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BSDEs with Two Reflecting Obstacles Driven by RCLL Martingales under Stochastic Lipschitz Coefficient

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Abstract. In this paper, we investigate a significant class of doubly reflected backward stochastic differential equations driven by a right continuous with left limits (RCLL) martingale with two completely separated RCLL barriers. Under the assumption of a stochastically Lipschitz coefficient and the complete separation of the barriers, we establish the existence and uniqueness of the solution using the notion of local solutions of reflected BSDEs. Notably, the analysis does not rely on Mokobodzki's condition or the regularity of the barriers.

1 Introduction

Pardoux and Peng [37] introduced the theory of nonlinear backward stochastic differential equations (BSDEs, for short). A solution to this equation, characterized by a terminal value ξ and a generator or driver function $f(\omega,t,y,z)$, consists of a pair of stochastic processes $(Y_t,Z_t)_{t\leq T}$, satisfying:

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s B_s, \quad 0\leq t\leq T, \text{ a.s.,} \tag{1.1}$$

where B is a standard Brownian motion and the couple $(Y_t,Z_t)_{t\leq T}$ is adapted to the natural filtration of B.

The authors established the existence and uniqueness of a solution to (1.1) under certain conditions, requiring the Lipschitz property with respect to (y,z) of the generator f and the square integrability of ξ and the process $(f(\omega,t,0,0))_{t\leq T}$. Since this groundbreaking achievement, BSDEs have proven to be a versatile tool for addressing various mathematical challenges. Applications span across finance, especially in the theory of pricing contingent claims (see e.g., [9, 41], and [3, 28, 42] for related studies), with notable contributions in stochastic control, differential games, and partial differential equations (see e.g., [21, 22, 38], among others).

The notion of doubly reflected BSDEs (DRBSDEs, for short) was initially introduced by Cvitanic and Karatzas [5] within the framework of a Brownian filtration as a direct extension of the one reflected case studied initially by El Karoui et al. in [7]. In DRBSDEs, the solution is constrained to remain bounded between the lower barrier L and the upper barrier U. This is achieved through the coordinated behavior of two continuous and increasing reflecting processes.

In their work, Cvitanic and Karatzas established the existence and uniqueness of the solution under specific conditions. Namely, the coefficient must be Lipschitz continuous, and either the barriers need to satisfy certain regularity conditions or meet the criteria of Mokobodski's condition. Mokobodski's condition, in essence, implies the existence of a difference between two non-negative supermartingales within the range defined by the lower and upper barriers. The regularity condition, on the other hand, ensures that the barriers can be uniformly approximated by Itô's processes.

The original assumptions on the data in [5] are often considered too restrictive for practical applications, making them challenging to verify in practice. Consequently, several efforts have been made to relax these assumptions and extend the theory along various paths.

In this regard, Hamadène and Hassani [17] proposed the concept of a local solution for DRBSDEs. A local solution is a solution to the equation but only within a specific range of comparable stopping times. Notably, the authors removed the requirement of Mokobodski's condition and demonstrated that if the barriers are continuous and completely separated (i.e., $L < U$), then the DRBSDEs possess a unique solution. This work represents a significant advancement in the theory, offering a more flexible framework for analyzing DRBSDEs under relaxed assumptions. Several notable contributions have been made in the pursuit of solving DRBSDEs, as evidenced by works such as [2, 20, 31]. Later on, the case of discontinuous barriers has also been studied by Hamadène et al. [19], where they actually show the existence of a solution when the obstacles and their left limits are completely separated.

In addition to the Brownian setting, the study of RBSDEs with jumps has been extended by several authors. This involves considering a filtration generated by a Brownian motion and an independent Poisson point process. Notably, Essaky et al. [14] employed the penalization approximation technique to address the related problem, while Hamadène and Hassani [18] focused on investigating the existence and uniqueness of local and global solutions for RBSDEs with reflecting obstacles occurring at inaccessible stopping times, indicating their lack of predictability.

Subsequently, Hamadène and Wang [24] addressed the same problem but considered completely separated barriers that allow for general jumps, including both predictable and totally inaccessible ones. They introduced a local solution for DRBSDEs using the convergence of increasing and decreasing penalization schemes. Building upon these results, they further constructed a global solution. Additionally, the authors applied their findings to determine the value of a related mixed zero-sum differential-integral game problem. These contributions enhance the understanding of RBSDEs with jumps and offer valuable insights for practical applications.

Furthermore, other notable works in this area include those by Abdallah et al. [1], El Otmani et al. [12], Essaky and Hassani [13], Ren and El Otmani [40], among others. These works further explore and advance the theory of DRBSDEs, providing additional valuable results in the field.

The theory of BSDEs driven by an RCLL martingale has been extensively studied. The seminal work by El Karoui and Huang [8] and subsequent research by Carbone et al. [4] have made significant contributions to this field. In particular, they have considered BSDEs under a more general filtration framework, where the filtration is assumed to be complete, right-continuous, and quasi-left continuous.

In this context, for a finite horizon time T, a square integrable martingale $(M_t)_{t\leq T}$, a terminal condition ξ, and a generator or driver $f(\omega, t, y, z)$, a solution to these equations consists of a triplet of stochastic processes $(Y_t, Z_t, N_t)_{t\leq T}$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) d\langle M\rangle_s - \int_t^T Z_s dM_s - \int_t^T dN_s, \quad 0 \leq t \leq T. \tag{1.2}$$

In this framework, the process N represents a square integrable martingale that is orthogonal to the martingale M. Notably, Øksendal and Zhang [36] have analyzed a specific class of BSDEs where the driver function f does not depend on the control variable z. They applied their findings to the field of insider finance. Furthermore, Liang et al. [30] have obtained results for a particular class of BSDEs (1.2) in the context of an arbitrary filtered probability space. In these studies, they establish the existence and uniqueness of solutions to (1.2) under the assumption of Lipschitz continuity of the function f, along with square integrability conditions on the given data (see also [10] for some general related study).

In a more general context, Nie and Rutkowski [35] have considered reflected BSDEs and doubly reflected BSDEs driven by an RCLL martingale. They have provided a proof for the existence of a solution in the case of one reflection, utilizing the Snell envelope notion. Additionally, for the case of doubly reflected BSDEs, the existence result is obtained under Mokobodzki's condition.

The objective of this paper is to address the problem of existence and uniqueness for a class of doubly reflected BSDEs driven by a broad class of RCLL martingales, along with two completely separated RCLL obstacles, within an arbitrary filtered probability space. Our investigation is conducted under the assumption of a stochastic Lipschitz condition on the driver, without assuming the existence of a difference of supermartingales between the obstacles. Notably, our study encompasses the previously mentioned results, while providing a more comprehensive framework. The main challenges in our problem formulation are as follows:

(i) *General Filtration*: Our filtration is characterized by its generality, satisfying the usual assumptions and a quasi-left continuity condition, without being restricted to being generated by or supporting a Brownian motion and an independent random Jump measure.

(ii) *Stochastic Lipschitz Condition:* The generator in our model satisfies a weak Lipschitz condition known as the stochastic Lipschitz condition.

(iii) *General Jump Structure:* The jumps of the obstacles in our setting can occur at either predictable or inaccessible stopping times, resulting in the state process of the solution featuring both types of jumps.

By addressing these challenges, we aim to provide a comprehensive analysis to this class of equations under a more flexible setting. Additionally, our analysis incorporates advanced technical results in stochastic calculus, enabling a rigorous examination of the problem and offering theoretical insights.

It's noteworthy that these equations find applications in various scenarios, including specific filtrations generated by a Brownian motion, a Brownian combined with an independent Poisson random measure, or those generated by a Lévy process in the realms of finance and stochastic games (see e.g. [5, 12, 16, 17, 18, 21, 24]).

This paper unfolds systematically, beginning with Section 2, where we establish the foundation by introducing essential properties, notations, definitions, and assumptions relevant to our problem. Moving on, we delve into the uniqueness result of our doubly reflected BSDE in Section 3. Due to the lack of integrability in the solution, we employ a localization procedure to address this challenge. Section 4 is dedicated to the specifics of the problem. We introduce increasing and decreasing penalization schemes designed for cases where the generator f does not depend on (y, z). Demonstrating the convergence of these schemes, we establish the local solution for the doubly reflected BSDE with two barriers. Our approach is inspired by techniques from [24], adapted to our specific context. Concluding the paper, Section 5 presents the main result, affirming the existence and uniqueness of the solution, particularly when obstacles and their left limits are completely separated. The approach involves two steps: addressing a coefficient that depends only on the y-variable using a fixed-point argument, and subsequently extending this result to the general case of a driver that depends on both the y- and z-variables.

2 Problem Formulation and assumptions

2.1 Setting and notations

Let $T > 0$ be a finite terminal time and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a complete filtered probability space where the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\leq T}$ is quasi-left continuous and satisfies the usual conditions of right-continuity and completeness, and $\mathcal{F}_T = \mathcal{F}$. The initial σ-field $\mathcal{F}_0$ is assumed to be trivial. The equality $X = Y$ between any two processes $(X_t)_{t\leq T}$ and $(Y_t)_{t\leq T}$ must be understood in the indistinguishably sens, meaning that $\mathbb{P}(\omega : X_t(\omega) = Y_t(\omega), \forall t \leq T) = 1$. The same signification holds for $X \leq Y$. For a given RCLL process $(Y_t)_{t\leq T}$, $Y_{t-} = \lim_{s\nearrow t} Y_s$ is the left limits of Y at t, we set $Y_{0-} = Y_0$ by convention. $Y_- = (Y_{t-})_{t\leq T}$ the left limited process, and $\Delta Y_t = Y_t - Y_{t-}$ the jump of Y at time t. Next, for given two locally square integrable $\mathbb{F}$-martingales M and N, we denote by $\langle M, N\rangle$ the predictable $\mathbb{F}$-dual projection of the quadratic co-variation process $[M, N]$, by M^c the continuous part of M. Finally, $\mathbb{E}^{\mathcal{F}_t}[\cdot]$ denotes conditional expectation with respect to $\mathcal{F}_t$, i.e., $\mathbb{E}^{\mathcal{F}_t}[\cdot] := \mathbb{E}[\cdot \mid \mathcal{F}_t]$. For $x \in \mathbb{R}$, we use the notation $x^+ = \max(x, 0)$ and $x^- = (-x)^+ = -\min(x, 0)$.

