)}{2} + m - \phi(m) - 1$. Hence from Theorem 3.3, the result follows. □

In the next result, we discuss the signless laplacian spectrum of strong power graphs of non-cyclic groups which is given by :

Theorem 3.8. *For any non-cyclic group G of order m, $Q(\mathscr{P}_S(G))$ is given by*

$$\begin{pmatrix} 2(m-1) & m-2 \\ 1 & m-1 \end{pmatrix}$$

Proof. Singh and Manilal in [17] proved that if G is a non-cyclic group with order m, then $\mathscr{P}_S(G)$ is a complete graph. Thus, $\mathscr{P}_S(G) \cong K_m$. Hence, from Proposition 2.3, the theorem holds. □

As a consequence of theorem 3.8, we have the following result :

Corollary 3.9. *If G is any non-cyclic group G with order m, $LE^+(\mathscr{P}_S(G)) = 2(m-1)$*

Proof. From Theorem 3.8, we see that $\mathscr{P}_S(G) \cong K_m$. Then the size of $\mathscr{P}_S(G) = \frac{m(m-1)}{2}$. Also by Theorem 3.8, we see that $2(m-1)$ and $m-2$ are the signless laplacian eigen values of $\mathscr{P}_S(G)$ with multiplicities 1 and $m-1$ respectively. Hence, $LE^+(\mathscr{P}_S(G)) = 2(m-1)$. □

3.1 Number of Spanning Trees of $\mathscr{P}_S(\mathbb{Z}_m)$

Bhuniya and Bera in [15] have determined the number of spanning trees using the laplacian spectrum of strong power graphs. Here, the authors have determined the same in an alternative way using signless laplacian spectrum. The following result is vital for the better understanding of the key result :

Proposition 3.10. *[16] For a graph Γ with order m, the number of spanning trees $\kappa(\Gamma)$ is computed using the rule $\kappa(\Gamma) = \dfrac{det(J+L)}{m^2}$ where J stands for the $m \times m$ matrix with all its entries $+1$.*

Remark 3.11. Singh and Manilal in [17] have characterized a necessary and sufficient condition for the connectedness for $\mathscr{P}_S(\mathbb{Z}_m)$ as : *$\mathscr{P}_S(\mathbb{Z}_m)$ is connected if and only if m is composite.* Hence it is meaningless to find the number of spanning trees of $\mathscr{P}_S(\mathbb{Z}_m)$ when m is prime.

Theorem 3.12. *If m is composite, the number of spanning trees of $\mathscr{P}_S(\mathbb{Z}_m)$ is given by*

$$m^{m-\phi(m)-2}(m-1)^{\phi(m)-1}(m-\phi(m)-1)$$

Proof. From equation 2.1, $Q = D + A \implies J + L = J + Q - 2A$.
Then by Proposition 3.10,

$$\kappa(\Gamma) = \frac{\det(J + Q - 2A)}{m^2} \tag{3.3}$$

Here $Q = Q(\mathscr{P}_S(\mathbb{Z}_m))$ and is given by

$$\begin{bmatrix} m-1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & m-1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & m-1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & m-2 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 & m-2 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & m-2 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & m-\phi(m)-1 \end{bmatrix}_{m\times m}$$

Then $J + Q - 2A$ is given by

$$\left[\begin{array}{cccc|ccccc} m & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & m & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & m & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & m-1 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & m-1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & m-1 & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & m-\phi(m) \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\\vdots\\0\end{array}}\right\} m-\phi(m)-1 \text{ rows} \\ \left.\vphantom{\begin{array}{c}0\\0\\\vdots\\0\\0\end{array}}\right\} \phi(m)+1 \text{ rows} \end{array}$$

$$= \left[\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array}\right]$$

Then $\det\,(J + Q - 2A) = \det X \cdot \det Y = m^{m-\phi(m)-1} \cdot \det Y$
Applying a series of elementary column and row operations

(i) $C_1 \to C_1 + C_2 + \ldots + C_{\phi(m)+1}$

(ii) $R_i \to R_i - R_1 \;\forall\, i = 2, 3, \ldots, \phi(m) + 1$

(iii) $R_{\phi(m)+1} \to R_{\phi(m)+1} - \frac{1}{m-1} R_j \;\forall\, j = 2, 3, \ldots, \phi(m)$ on Y,

we get $\det Y = m(m - \phi(m) - 1)(m - 1)^{\phi(m)-1}$

$$\text{Thus, } \det\,(J + Q - 2A) = m^{m-\phi(m)} \cdot (m - \phi(m) - 1) \cdot (m - 1)^{\phi(m)-1}$$

Substituting in equation 3.3, we get the desired result. □

4 Conclusion remarks

In this paper, we have discussed the signless laplacian spectrum of strong power graphs of any cyclic group of finite order. We have also analyzed the cases when the strong power graph is Signless Laplacian integral. Also, we have evaluated the signless Laplacian energy of strong power graphs (when the graphs are Signless Laplacian integral). We have also determined the number of spanning trees of $\mathscr{P}_S(\mathbb{Z}_m)$ when m is composite. When m is composite and $m \neq 2^r 3^s$, the strong power graph is not signless Laplacian integral. Evaluating the bounds of the signless Laplacian spectral radius of $\mathscr{P}_S(\mathbb{Z}_m)$ lies in the future scope of this work.

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