We denote by $\mathcal{T}_{\gamma_1}^{\gamma_2}$ the set of $[0, T]$-valued $\mathbb{F}$-stopping times γ such that $\gamma_1 \leq \gamma \leq \gamma_2$, a.s. for two $[0, T]$-valued $\mathbb{F}$-stopping times γ_1 and γ_2 such that $\gamma_1 \leq \gamma_2$, a.s. An $\mathcal{F}_t$-adapted process $\mathcal{X}$ is said to be in the class $\mathcal{D}([\gamma_1, \gamma_2])$ for two stopping times $\gamma_2 \in \mathcal{T}_0^T$ and $\gamma_2 \in \mathcal{T}_0^{\gamma_2}$ if the set of random $\{\mathcal{X}_\eta, \eta \in \mathcal{T}_{\gamma_1}^{\gamma_2}\}$ is uniformly integrable. For $\mathcal{F}_t$-progressively measurable RCLL processes $(Y^n)_{n\in\mathbb{N}}$ and Y, we say that $Y^n \to Y$ in **ucp** (uniformly on compacts in probability) if $\sup_{0\le s\le t}|Y^n_s - Y_s|^2 \to 0$ in probability $\mathbb{P}$ for every $t \in [0,T]$.

Let $M = (M_t)_{t\le T}$ be an $\mathbb{R}$-dimensional, square-integrable, $\mathbb{F}$-martingale. It is presumed that M is an RCLL process because the filtration $\mathbb{F}$ is right-continuous and an RCLL modification of any $\mathbb{F}$-martingale is known to exist (see Theorem I.9 in [39]). Let us recall that a filtration $\mathbb{F}$ is called quasi-left continuous if for every sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\in\mathbb{N}} \subset \mathcal{T}_0^T$ such that $\tau_n \nearrow \tau$, we have $\bigvee_{n\in\mathbb{N}} \mathcal{F}_{\tau_n} = \mathcal{F}_\tau$.

In order to make the notation easier to understand, we exclude any reference to the dependence on ω for a given process or random function, and it is customary to assume that all brackets and stochastic integrals have a value of zero at time zero.

Next, we introduce the following processes and spaces to describe the parameters and the solution of our equation.

2.2 Spaces

Let $\beta > 0$, $(\alpha_t)_{t\le T}$ be a non-negative $\mathcal{F}_t$-adapted process, and $(A_t)_{t\le T}$ be the increasing continuous process defined as $A_t := \int_0^t \alpha_s^2 d\langle M\rangle_s$ for all $t \in [0,T]$. We then introduce the following spaces:

- $\mathcal{S}^2$: the space of one-dimensional $\mathcal{F}_t$-predictable RCLL increasing processes $(K_t)_{t\le T}$ such that $K_0 = 0$ and
$$\| K \|^2_{\mathcal{S}^2} = \mathbb{E}\left[|K_T|^2\right] < \infty.$$

- $\mathbb{L}^2_\beta$: the set of one-dimensional $\mathcal{F}_T$-measurable random variables ξ such that
$$\|\xi\|^2_\beta := \mathbb{E}\left[e^{\beta A_T}|\xi|^2\right] < \infty.$$

- $\mathcal{S}^2_\beta$: the space of one-dimensional $\mathcal{F}_t$-adapted RCLL processes $(Y_t)_{t\le T}$ such that
$$\| Y \|^2_{\mathcal{S}^2_\beta} = \mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_t}|Y_t|^2\right] < \infty.$$

- $\mathcal{S}^{2,\alpha}_\beta$: the space of one-dimensional $\mathcal{F}_t$-adapted RCLL processes $(Y_t)_{t\le T}$ such that
$$\| Y \|^2_{\mathcal{S}^{2,\alpha}_\beta} = \mathbb{E}\left[\int_0^T e^{\beta A_s}|\alpha_s Y_s|^2 d\langle M\rangle_s\right] < \infty.$$

- $\mathcal{H}^2_\beta$: the space of one-dimensional $\mathcal{F}_t$-predictable processes $(Z_t)_{t\le T}$ such that
$$\| Z \|^2_{\mathcal{H}^2_\beta} = \mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s|^2 d\langle M\rangle_s\right] < \infty.$$

- $\mathcal{M}^2_\beta$: the space of one-dimensional square-integrable $\mathbb{F}$-martingale $(N_t)_{t\le T}$ orthogonal to M such that
$$\| N \|^2_{\mathcal{M}^2_\beta} = \mathbb{E}\left[\int_0^T e^{\beta A_s} d[N]_s\right] < \infty.$$

- $\mathfrak{A}^2_\beta := \left(\mathcal{S}^2_\beta \cap \mathcal{S}^{2,\alpha}_\beta\right) \times \mathcal{H}^2_\beta \times \mathcal{S}^2 \times \mathcal{M}^2_\beta$.

Remark 2.1.
- For a more detailed explanation of the orthogonality between two locally square-integrable martingales, readers can refer to Section 4.a in [26].
- The filtration $\mathbb{F}$ being quasi-left continuous implies that every uniformly integrable $\mathbb{F}$-martingale is also quasi-left continuous (see Theorem 5.36, pp. 155 in [25]). This means that such martingales cannot exhibit jumps at predictable $\mathbb{F}$-stopping times.

- In general, if the underlying filtration $\mathbb{F}$ is not quasi-left continuous, the sharp bracket $\langle M\rangle$ may not exhibit continuity. However, following the approach in [34] (Assumption 2.1 and Remark 2.1, pp. 3), we can make certain assumptions in the case of an n-dimensional, square-integrable martingale $M_t = \left\{\left(M_t^1, M_t^2, \ldots, M_t^n\right)^*, 0 \le t \le T\right\}$, where * denotes the transpose operator. Specifically, we assume that $d\langle M\rangle_t = m_t m_t^\star dQ_t$, where Q is a bounded, $\mathcal{F}_t$-adapted, continuous, non-decreasing process with $Q_0 = 0$, and $(m_t)_{t\le T}$ is an $\mathbb{R}^{n\times n}$-dimensional, $\mathcal{F}_t$-predictable process. Additionally, we can assume that m is a symmetric $\mathbb{R}^{n\times n}$-matrix. These assumptions allow us to handle the lack of continuity in the sharp bracket and facilitate the analysis in our more general setting.

2.3 Problem Statement

This paper is centered on the exploration of a doubly reflected backward stochastic differential equation (DRBSDE) driven by the square-integrable, right-continuous, left-limited martingale M on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\le T}, \mathbb{P})$. Specifically, we aim to investigate the DRBSDE expressed in the following form:

$$\begin{cases} \text{(i) } Y_t = \xi + \displaystyle\int_t^T f(s, Y_s, Z_s)d\langle M\rangle_s + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dM_s - \int_t^T dN_s. \\ \text{(ii) } L_t \le Y_t \le U_t,\ 0 \le t \le T \\ \text{(iii) If } K^{c,\pm} \text{ is the continuous part of } K^\pm, \text{ then} \\ \qquad \displaystyle\int_0^T (Y_t - L_t)dK_t^{c,+} = \int_0^T (U_t - Y_t)dK_t^{c,-} = 0. \\ \text{(iv) If } K^{d,\pm} \text{ is the purely discontinuous part of } K^\pm, \text{ then } K_t^{d,+} = \displaystyle\sum_{0<s\le t} (Y_s - L_{s-})^- \\ \quad \text{and } K_t^{d,-} = \displaystyle\sum_{0<s\le t} (Y_s - U_{s-})^+. \end{cases} \tag{2.1}$$

By observing the form of the DRBSDE (2.1), we can note the following:

Remark 2.2. (i) The state process Y in the RBSDE (2.1) exhibits two types of jumps. The first type of jumps are the totally inaccessible ones that arise from its martingale part (see Remark 2.1), namely, the two $\mathbb{F}$-martingales $\left(\int_0^t Z_s dM_s\right)_{t\le T}$ and $\left(\int_0^t dN_s\right)_{t\le T}$. This is due to the quasi-left continuity of the filtration $\mathbb{F}$, which implies that the stochastic integrals $\left(\int_0^t Z_s dM_s\right)_{t\le T}$ and $\left(\int_0^t dN_s\right)_{t\le T}$ cannot jump at an $\mathbb{F}$-predictable stopping time. The second type of jumps are predictable jumps that arise from the negative jumps of the lower obstacle L and the positive predictable jumps of the upper reflecting barrier U. Therefore, it is necessary to introduce some predictable jump processes $K^{d,\pm}$ to characterize the predictable jumps of Y.

(ii) The predictable jump process $K^{d,\pm}$ for $K^\pm$ can be expressed as (see e.g. [24], Remark 2.1): $\forall t \le T$,

$$K_t^{d,+} = \sum_{0<s\le t} (Y_s - L_{s-})^- \mathbb{1}_{\{\Delta L_s<0\}} = \sum_{0<s\le t} (Y_s - L_{s-})^- \mathbb{1}_{\{Y_{s-}=L_{s-}\}},$$
$$K_t^{d,-} = \sum_{0<s\le t} (Y_s - U_{s-})^+ \mathbb{1}_{\{\Delta U_s>0\}} = \sum_{0<s\le t} (Y_s - U_{s-})^+ \mathbb{1}_{\{Y_{s-}=U_{s-}\}}.$$

(iii) The Skorokhod condition (2.1)-(iii)-(iv) is equivalent to (see e.g. [11], Remark 4.1)

$$\int_0^T (Y_{s-} - L_{s-})\, dK_s^+ = \int_0^T (U_{s-} - Y_{s-})\, dK_s^- = 0, \quad \mathbb{P}\text{-a.s.} \tag{2.2}$$

(iv) Note that the minimality condition (2.2) was first introduced by Hamadène [15] and then explicitly stated by Lepeltier and Xu [29] for RBSDEs with a single discontinuous RCLL barrier within the framework of a Brownian filtration.

Hypothesis on the data of the RBSDE (2.1): The quadruplet (ξ, f, L, U) is such that:

(H1) The terminal variable $\xi \in \mathbb{L}^2_\beta$.

(H2) The driver $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that:

(i) For all (y, z) the stochastic process $f(\cdot, y, z)$ is $\mathcal{F}_t$-progressively measurable,

(ii) There exists two non-negative $\mathcal{F}_t$-adapted processes $(\kappa_t)_t$ and (γ_t) such that

* for all $t \in [0, T]$, y, $y' \in \mathbb{R}$ and z, $z' \in \mathbb{R}$

$$|f(t, y, z) - f(t, y', z')| \leq \kappa_t |y - y'| + \gamma_t |z - z'|$$

* There exists $\epsilon > 0$ such that $\alpha_s^2 := \kappa_s + \gamma_s^2 \geq \epsilon$ and $\frac{f(\cdot,0,0)}{\alpha.} \in \mathcal{H}^2_\beta$.

(H3) The barriers $(L_t)_{t\leq T}$ and $(U_t)_{t\leq T}$ are real-valued $\mathcal{F}_t$-progressively measurable RCLL processes satisfying

(i) $L_T \leq \xi \leq U_T$, $\mathbb{P}$-a.s.

(ii) $\mathbb{E}\left[\sup_{0\leq t\leq T} \left|e^{\beta A_t} L_t^+\right|^2\right] < \infty$ and $\mathbb{E}\left[\sup_{0\leq t\leq T} \left|e^{\beta A_t} U_t^-\right|^2\right] < \infty$,

(iii) $\mathbb{P}$-a.s., $\forall t \leq T$, $L_t \leq U_t$.

Let's now delve into the definition of solutions and the uniqueness criterion for DRBSDE (2.1).

Definition 2.3. • Consider $\beta > 0$ and $(\alpha_t)_{t\leq T}$ as a non-negative $\mathcal{F}_t$-adapted process. A solution to the DRBSDE (2.1) associated with parameters (ξ, f, L, U) is represented by a quintuplet of processes (Y, Z, K^+, K^-, N) satisfying (2.1). Here, Y belongs to $\mathcal{S}^2_\beta \cap \mathcal{S}^{2,\alpha}_\beta$, $K_0^\pm = 0$, and (Z, K^+, K^-, N) satisfies

$$\int_0^T e^{\beta A_s} \left\{ |Z_s|^2 \, d\langle M\rangle_s + d\langle N\rangle_s \right\} + K_T^+ + K_T^- < \infty, \quad \mathbb{P}\text{-a.s.}$$

• A solution to (2.1) is deemed unique if, given any two solutions $\left(Y, Z, K^+, K^-, N\right)$ and $\left(Y', Z', K'^{,+}, K'^{,-}, N'\right)$ of (2.1), the following conditions hold: $(Y, Z, N) = (Y', Z'^{,+}, N')$, $K^+ - K^- = K'^{,+} - K'^{,-}$, and $K^{d,\pm} = K'^{d,\pm}$. Additionally, if, for any $t < T$, $L_t < U_t$, then we also have $K^{c,\pm} = K'^{c,\pm}$.

Remark 2.4.

(i) Recall that for any pair of RCLL $\mathbb{F}$-semimartingales S^1 and S^2, the operation $(S^1, S^2) \to [S^1, S^2]$ is bilinear and symmetric (i.e. $[S^1, S^2] = [S^2, S^1]$), therefore, we have the following polarization identities:

$$[S^1 + S^2] = [S^1] + 2[S^1, S^2] + [S^2], \quad [S^1 - S^2] = [S^1] - 2[S^1, S^2] + [S^2].$$

Applying the first identity to the dynamic of the process $(Y_t)_{t\leq T}$ given by (2.1)-(i) with $S^1 = \int_0^\cdot Z_s dM_s + N$, $S^2 = K^+ - K^-$, and the second to the latest, taking into account the strict separability of the barriers (which implies in particular $\Delta K^{d,+} \Delta K^{d,-} = 0$), yields to

$$[Y] = \sum_{0<s\leq \cdot} \left(\Delta K_s^{d,+}\right)^2 + \sum_{0<s\leq \cdot} \left(\Delta K_s^{d,-}\right)^2 + \int_0^\cdot |Z_s|^2 \, d[M]_s + 2\int_0^\cdot Z_s d[M, N]_s + \int_0^\cdot d[N]_s.$$

Thus, the jump part of the process $[Y]$ is described by

$$\sum_{0<s\leq \cdot} (\Delta Y_s)^2$$

$$= \sum_{0<s\leq \cdot} \left(\Delta K_s^{d,+}\right)^2 + \sum_{0<s\leq \cdot} \left(\Delta K_s^{d,-}\right)^2 + \sum_{0<s\leq \cdot} |Z_s|^2 (\Delta M_s)^2 + 2\sum_{0<s\leq \cdot} Z_s \Delta M_s \Delta N_s + \sum_{0<s\leq \cdot} (\Delta N_s)^2, \quad (2.3)$$

and the path-by-path continuous part of $t \mapsto [Y]_t$ is given by:

$$[Y]^c = \int_0^{\cdot} |Z_s|^2 \, d\langle M\rangle_s + 2\int_0^{\cdot} Z_s d\langle M^c, N^c\rangle_s + \int_0^{\cdot} d\langle N^c\rangle_s . \tag{2.4}$$

Formulas (2.3) and (2.4) will be needed when dealing with the quadratic variation that arises in Itô's formula. For a path-wise decomposition of the quadratic variation, the reader is referred to page 70 in [39] (see also Chapter I, Section 4 in [26] for a general study).

(ii) For any process Z that belongs to $\mathcal{H}^2$, the process $\left(\int_0^{\cdot} Z_s dM_s\right)^2 - \int_0^{\cdot} |Z_s|^2 \, d[M]_s$ is an $\mathbb{F}$-martingale (Theorem 27 in [39], pp. 71), and we have

$$\mathbb{E}\left[\left(\int_0^T Z_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^T |Z_s|^2 \, d[M]_s\right] = \mathbb{E}\left[\int_0^T |Z_s|^2 \, d\langle M\rangle_s\right]. \tag{2.5}$$

(iii) Using the sharp bracket version of the Kunita-Watanabe inequality ([39], pp. 148), for a jointly measurable process A such that $\frac{A}{\alpha} \in \mathcal{H}_b^2$, for some $b > 0$, we have

$$\left(\int_{t_1}^{t_2} \chi_s d\langle M\rangle_s\right) \leq \frac{1}{\sqrt{b}} \left(e^{-bA_{t_1}} - e^{-bA_{t_2}}\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} e^{bA_s} \left|\frac{\chi_s}{\alpha_s}\right|^2 d\langle M\rangle_s\right)^{\frac{1}{2}}. \tag{2.6}$$

(iv) We point out that, since $([M,N] - \langle M,N\rangle)$ is a martingale (see Proposition 4.50-b, pp.53 in [26]), and if Z is an element of $\mathcal{H}_\beta^2$ orthogonal to M, we have

$$\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T e^{\beta A_s} Z_s d[M,N]_s\right] = \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T e^{\beta A_s} Z_s d\langle M,N\rangle_s\right] = 0, \tag{2.7}$$

and

$$\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T e^{\beta A_s} |Z_s|^2 \, d[M]_s\right] = \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T e^{\beta A_s} |Z_s|^2 \, d\langle M\rangle_s\right]. \tag{2.8}$$

We will start by focusing on the uniqueness of the solution to the DRBSDE (2.1).

3 Uniqueness result

To establish the uniqueness of the solution, we require the following auxiliary result:

Lemma 3.1. *For any locally square-integrable $\mathbb{F}$-local martingale $(N_t)_{t\geq 0}$ and any locally bounded $\mathcal{F}_t$-predictable process $(H_t)_{t\leq T}$, the stochastic integral $\left(\int_0^t H_s d\langle N\rangle_s\right)_{t\leq T}$ is continuous. In particular, the process $(\langle M\rangle_t)_{t\leq T}$ has continuous paths over the time interval $[0,T]$.*

Proof. From Corollary I.2.31 in [26], the quasi-left continuity of $\mathbb{F}$, and Proposition 10.19 in [27], we deduce that $\Delta\mathfrak{M}_\eta = 0$ for any $\mathbb{F}$-local martingale $(\mathfrak{M}_t)_{t\leq T}$ and any predictable stopping time $\eta \in \mathcal{T}_0^T$. Moreover, as $\int_0^{\cdot} H_s d\langle N\rangle_s$ is the $\mathcal{F}_t$-predictable compensator of the finite variation process $\int_0^{\cdot} H_s d[N]_s$ (see Proposition I.4.50 in [26]), then, from Theorem II.13 and Theorem II.22 in [39], and Theorem 5.29 in [25], we obtain, for any $\mathbb{F}$-predictable stopping time $\eta \in \mathcal{T}_0^T$,

$$\Delta\left(\int_0^\eta H_s d\langle N\rangle_s\right) = \mathbb{E}^{\mathcal{F}_\eta}\left[\Delta\left(\int_0^\eta H_s [N]_s\right)\right] = \mathbb{E}^{\mathcal{F}_\eta}\left[H_\eta \Delta [N]_\eta\right] = \mathbb{E}^{\mathcal{F}_\eta}\left[H_\eta (\Delta N_\eta)^2\right] = 0.$$

As the process $\Delta\left(\int_0^{\cdot} H_s d\langle N\rangle_s\right)$ is $\mathcal{F}_t$-predictable, and the above equality holds for an arbitrary $\mathbb{F}$-predictable stopping time, then using the predictable version of the section theorem (see Theorems 4.8 and 4.10 in [25]), we derive that the process $\Delta\left(\int_0^{\cdot} H_s d\langle N\rangle_s\right)$ is indistinguishable from the zero process. Thus, we have the desired result. □

Now, let's consider a set of data (ξ, f, L, U) where the generator f satisfies hypothesis **(H2)**. Let $\left(Y_t, Z_t, K_t^+, K_t^- N_t\right)_{t\leq T}$ and $\left(Y'_t, Z'_t, K_t^{\prime,+}, K_t^{\prime,-} N'_t\right)_{t\leq T}$ denote a solution of the DRBSDE (2.1) with data (ξ, f, L, U) in the sense of Definition 2.3. Then, we have the following proposition:

Proposition 3.2. *There exists at most one solution (Y, Z, K^+, K^-, N) of the DRBSDE (2.1) associated with data (ξ, f, L, U).*

Proof. Due to the lack integrability of the processes (Z, N) and (Z', N') (see Definition 2.3), we will use a localization procedure in order to get the martingale characterization in Itô's formula. To this end, for $k \geq 1$, we define the sequence of stopping times $\{\sigma_k\}_{k\geq 1}$ as follows:

$$\sigma_k := \inf\left\{t \geq 0, \int_0^t e^{\beta A_s}\left(|Z_s|^2 + |Z'_s|^2\right) d\langle M\rangle_s + \int_0^t e^{\beta A_s}\left(d\langle N\rangle_s + d\langle N'\rangle_s\right) \geq k\right\} \wedge T.$$

From the definition of the solution (see Definition 2.3), we deduce that the sequence $\{\sigma_k\}_{k\geq 1}$ is non-decreasing, of stationary type that converges $\mathbb{P}$-almost surely to T.

Next, using Itô's formula on $[t\wedge\sigma_k, \sigma_k]$ we can write for $\beta > 0$ and for any $t \in [0, T]$,

$$\begin{aligned}
&e^{\beta A_{t\wedge\sigma_k}}\left|Y_{t\wedge\sigma_k} - Y'_{t\wedge\sigma_k}\right|^2 + \beta\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}|Y_s - Y'_s|^2 d\langle M\rangle_s + \int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}|Z_s - Z'_s|^2 d\langle M^c\rangle_s \\
&+ 2\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}|Z_s - Z'_s|^2 d\langle M^c, N^c\rangle_s + \int_{t\wedge\sigma_k}^{\sigma_k} e^{2A_s} d\langle N^c\rangle_s + \sum_{t\wedge\sigma_k < s \leq \sigma_k} e^{\beta A_s}(\Delta Y_s)^2 \\
&= e^{\beta A_{\sigma_k}}\left|Y_{\sigma_k} - Y'_{\sigma_k}\right|^2 + 2\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}(Y_s - Y'_s)\left(f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)\right) d\langle M\rangle_s \\
&+ 2\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left(Y_{s-} - Y'_{s-}\right)\left(dK^+_s - dK'^{,+}_s\right) - 2\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left(Y_{s-} - Y'_{s-}\right)\left(dK^-_s - dK'^{,-}_s\right) \\
&- 2\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left(Y_{s-} - Y'_{s-}\right)(Z_s - Z'_s)\, dM_s - 2\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left(Y_{s-} - Y'_{s-}\right)(dN_s - dN'_s).
\end{aligned} \tag{3.1}$$

Note that, from Lemma 3.1 and [26, Ch I. Theorem 4.40], we deduce that the stochastic integrals $\left(\int_0^{t\wedge\sigma_k} e^{\beta A_s} Z_s dM_s\right)_{t\leq T}$, $\left(\int_0^{t\wedge\sigma_k} e^{\beta A_s} Z'_s dM_s\right)_{t\leq T}$, $\left(\int_0^{t\wedge\sigma_k} e^{\beta A_s} dN_s\right)_{t\leq T}$, and $\left(\int_0^{t\wedge\sigma_k} e^{\beta A_s} dN'_s\right)_{t\leq T}$ are RCLL square integrable $\mathbb{F}$-martingales. Moreover, due to the integrability condition satisfied by the first component Y of the solution of the DRBSDE (2.1), we deduce that the two terms in the last line of (3.1) are uniformly integrable $\mathbb{F}$-martingales.

Next, using assumption **(H2)**-(ii), the fact that $\alpha_s^2 = \kappa_s + \gamma_s^2$, and inequality $2ab \leq 2a^2 + \frac{1}{2}b^2$, we get

$$\begin{aligned}
2(Y_s - Y'_s)(f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) &\leq 2\kappa_s|Y_s - Y'_s|^2 + 2\gamma_s|Y_s - Y'_s||Z_s - Z'_s| \\
&\leq 2\alpha_s^2|Y_s - Y'_s|^2 + \frac{1}{2}|Z_s - Z'_s|^2.
\end{aligned} \tag{3.2}$$

On the other hand, thanks to the Skorokhod condition (2.1)-(iii)-(iv), we obtain

$$\begin{aligned}
&\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left(Y_{s-} - Y'_{s-}\right)\left(dK^+_s - dK'^{,+}_s\right) \\
&= \int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}(Y_{s-} - L_{s-})\, dK^+_s + \int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}(L_{s-} - Y_{s-})\, dK'^{,+}_s \\
&+ \int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left(L'_{s-} - Y'_{s-}\right) K^+_s + \int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left(Y'_{s-} - L'_{s-}\right) dK'^{,+}_s \\
&\leq 0.
\end{aligned} \tag{3.3}$$

Similarly, we may show that

$$\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left(Y_{s-} - Y'_{s-}\right)\left(dK^-_s - dK'^{,-}_s\right) \geq 0, \ \mathbb{P}\text{-a.s.} \tag{3.4}$$

Plugging (2.3), (2.4), (2.5), (2.7), (2.8), (3.2), (3.3) and (3.4) into (3.1), we obtain, after taking the expectation on both sides,

$$\mathbb{E}\left[e^{\beta A_t}\left|Y_{t\wedge\sigma_k}-Y'_{t\wedge\sigma_k}\right|^2\right]+(\beta-2)\mathbb{E}\left[\int_{t\wedge\sigma_k}^{\sigma_k}e^{\beta A_s}\alpha_s^2\left|Y_s-Y'_s\right|^2d\langle M\rangle_s\right]$$
$$+\frac{1}{2}\mathbb{E}\left[\int_{t\wedge\sigma_k}^{\sigma_k}e^{\beta A_s}\left|Z_s-Z'_s\right|^2d\langle M\rangle_s\right]+\mathbb{E}\left[\int_{t\wedge\sigma_k}^{\sigma_k}e^{\beta A_s}d\left[N-N'\right]_s\right]$$
$$\leq\mathbb{E}\left[e^{\beta A_{\sigma_k}}\left|Y_{\sigma_k}-Y'_{\sigma_k}\right|^2\right]$$

Choosing $\beta>2$, after that, using Fatou's Lemma, the monotonic convergence theorem, Lebesgue dominated convergence theorem, and the continuity of the process $(A_t)_{t\leq T}$, all with respect to k, we obtain that $Y=Y'$, $Z=Z'$, $N=N'$ and $K^+-K^-=K'^{,+}-K'^{,-}$. On the one hand, the expression of $K^{d,\pm}$ and $K'^{d,\pm}$ by means of Y and Y', respectively, implies that $K^{d,\pm}=K'^{d,\pm}$ (see Remark 2.2-(ii)). Additionally, if $L<U$ is satisfied on $[0,T]$, then, we may easily deduce that from $(U_s-L_s)\left(dK_s^{c,\pm}-dK_s'^{c,\pm}\right)=0$ that $K^{c,\pm}=K'^{c,\pm}$. Henceforth, $\left(Y_t,Z_t,K_t^+,K_t^-N_t\right)=\left(Y'_t,Z'_t,K_t'^{,+},K_t'^{,-}N'_t\right)$, and this completes the proof of uniqueness. □

4 Local BSDE solutions with two general RCLL reflecting barriers

We will now demonstrate the existence of a process Y that fulfills the DRBSDE (2.1) under the general assumptions **(H1)-(H3)** locally, this latter means that, for any $\mathbb{F}$-stopping time τ, it is possible to find an $\mathbb{F}$-stopping time θ_τ that is bigger than τ, such that on $[\tau,\theta_\tau]$, Y satisfies the DRBSDE (2.1) with terminal condition Y_{θ_τ}. The process Y will be built as the limit of two different types of penalization schemes, increasing and decreasing.

To begin, we assume that the driver f does not depend on (y,z), i.e., $\mathbb{P}$-a.s., $f(\omega,t,y,z)=:g(\omega,t)$ for any t, y, and z. It is noteworthy that the driver process g belongs to $\mathcal{H}^2_\beta$ (Assumption **(H2)**-(ii)).

Let us now look at the increasing penalization schemes.

4.1 The increasing penalization schemes

For $n\geq 0$, let $(Y_t^n,Z_t^n,K_t^n,N_t^n)_{t\leq T}$ be the quadruple of processes with values in $\mathbb{R}^2\times\mathbb{R}^+\times\mathbb{R}$ such that: $\forall t\in[0,T]$ a.s.

$$\begin{cases}\text{(i) } (Y_t^n,Z_t^n,K_t^n,N_t^n)_{t\leq T}\in\mathfrak{A}^2_\beta.\\ \text{(ii) } Y_t^n=\xi+\int_t^T g(s)d\langle M\rangle_s+n\int_t^T(L_s-Y_s^n)^+ds-(K_T^n-K_t^n)-\int_t^T Z_s^n dM_s-\int_t^T dN_s^n.\\ \text{(iii) } Y_t^n\leq U_t.\\ \text{(iv) If } K^{n,c} \text{ is the continuous part of } K^\pm, \text{ then } \int_0^T(Y_t-U_t)dK_t^{n,c}=0.\\ \text{(v) If } K^{n,d} \text{ is the purely discontinuous part of } K^n, \text{ then } K_t^{n,d}=\sum_{0<s\leq t}(Y_s^n-U_{s-})^+.\end{cases}\tag{4.1}$$

We denote $K_t^{n,+}:=n\int_0^t(L_s-Y_s^n)^+ds$.

The existence of the process $(Y_t^n,Z_t^n,K_t^n,N_t^n)_{t\leq T}$ is due to Corollary .6. Recall also that any $\mathcal{F}_t$-predictable increasing process K^n admits a classical decomposition $K^n=K^{n,c}+K^{n,d}$, where $K^{n,c}$ the continuous part of K^n and $K^{n,d}$ is the predictable purely-discontinuous part.

Remark 4.1. Let us notice that, rewriting equation (4.1)-(i) forwardly, we easily deduce that, for each $n\geq 0$ in the state process Y^n has only positive predictable jumps, and they are described by $\Delta Y_\delta^n=(Y_\delta^n-U_{\delta-})^+\mathbb{1}_{\{Y_{\delta-}^n=U_{\delta-}\}}$ for every $\mathbb{F}$-predictable jump time $\delta\in\mathcal{T}_0^T$. Therefore, when Y^n jumps positively at a predictable time δ, we must have $Y_{\delta-}^n=U_{\delta-}$ and $\Delta U_\delta>0$.

It is worth noting that throughout our discussion, we consistently denote $C_\beta>0$ as a constant whose value depends only on the parameter $\beta>0$ and may vary from line to line.

The comparison Theorem .5 implies that, $U \geq Y^{n+1} \geq Y^n$, for all $n \geq 0$, then we deduce the existence of an $\mathcal{F}_t$-optional process $Y := (Y_t)_{t \leq T}$ such that $\mathbb{P}$-a.s., for any $t \leq T$, $Y_t^n \nearrow Y_t$ as $n \to \infty$. Moreover, from the fact that U is right-continuous, we also get that the process Y is right-lower semi-continuous over $[0, T]$. In the one hand, from this monotonic property, we have $Y^0 \leq Y^n \leq U$, $\forall n \in \mathbb{N}$. Thus, $Y \leq U$.

Next, we consider the penalization equation associated with the one reflected BSDE (.2)

$$\bar{Y}_t^n = \xi + \int_t^T g(s) d\langle M\rangle_s + n \int_t^T (L_s - \bar{Y}_s)^+ ds - \int_0^T \bar{Z}_t^n dM_s - \int_t^T d\bar{N}_s^n.$$

From the comparison Theorem .2, we deduce that $Y_t^0 \leq Y_t^n \leq \bar{Y}_t^n$ for all $t \leq T$. Moreover, from Remark .4, we get $\mathbb{E}\left[\sup_{0 \leq t \leq T} e^{\beta A_t} \left|\bar{Y}_t^n\right|^2\right] \leq C_\beta$, for any $n \geq 0$, for some constant C_β independent of n. Thus

$$\sup_{n \in \mathbb{N}} \left\{\mathbb{E}\left[\sup_{0 \leq t \leq T} e^{\beta A_t} |Y_t^n|^2\right]\right\} \leq \max\left(\mathbb{E}\left[\sup_{0 \leq t \leq T} e^{\beta A_t} \left|Y_t^0\right|^2\right], \mathbb{E}\left[\sup_{0 \leq t \leq T} e^{\beta A_t} \left|\bar{Y}_t^n\right|^2\right]\right) \leq C_\beta. \tag{4.2}$$

Then using Fatou's lemma, we obtain

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} e^{\beta A_t} |Y_t|^2\right] \leq \liminf_{n \to +\infty} \mathbb{E}\left[\sup_{0 \leq t \leq T} e^{\beta A_t} |Y_t^n|^2\right] \leq C_\beta. \tag{4.3}$$

On the other hand, following a similar argument as in (4.2), we get

$$\sup_{n \in \mathbb{N}} \left\{\mathbb{E}\left[\int_0^T e^{\beta A_s} |\alpha_s Y_s^n|^2 d\langle M\rangle_s\right]\right\} \leq \mathbb{E}\left[\int_0^T e^{\beta A_s} \left(\left|\bar{Y}_s^n\right|^2 \vee \left|Y_s^0\right|^2\right) dA_s\right] \leq C_\beta.$$

Once more, using Fatou's Lemma and the dominated convergence theorem, we obtain

$$\mathbb{E}\left[\int_0^T e^{\beta A_s} |\alpha_s Y_s|^2 d\langle M\rangle_s\right] \leq \liminf_{n \to +\infty} \mathbb{E}\left[\int_0^T e^{\beta A_s} |\alpha_s Y_s^n|^2 d\langle M\rangle_s\right] \leq C_\beta.$$

Henceforth, the sequence $\{Y^n\}_{n \in \mathbb{N}}$ converges to Y in $\mathcal{S}_\beta^{2,\alpha}$ as a consequence of the dominated convergence theorem, and the process Y belongs to $\mathcal{S}_\beta^2 \cap \mathcal{S}_\beta^{2,\alpha}$.

Now, for any stopping time τ, let us set

$$\begin{aligned} \delta_\tau^n &:= \inf\{s \geq \tau : K_s^n - K_\tau^n > 0\} \wedge T \\ &= \inf\left\{s \geq \tau : K_s^{n,d} - K_\tau^{n,d} > 0\right\} \wedge \inf\{s \geq \tau : K_s^{n,c} - K_\tau^{n,c} > 0\} \wedge T. \end{aligned}$$

Once more, using the comparison results (Theorem .5), we deduce that, $K_t^{n+1} - K_s^{n+1} \geq K_t^n - K_s^n$, $0 \leq s \leq t \leq T$. Therefore $(\delta_\tau^n)_{n \geq 0}$ is a decreasing sequence of $\mathbb{F}$-stopping times and then it converges toward another $\mathbb{F}$-stopping time $\delta_\tau := \lim_{n \to +\infty} \delta_\tau^n \geq \tau$, $\mathbb{P}$-a.s. Furthermore, keep in mind that for any $t \in [\tau, \delta_\tau[$, $K_t^{n,d} = K_\tau^{n,d}$, $\mathbb{P}$-a.s., for all $n \geq 0$.

The processes Y satisfies the following properties:

Proposition 4.2. *(i) For any stopping time τ, we have, $\mathbb{P}$-a.s.*

$$Y_{\delta_\tau} \mathbb{1}_{\{\delta_\tau < T\}} \geq \left(U_{\delta_\tau} - \mathbb{1}_{\{\delta_\tau > \tau\}} (\Delta U_{\delta_\tau})^+\right) \mathbb{1}_{\{\delta_\tau < T\}}.$$

(ii) $L_t \leq Y_t \leq U_t$, $\forall t \in [0, T]$, $\mathbb{P}$-a.s.

Proof. (i) On one hand, recall that the role of the process $K_t^{n,d} = \sum_{0 < s \leq t} (Y_s^n - U_{s-})^+ \mathbb{1}_{\{\Delta U_s > 0\}}$ is to make the necessary jump when U has a positive predictable jump to keep it below U.

In this case, Y^n and $K^{n,d}$ have the same jump size, so we need $Y^n_- = U_-$. On the other hand, the continuity of $K^{n,c}$ combined with the definition of δ^n_τ and (4.1-(i)) yields

$$Y^n_t = Y^n_{\delta^n_\tau} + \int_t^{\delta^n_\tau} g(s)d\langle M\rangle_s + n\int_t^{\delta^n_\tau}(L_s - Y^n_s)^+ ds - (K^{n,d}_{\delta^n_\tau} - K^{n,d}_t) - \int_t^{\delta^n_\tau} Z^n_s dM_s - \int_t^{\delta^n_\tau} dN^n_s, \quad \forall t \in [\tau, \delta^n_\tau]. \tag{4.4}$$

In the equation (4.4), the term $K^{n,d}_{\delta^n_\tau} - K^{n,d}_t$ remains because the process $K^{n,d}$ could experience a jump at δ^n_τ. In this situation, we have $K^{n,d}_{\delta^n_\tau} - K^{n,d}_t > 0$ for all $t \in [\tau, \delta^n_\tau[$. Additionally, from the definition of $K^{n,d}$, we obtain

$$\forall t \in [\tau, \delta^n_\tau], \quad K^{n,d}_{\delta^n_\tau} - K^{n,d}_t \leq \left(Y^n_{\delta^n_\tau} - U_{\delta_\tau -}\right)^+ \mathbb{1}_{\{t<\delta^n_\tau\}\cap\{Y^n_{\delta^n_\tau -}=U_{\delta^n_\tau -}\}\cap\{\Delta U_{\delta^n_\tau -}>0\}}. \tag{4.5}$$

This, combined with (4.2), the inequality $(a-b)^+ \leq a^+ + b^-$, and assumption **(H3)**-(ii), gives

$$\sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left[\sup_{\tau\leq t\leq\delta^n_\tau} e^{\beta A_s}\left|K^n_{\delta^n_\tau} - K^n_t\right|^2\right]\right\} \leq C_\beta \mathbb{E}\left[\sup_{0\leq t\leq T} e^{2\beta A_t}\left(U^-_t\right)^2\right] \tag{4.6}$$

Using estimation (4.6), we may show the following lemma:

Lemma 4.3. *There exists a positive constant C_β depending only on β (but not on n), such that*

$$\sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left[\int_\tau^{\delta^n_\tau} e^{\beta A_s}|Y^n_s\alpha_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_\tau^{\delta^n_\tau} e^{\beta A_s}|Z^n_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_\tau^{\delta^n_\tau} e^{\beta A_s} d[N^n]_s\right] + \mathbb{E}\left[\left|K^{n,+}_{\delta^n_\tau} - K^{n,+}_\tau\right|^2\right]\right\} \leq C_\beta. \tag{4.7}$$

Proof. By Itô's formula, we obtain: For any $t \in [\tau, \delta^n_\tau]$

$$\begin{aligned}
&e^{\beta A_t}|Y^n_t|^2 + \beta\int_t^{\delta^n_\tau} e^{\beta A_s}|Y^n_s|^2 dA_s + \int_t^T e^{\beta A_s}|Z^n_s|^2 d\langle M^c\rangle_s + \int_t^{\delta^n_\tau} e^{\beta A_s} d\langle (N^n)^c\rangle_s \\
&\quad + \int_t^T e^{\beta A_s} Z^n_s d\langle M^c, (N^n)^c\rangle_s + \sum_{t<s\leq\delta^n_\tau} e^{\beta A_s}(\Delta Y^n_s)^2 \\
&= e^{\beta A_{\delta^n_\tau}}\left|Y^n_{\delta^n_\tau}\right|^2 + 2\int_t^{\delta^n_\tau} e^{\beta A_s} Y^n_s g(s) d\langle M\rangle_s + 2n\int_t^{\delta^n_\tau} e^{\beta A_s} Y^n_s (Y^n_s - L_s)^- d\langle M\rangle_s \\
&\quad - 2\int_t^{\delta^n_\tau} e^{\beta A_s} Y^n_{s-} dK^n_s - 2\int_t^{\delta^n_\tau} e^{\beta A_s} Y^n_{s-} Z^n_s dM_s - 2\int_t^{\delta^n_\tau} e^{\beta A_s} Y^n_{s-} dN^n_s.
\end{aligned} \tag{4.8}$$

Taking the expectation on both sides, and using equalities (2.3), (2.4), (2.7), (2.8), along with the inequalities $a(a-b)^- \leq b^+(a-b)^-$, $2ab \leq \frac{1}{\epsilon}a^2 + \epsilon b^2$ for $\epsilon > 0$, the sharp bracket version of the Kunita-Watanabe inequality, and the fact that the stochastic integrals appearing on the right-hand side of (4.8) are all square-integrable $\mathbb{F}$-martingales, yields

$$\begin{aligned}
&\beta\mathbb{E}\left[\int_t^{\delta_\tau^n} e^{\beta A_s}\left|Y_s^n\alpha_s\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_t^T e^{\beta A_s}\left|Z_s^n\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_t^{\delta_\tau^n} e^{\beta A_s} d\left[N^n\right]_s\right]\\
&\leq \mathbb{E}\left[e^{\beta A_{\delta_\tau^n}}\left|Y_{\delta_\tau^n}^n\right|^2\right]+\mathbb{E}\left[\int_t^{\delta_\tau^n} e^{\beta A_s}\left|\alpha_s Y_s^n\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\\
&\quad+\epsilon\mathbb{E}\left[\sup_{0\leq t\leq T} e^{2\beta A_t}\left(L_t^+\right)^2\right]+\frac{1}{\epsilon}\mathbb{E}\left[\left|K_{\delta_\tau^n}^{n,+}-K_t^{n,+}\right|^2\right]\\
&\quad+\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t}\left|Y_t^n\right|^2\right]+\mathbb{E}\left[\sup_{t\leq s\leq\delta_\tau^n} e^{\beta A_t}\left|K_{\delta_\tau^n}^n-K_t^n\right|^2\right].
\end{aligned} \tag{4.9}$$

Furthermore, since the process $\left(\int_0^\cdot Z_s^n dM_s\right)\left(\int_0^\cdot dN_s^n\right)-\int_0^\cdot Z_s^n d\left[M,N^n\right]_s$ is a uniformly integrable martingale starting from zero (Proposition 4.50-(a), pp. 53 in [26]), the isometric formula, and the orthogonality property of the sequence of martingales $\{N^n\}_{n\in\mathbb{N}}$, implies

$$\begin{aligned}
\mathbb{E}\left[\left(\int_t^{\delta_\tau^n} Z_s^n dM_s\right)\left(\int_t^{\delta_\tau^n} dN_s^n\right)\right]&=\mathbb{E}\left[\int_t^{\delta_\tau^n} Z_s^n d\left[M,N^n\right]_s\right]\\
&=\mathbb{E}\left[\int_t^{\delta_\tau^n} Z_s^n d\left\langle M,N^n\right\rangle_s\right]=0.
\end{aligned} \tag{4.10}$$

From equation (4.4), by squaring and taking into account (4.10) and inequality (2.6) with $b=\beta$, we conclude that

$$\begin{aligned}
&\mathbb{E}\left[\left|K_{\delta_\tau^n}^{n,+}-K_t^{n,+}\right|^2\right]\\
&\leq 6\Bigg(\mathbb{E}\left[\left|Y_{\delta_\tau^n}^n\right|^2+\left|Y_t^n\right|^2\right]+\frac{1}{\beta}\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\left|K_{\delta_\tau^n}^{n,d}-K_t^{n,d}\right|^2\right]\\
&\quad+\mathbb{E}\left[\int_t^T\left|Z_s^n\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_t^{\delta_\tau^n} d\left[N^n\right]_s\right]\Bigg).
\end{aligned} \tag{4.11}$$

Choosing $\beta>1>$, $\epsilon>\frac{6}{(\beta-1)\wedge 1}$, using inequalities (4.2), (4.6), and finally evolving inequality (4.11) to (4.9), we deduce the existence a constant C_β such that

$$\begin{aligned}
&\mathbb{E}\left[\int_\tau^{\delta_\tau^n} e^{\beta A_s}\left|Y_s^n\alpha_s\right|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_\tau^{\delta_\tau^n} e^{\beta A_s}\left|Z_s^n\right|^2 d\langle M\rangle_s\right]\\
&\quad+\mathbb{E}\left[\int_\tau^{\delta_\tau^n} e^{\beta A_s} d\left[N^n\right]_s\right]+\mathbb{E}\left[\left|K_{\delta_\tau^n}^{n,+}-K_\tau^{n,+}\right|^2\right]\\
&\leq C_\beta\Bigg(\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t}\left|Y_t^0\right|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\\
&\quad+\mathbb{E}\left[\sup_{0\leq t\leq T} e^{2\beta A_t}\left(L_t^+\right)^2\right]+\mathbb{E}\left[\sup_{0\leq t\leq T} e^{2\beta A_t}\left(U_t^-\right)^2\right]\Bigg).
\end{aligned}$$

which complete the proof. □

Next, from (4.7), the fact that $n(Y_s^n-L_s)^-\geq 0$ and $Y_{\delta_\tau}^n\mathbb{1}_{\{\delta_\tau<T\}}\in\mathcal{F}_{\delta_\tau}$, (4.4), and (4.5), it follows that:

$$\begin{aligned}
Y_{\delta_\tau}^n\mathbb{1}_{\{\delta_\tau<T\}}\geq\mathbb{E}^{\mathcal{F}_{\delta_\tau}}&\left[\left(Y_{\delta_\tau^n}^n-\mathbb{1}_{\{\delta_\tau<\delta_\tau^n\}}\left(Y_{\delta_\tau^n}^n-U_{\delta_\tau^n-}\right)^+\right)\mathbb{1}_{\{\delta_\tau<T\}}\right]\\
&-\mathbb{E}^{\mathcal{F}_{\delta_\tau}}\left[\int_{\delta_\tau}^{\delta_\tau^n}|g(s)|\,d\langle M\rangle_s\right]
\end{aligned} \tag{4.12}$$

But it holds true that $Y^n_{\delta^n_\tau}\mathbb{1}_{\{\delta^n_\tau<T\}} \geq U_{\delta^n_\tau}\mathbb{1}_{\{\delta^n_\tau<T\}} - \mathbb{1}_{\{\tau<\delta^n_\tau<T\}}(\Delta U_{\delta^n_\tau})^+$. Actually, on the set $\{\delta^n_\tau > \tau\}\cap\{\delta^n_\tau < T\}$, either of the two processes $K^{n,c}$ or $K^{n,d}$ may increases at δ^n_τ, then we have either $\{Y_{\delta^n_\tau -} = U_{\delta^n_\tau -}$ and $Y_{\delta^n_\tau} > U_{\delta^n_\tau -}\}$ or $\{Y_{\delta^n_\tau} = U_{\delta^n_\tau}\}$. Hence, $Y^n_{\delta^n_\tau} \geq U_{\delta^n_\tau} - (\Delta U_{\delta^n_\tau})^+$. Now on $\{\delta^n_\tau = \tau\}\cap\{\delta^n_\tau < T\}$, there exists a decreasing sequence of real numbers $(t^n_k)_{k\geq 0}$ converging to τ such that $Y^n_{t^n_k} \geq U_{t^n_k} - (\Delta U_{t^n_k})^+$. Taking the limit as $k\to\infty$ gives $Y^n_\tau \geq U_\tau$ since Y^n and U are RCLL.

Returning to (4.12), we get

$$
\begin{aligned}
&Y^n_{\delta_\tau}\mathbb{1}_{\{\delta_\tau<T\}}\\
&\geq \mathbb{E}^{\mathcal{F}_{\delta_\tau}}\left[\left(\left(U_{\delta^n_\tau} - \mathbb{1}_{\{\delta^n_\tau>\tau\}}(\Delta U_{\delta^n_\tau})^+\right)\mathbb{1}_{\{\delta^n_\tau<T\}} - \mathbb{1}_{\{\delta_\tau<\delta^n_\tau\}}\left(Y^n_{\delta^n_\tau} - U_{\delta^n_\tau -}\right)^+\right)\mathbb{1}_{\{\delta_\tau<T\}}\right]\\
&\quad + \mathbb{E}^{\mathcal{F}_{\delta_\tau}}\left[\xi\mathbb{1}_{\{\delta^n_\tau=T\}\cap\{\delta_\tau<T\}}\right] - \mathbb{E}^{\mathcal{F}_{\delta_\tau}}\left[\int_{\delta_\tau}^{\delta^n_\tau}|g(s)|\,d\langle M\rangle_s\right]
\end{aligned}
\tag{4.13}
$$

We examine now the terms on the right-hand side of (4.13). In the space $\mathbb{L}^1(\Omega)$, since $\delta^n_\tau \searrow \delta_\tau$, we have $\mathbb{E}^{\mathcal{F}_{\delta_\tau}}\left[\xi\mathbb{1}_{\{\delta^n_\tau=T\}\cap\{\delta_\tau<T\}}\right] \to 0$ as $n\to+\infty$. Moreover, by applying the dominated convergence theorem for stochastic integrals (Theorem 32 in [39], pp. 174), we get $\lim_{n\to+\infty}\int_{\delta_\tau}^{\delta^n_\tau}|g(s)|\,d\langle M\rangle_s = 0$ in **ucp**. Hence, we can assume, passing to a sub-sequence if necessary, that $\int_{\delta_\tau}^{\cdot}\mathbb{1}_{\{s\leq\delta^n_\tau\}}|g(s)|\,d\langle M\rangle_s$ converges to 0, $\mathbb{P}$-a.s. But, the sequence $\{\delta^n_\tau\}_{n\in\mathbb{N}}$ is decreasing; thus, we have convergence for the entire sequence rather than just a sub-sequence. We just need to use the classical Lebesgue-dominate convergence theorem since $g\in\mathcal{H}^2_\beta$ by assumption, which implies $\int_{\delta_\tau}^{\delta^n_\tau}|g(s)|\,ds \to 0$ as $n\to+\infty$ in $\mathbb{L}^2(\Omega)$. Otherwise, let us set $A = \cap_{n\geq 0}\{\delta_\tau < \delta^n_\tau\}$. For n large enough, we have

$$\mathbb{1}_{\{\delta_\tau<\delta^n_\tau\}}(Y^n_{\delta^n_\tau} - U_{\delta^n_\tau -})^+ = \mathbb{1}_A(Y^n_{\delta^n_\tau} - U_{\delta^n_\tau -})^+.$$

Therefore,

$$\limsup_{n\to\infty}\mathbb{1}_{\{\delta_\tau<\delta^n_\tau\}}(Y^n_{\delta^n_\tau} - U_{\delta^n_\tau -})^+ \leq \mathbb{1}_A\limsup_{n\to\infty}(Y_{\delta^n_\tau} - U_{\delta^n_\tau -})^+ = 0,$$

and then

$$\lim_{n\to\infty}\mathbb{1}_{\{\delta_\tau<\delta^n_\tau\}}(Y^n_{\delta^n_\tau} - U_{\delta^n_\tau -})^+ = 0.$$

This, with the right continuity of the barrier U, yields

$$\lim_{n\to\infty}\left(U_{\delta^n_\tau} - \mathbb{1}_{\{\delta^n_\tau>\tau\}}(\Delta U_{\delta^n_\tau})^+\right) \geq U_{\delta_\tau} - \mathbb{1}_{\{\delta_\tau>\tau\}}(\Delta U_{\delta_\tau})^+,$$

and $\mathbb{1}_{\{\delta^n_\tau<T\}\cap\{\delta_\tau<T\}} \to \mathbb{1}_{\{\delta_\tau<T\}}$ as $n\to\infty$. Then, on the set $\{\delta_\tau < T\}$, after extracting a sub-sequence that converges $\mathbb{P}$-a.s. and taking the limit, we deduce that

$$Y_{\delta_\tau} \geq U_{\delta_\tau} - \mathbb{1}_{\{\delta_\tau>\tau\}}(\Delta U_{\delta_\tau})^+, \quad \mathbb{P}\text{-a.s.}$$

Or equivalently,

$$Y_{\delta_\tau}\mathbb{1}_{\{\delta_\tau<T\}} \geq U_{\delta_\tau}\mathbb{1}_{\{\delta_\tau<T\}} - \mathbb{1}_{\{\tau<\delta_\tau<T\}}(\Delta U_{\delta_\tau})^+, \quad \mathbb{P}\text{-a.s.}$$

Let consider the following unconstrained BSDE:

$$
\begin{aligned}
\bar{Y}^{n,*}_t = Y^n_{\delta_\tau} + \int_t^{\delta_\tau} g(s)d\langle M\rangle_s + n\int_t^{\delta_\tau}(L_s - \bar{Y}^{n,*}_s)ds - \int_t^{\delta_\tau} dK^{n,d}_s\\
- \int_t^{\delta_\tau}\bar{Z}^{n,*}_s dM_s - \int_t^{\delta_\tau} d\bar{N}^{n,*}_s,\ \forall t\in[\tau,\delta_\tau].
\end{aligned}
$$

The comparison Theorem .5 implies that, for any $n \geq 0$, $\bar{Y}_t^{n,*} \leq Y_t^n$, $\forall t \in [\tau, \delta_\tau]$. Next, applying the integration by part formula (see Corollary 2, pp. 68 in [39]) to $e^{-n(s-\tau)}\bar{Y}_s^{n,*}$ on the interval between τ and δ_τ, we obtain

$$\begin{aligned}\bar{Y}_\tau^{n,*} &= e^{-n(\delta_\tau-\tau)}Y_{\delta_\tau}^n + \int_\tau^{\delta_\tau} e^{-n(s-\tau)}g(s)d\langle M\rangle_s + n\int_\tau^{\delta_\tau} e^{-n(s-\tau)}L_s ds \\ &\quad - \int_\tau^{\delta_\tau} e^{-n(s-\tau)}dK_s^{n,d} - \int_\tau^{\delta_\tau} e^{-n(s-\tau)}\bar{Z}_s^{n,*}dM_s - \int_\tau^{\delta_\tau} e^{-n(s-\tau)}d\bar{N}_s^{n,*}.\end{aligned}$$

Then, taking the conditional expectation with respect to $\mathcal{F}_\tau$, we can write

$$\begin{aligned}\bar{Y}_\tau^{n,*} = \mathbb{E}^{\mathcal{F}_\tau}\Bigg[&e^{-n(\delta_\tau-\tau)}Y_{\delta_\tau}^n + \int_\tau^{\delta_\tau} e^{-n\int_\tau^s d\langle M\rangle_r}g(s)d\langle M\rangle_s \\ &+n\int_\tau^{\delta_\tau} e^{-n(s-\tau)}L_s ds - \int_\tau^{\delta_\tau} e^{-n(s-\tau)}dK_s^{n,d}\Bigg].\end{aligned} \tag{4.14}$$

Furthermore, from the dynamic of the process $K^{n,d}$ given in (4.5) over $[\tau, \delta_\tau]$, we have

$$-\int_\tau^{\delta_\tau} e^{-n(s-\tau)}dK_s^{n,d} \geq -e^{-n(\delta_\tau-\tau)}\left(Y_{\delta_\tau}^n - U_{\delta_\tau-}\right)^+ \mathbb{1}_{\{\delta_\tau>\tau\}}.$$

Plugging this into (4.14), we get

$$\begin{aligned}\bar{Y}_\tau^{n,*} \geq\ &\mathbb{E}^{\mathcal{F}_\tau}\left[\left\{Y_{\delta_\tau}^n - \left(Y_{\delta_\tau}^n - U_{\delta_\tau-}\right)^+ \mathbb{1}_{\{\delta_\tau>\tau\}}\right\}e^{-n(s-\tau)}\right] \\ &+\mathbb{E}^{\mathcal{F}_\tau}\left[\int_\tau^{\delta_\tau} e^{-n\int_\tau^s d\langle M\rangle_r}g(s)d\langle M\rangle_s + n\int_\tau^{\delta_\tau} e^{-n(s-\tau)}L_s ds\right].\end{aligned}$$

Using the fact that $Y_{\delta_\tau}^n \to Y_{\delta_\tau}$, the integrability condition satisfied by Y^n and L, combined with the right continuity of it's trajectories, we can easily deduce that the following convergences hold in $\mathbb{L}^1(\Omega)$: $\left\{n\int_\tau^{\delta_\tau} e^{-n(s-\tau)}L_s ds + \int_\tau^{\delta_\tau} e^{-n(s-\tau)}g(s)\mathrm{d}\langle M\rangle_s\right\}_{n\geq 0}$ converge to $L_\tau \mathbb{1}_{\{\delta_\tau>\tau\}}$ and the sequence $\left(\{Y_{\delta_\tau}^n - \mathbb{1}_{\{\delta_\tau>\tau\}}(Y_{\delta_\tau}^n - U_{\delta_{\tau-}})^+\}e^{-n(\tau-\delta_\tau)}\right)_{n\geq 0}$ converges to $Y_{\delta_\tau}\mathbb{1}_{\{\delta_\tau=\tau\}}$.

Therefore, after extracting a subsequence (if necessary), we have

$$Y_\tau = \lim_{n\to\infty} Y_\tau^n \geq \lim_{n\to\infty} \bar{Y}_\tau^{n,*} \geq L_\tau \mathbb{1}_{\{\delta_\tau>\tau\}} + Y_{\delta_\tau}\mathbb{1}_{\{\delta_\tau=\tau\}}. \tag{4.15}$$

Using **(H3)**-(iii), we obtain

$$\begin{aligned}Y_{\delta_\tau}\mathbb{1}_{\{\delta_\tau=\tau\}} &\geq U_{\delta_\tau}\mathbb{1}_{\{\delta_\tau=\tau\}\cap\{\delta_\tau<T\}} + \xi\mathbb{1}_{\{\delta_\tau=\tau\}\cap\{\delta_\tau=T\}} \\ &\geq L_{\delta_\tau}\mathbb{1}_{\{\delta_\tau=\tau\}\cap\{\delta_\tau<T\}} + L_{\delta_\tau}\mathbb{1}_{\{\delta_\tau=\tau\}\cap\{\delta_\tau=T\}} = L_{\delta_\tau}\mathbb{1}_{\{\delta_\tau=\tau\}}.\end{aligned}$$

Thus, putting this together with (4.15), we get $Y_\tau \geq L_\tau$, for every stopping time $\tau \in \mathcal{T}_0^T$. Finally, since Y and L are $\mathbb{F}$-optional process, we deduce that $Y \geq L$ (see Theorem 4.10 in [25], pp. 116).

□

Next, we will show that the limited process Y satisfies locally a generalized BSDE.

Proposition 4.4. *For any stopping time τ there is $(Z', V', K'^{+}, K'^{d,-}, N')$ such that:*

(i) The sextuplet $(Y, Z', V', K'^{+}, K'^{d,-}, N')$ *satisfies the following BSDE :*

$$\begin{cases} (a)\ Z' \in \mathcal{H}^2_\beta, \quad K'^{,+} \in \mathcal{S}^2, \quad K'^{d,-} \in \mathcal{S}^2 \text{ and } N' \in \mathcal{M}^2_\beta, \\ (b)\ Y_t = Y_{\delta_\tau} + \int_t^{\delta_\tau} g(s)d\langle M\rangle_s - (K'^{d,-}_{\delta_\tau} - K'^{d,-}_t) + (K'^{+}_{\delta_\tau} - K'^{+}_t) \\ \qquad\qquad - \int_t^{\delta_\tau} Z'_s dM_s - \int_t^{\delta_\tau} dN'_s, \quad \forall t \in [\tau, \delta_\tau]. \\ (c)\ K'^{+}_\tau = 0, \quad \int_\tau^{\delta_\tau} (Y_s - L_s)dK'^{+,c}_s = 0 \quad \text{and} \quad K'^{+,d}_t = \sum_{\tau < s \le t} (L_{s-} - Y_s)^+, \text{ where} \\ K'^{+,c} \text{ is the continuous part of } K'^{+} \text{ and } K'^{+,d} \text{ it's predictable purely-discontinuous part.} \\ (d)\ K'^{d,-} \text{ is predictable and purely-discontinuous, } K'^{d,-}_\tau = 0,\ K'^{d,-}_t = 0,\ \forall t \in [\tau, \delta_\tau[, \\ \quad \text{and if } K'^{d,-}_{\delta_\tau} > 0 \text{ then } Y_{\delta_\tau -} = U_{\delta_\tau -} \text{ and } K'^{d,-}_{\delta_\tau} = (Y_{\delta_\tau} - U_{\delta_\tau -})^+. \end{cases} \tag{4.16}$$

(ii) The limiting process process $(Y_t)_{t\le T}$ *has right-continuous with left limits paths on on* $[\tau, \delta_\tau]$.

Proof. In establishing our result, we draw inspiration from the proof technique presented in Proposition 3.2 by Hamadène and Wang [24]. We make necessary adaptations to suit our specific context, ensuring the reliability of the proof. The proof will be presented in three parts:

1. In the first part, we consider a sequence of RCLL $\mathbb{F}$-supermartingales defined on $[\tau, \delta_\tau]$ from the BSDE (4.1)-(i). We demonstrate the increasing property of this sequence, leading to its point-wise convergence to another RCLL $\mathbb{F}$-supermartingale.
2. In the second part, we construct the $\mathcal{F}_t$-predictable purely-discontinuous process $K'^{d,-}$ as a limit (in a precise sense) of the sequence $\left\{K^{n,d}\right\}_{n\in\mathbb{N}}$. By leveraging the results from the first two parts, we establish the regularity of the process $(Y_t)_{t\le T}$ on $[\tau, \delta\tau]$, indicating that the process $(Y_t)_{t\le \delta\tau}$ is RCLL.
3. Finally, in the third part, we adapt increasing penalized schemes of reflected BSDEs, constructed from the penalized equations (4.1)-(i). By incorporating the characterization of the solution of reflected BSDEs given in Theorem .3 and utilizing the monotonic property satisfied by the Snell envelope notion for stochastic processes, we establish the first three claims in (4.16).

This adaptation of the proof technique from [24] allows us to present the derivation in a way that is convenient and accessible to the reader, facilitating a thorough understanding of our analysis based on the cited work.

Part 1: By definition, we have $\delta_\tau \le \delta^n_\tau$, then writing equation (4.1)-(i) forwardly, we get

$$Y^n_t = Y^n_\tau - \int_\tau^t g(s)d\langle M\rangle_s - \int_\tau^t dK^{n,+}_s + \int_\tau^t dK^{n,d}_s + \int_\tau^t Z^n_s dM_s + \int_\tau^t dN^n_s, \quad \forall t \in [\tau, \delta_\tau]. \tag{4.17}$$

Therefore, the sequence $\{S^n\}_{n\in\mathbb{N}} := \left\{Y^n_t + \int_\tau^t g(s)d\langle M\rangle_s - \left(K^{n,d}_t - K^{n,d}_\tau\right)\right\}_{n\in\mathbb{N}}$ is a sequence of RCLL $\mathbb{F}$-supermartingale, that satisfies the following BSDE:

$$S^n_t = S^n_{\delta_\tau} + n\int_t^{\delta_\tau} (L_s - Y^n_s)^+ ds - \int_t^{\delta_\tau} Z^n_s dM_s - \int_t^{\delta_\tau} dN^n_s, \quad \forall t \in [\tau, \delta_\tau].$$

and $S^n_{\delta_\tau} = Y^n_{\delta_\tau} + \int_\tau^{\delta_\tau} g(s)d\langle M\rangle_s - \left(K^{n,d}_{\delta_\tau} - K^{n,d}_\tau\right)$.

• *The sequence of processes* $\{S^n\}_{n\in\mathbb{N}}$ *is increasing:*

Now, we aim to demonstrate the increasing nature of the sequence $\{S^n\}_{n\in\mathbb{N}}$ with respect to n. To establish this, we analyze various cases, primarily related to the time interval $[\tau, \delta_\tau]$ and the jump properties of the process $K^{n,d}$ during this stochastic period. First of all, note that if $\tau = \delta_\tau$ (initial time), then $S^n_\tau = Y^n_\tau$, making it unnecessary to prove anything. Next, if $t \in [\tau, \delta_\tau[\cap\{\tau < \delta_\tau\}$, then $S^n_t = Y^n_t + \int_\tau^t g(s)d\langle M\rangle_s \geq S^{n+1}_t$ from the monotonic property of $\{Y^n\}_{n\in\mathbb{N}}$. Finally, we are interested in studying the jumps of the purely-discontinuous $\mathcal{F}_t$-predictable process $K^{n+1,d}$ on the set $\{\delta_\tau = t\} \cap \{\tau < \delta_\tau\}$ for an arbitrary element $t \in [\tau, \delta_\tau]$. Two cases can be observed, depending on whether or not a jump of the process $K^{n+1,d}$ occurs at the stopping time δ_τ. First, If $K^{n+1,d}$ has no jumps at the terminal time δ_τ, then $K^{k,d}_{\delta_\tau} - K^{k,d}_\tau \leq K^{n+1,d}_{\delta_\tau} - K^{n+1,d}_\tau = 0$, $\forall k \leq n$, since, in this case, the process $K^{n+1,d}$ does not increase. In particular, we have $K^{n,d}_{\delta_\tau} - K^{n,d}_\tau = 0$ and $S^n_{\delta_\tau} = Y^n_{\delta_\tau} + \int_\tau^{\delta_\tau} g(s)d\langle M\rangle_s \leq S^{n+1}_{\delta_\tau}$. Secondly, if the process $K^{n+1,d}$ jump at δ_τ, then by definition, we have $K^{n+1,d}_{\delta_\tau} - K^{n+1,d}_\tau > 0$ and necessary δ_τ avoids all $\mathbb{F}$-totally inaccessible stopping time, meaning that, $\mathbb{P}(\omega : \delta_\tau(\omega) = \mu(\omega)) = 0$ for any totally inaccessible $\mathbb{F}$-stopping time μ, we may find a sequence of $\mathbb{F}$-predictable stopping times $\{\mathfrak{t}^\tau_n\}_{n\in\mathbb{N}}$ such that $[\![\delta_\tau]\!] \subset \cup_{n\in\mathbb{N}}[\![\mathfrak{t}^\tau_n]\!]$ (see Theorem 3.33 in [25], pp. 96-97). On the other hand, from the Skorokhod condition (4.1)-(iv), we deduce that $K^{n+1,d}_{\delta_\tau} - K^{n+1,d}_\tau = \left(Y^{n+1}_{\delta_\tau} - U_{\delta_\tau-}\right)^+ \mathbb{1}_{\{Y^{n+1}_{\delta_\tau-}=U_{\delta_\tau-}\}\cap\{\Delta U_{\delta_\tau}>0\}}$, and then $S^{n+1}_{\delta_\tau} = Y^{n+1}_{\delta_\tau-} + \int_\tau^{\delta_\tau} g(s)d\langle M\rangle_s = U_{\delta_\tau-} + \int_\tau^{\delta_\tau} g(s)d\langle M\rangle_s$. So if $K^{n,d}_{\delta_\tau} - K^{n,d}_\tau > 0$, we get $S^{n+1}_{\delta_\tau} = S^n_{\delta_\tau}$. Otherwise, if $K^{n,d}_{\delta_\tau} - K^{n,d}_\tau = 0$, in virtue of (4.17) and the quasi-left continuity of the filtration, we conclude that the sequence $\{Y^n\}_{n\in\mathbb{N}}$ can jump only at inaccessible $\mathbb{F}$-stopping times. But, as we already mentioned, δ_τ cannot be such a kind of stopping time, then $\Delta Y^n_{\delta_\tau} = 0$ and Y^n has continuous paths at δ_τ. This implies that $S^n_{\delta_\tau} \leq S^{n+1}_{\delta_\tau}$. In conclusion, $S^{n+1}_t \geq S^n_t$, $\forall t \in [\tau, \delta_\tau]$.

• *The sequence of processes $\{S^n\}_{n\in\mathbb{N}}$ converges toward another RCLL $\mathbb{F}$-supermartingale S on $[\tau, \delta_\tau]$:*

Since $\{S^n\}_{n\in\mathbb{N}}$ is an increasing sequence of RCLL $\mathbb{F}$-supermartingales, then a direct application of Theorem 18 in [6] (pp. 86) gives $\{S^n\}_{n\in\mathbb{N}} \nearrow S$, where S another RCLL $\mathbb{F}$-supermartingale on $[\tau, \delta_\tau]$. In other word, $\forall t \in [\tau, \delta_\tau]$, the limiting process $S_t := \lim_{n\to+\infty} S^n$ is an RCLL $\mathbb{F}$-supermartingale on $[\tau, \delta_\tau]$.

Part 2: Now, we will construct the process $K'^{d,-}$ on $[\tau, \delta_\tau]$, appearing in the third term on the right-hand side of (4.16), from the sequence $\left\{K^{n,d}\right\}_{n\in\mathbb{N}}$ defined on $[\tau, \delta_\tau]$.

• *Construction of the process $K'^{d,-}$ satisfying (4.16)-(d) via $\left\{K^{n,d}\right\}_{n\in\mathbb{N}}$ on $[\tau, \delta_\tau]$:*

For $n \geq 0$ and $t \in [0, T]$, let us set $\Delta^{n,d}_t := K^{n,d}_{(t\vee\tau)\wedge\delta_\tau} - K^{n,d}_\tau$. The process $\Delta^{n,d}$ is purely-discontinuous and predictable. The purely-discontinuity is obvious since $K^{n,d}$ it is, so we just focus on the predictable property. Actually for any totally inaccessible stopping time ζ the process $\Delta^{n,d}$ cannot jump at ζ since the jumping time of $K^{n,d}$ are exhausted by a countable set of disjunctive graphs of predictable stopping times. Additionally, for any predictable stopping time η, we have $\Delta^{n,d}_\eta = \mathbb{1}_{\{\tau<\eta\}} K^{nd,-}_{\{\eta\wedge\delta_\tau\}} - \mathbb{1}_{\{\tau<\eta\}} K^{nd,-}_{\tau\wedge\eta} \in \mathcal{F}_{\eta-}$ because $\Delta^{n,d}_\eta \mathbb{1}_{\{\tau\geq\eta\}} = 0$ and a stopped predictable process remains predictable (see Corollary 3.24 in [25], pp. 91). Hence $\Delta^{n,d}$ is a predictable process (see Proposition 7.7 in [33], pp. 31). Finally, for $t \in [\tau, \delta_\tau[$, $\Delta^{n,d}_t = 0$ ($K^{nd,-}_t = K^{nd,-}_\tau$, $\forall t \in [\tau, \delta_\tau[$), then we deduce from (4.5) that, for any $t \in [\tau, \delta_\tau]$, $\Delta^{n,d}_t \leq \mathbb{1}_{\{t<\delta_\tau\}\cap\{Y^n_{\delta_\tau-}=U_{\delta_\tau-}\}}(Y^n_{\delta_\tau} - U_{\delta_\tau-})^+$. From the comparison Theorem .5, we get for any $n \geq 0$, $\Delta^{n,d}_t \leq \Delta^{n+1,d}_t$, $\forall t \leq T$. By combining all this arguments, it follows that $\{\Delta^{n,d}_t\}_{n\in\mathbb{N}}$ converges point-wisely for $d\mathbb{P}\otimes dt$-almost all (ω, t) to a purely discontinuous predictable RCLL process $(K'^{d,-})_{t\in[0,T]}$.

• *Claim (4.16)-(d):*

Clearly, the process $(K'^{d,-})_{t\in[0,T]}$ satisfies $K'^{d,-}_\tau = 0$ and for any $t \in [\tau, \delta_\tau[$, $K'^{d,-}_t = 0$. Moreover, the Lebesgue dominate convergence theorem implies that $\{\Delta^{n,d}_{\delta_\tau}\}_{n\in\mathbb{N}}$ convergence to

$K'^{d,-}_{\delta_\tau}$ in $\mathbb{L}^2(\Omega, \mathcal{F}_{\delta_\tau}, \mathbb{P}; \mathbb{R}^+)$, which specifically gives

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\Delta^{n,d}_{\delta_\tau} - K'^{d,-}_{\delta_\tau}\right|^2\right] = \lim_{n\to\infty} \mathbb{E}\left[\left|\int_\tau^{\delta_\tau} dK^{n,d}_s - \int_\tau^{\delta_\tau} dK'^{d,-}_s\right|^2\right] = 0.$$

Let ω is such that $K'^{d,-}_{\delta_\tau(\omega)}(\omega) > 0$ (then $\delta_\tau(\omega) > \tau(\omega)$). Since $\{\Delta^{n,d}_{\delta_\tau(\omega)}(\omega)\}_{n\in\mathbb{N}}$ is an increasing sequence, there exists some integer $n_0(\omega)$ such that, for any $n \geq n_0(\omega)$, we have $\Delta^{n,d}_{\delta_\tau(\omega)}(\omega) > 0$. Thus, from (4.1)-(iv), we deduce that the state process Y^n has a positive predictable jump at δ_τ, then $Y^n_{\delta_\tau(\omega)-}(\omega) = U_{\delta_\tau(\omega)-}(\omega)$ and $\Delta^{n,d}_{\delta_\tau(\omega)}(\omega) = (Y^n_{\delta_\tau(\omega)} - U_{\delta_\tau(\omega)-})^+(\omega)$ for any $n \geq n_0(\omega)$. Consequently, passing to the limit as $n \to +\infty$, we obtain $K'^{d,-}_{\delta_\tau(\omega)}(\omega) = (Y_{\delta_\tau(\omega)} - U_{\delta_\tau(\omega)-})^+$ and $Y_{\delta_\tau(\omega)-}(\omega) = U_{\delta_\tau(\omega)-}(\omega)$.

• *The process Y is RCLL on $[\tau, \delta_\tau]$:*

From the fact that $Y^n \leq Y \leq U$ and $\{Y^n\}_{n\in\mathbb{N}}$, U are RCLL, we deduce that the left and the right limit of $Y(\omega)$ at $\delta_\tau(\omega)$ exists. Furthermore, recall that $S^n_t = Y^n_t + \int_\tau^t g(s)d\langle M\rangle_s - \Delta^{n,d}_t$ for $t \in [\tau, \delta_\tau]$, then passing to the limit in $\mathbb{L}^2(\Omega \times [0,T])$ as $n \to +\infty$, we obtain $S_t = Y_t + \int_\tau^t g(s)d\langle M\rangle_s - K'^{d,-}_t$, then Y is an RCLL process on $[\tau, \delta_\tau]$ since $K'^{d,-}_t$ and S are. Which completes the proof of the claim (4.16)-(d).

Part 3: Claims (4.16)-(a)-(b)-(c). First let us set $\tilde{L}^n_t := L_t - \Delta^{n,d}_t$, $t \in [\tau, \delta_\tau]$.

Using (4.6), Assumption **(H3)**-(ii) and the basic inequality $(a-b)^+ \leq a^+ + |b|$, we get

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left[\int_\tau^{\delta_\tau} e^{\beta A_s}\left|\left(\tilde{L}^n_s\right)^+\right|^2 ds\right]\right\}\\
&= \sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left[\int_\tau^{\delta_\tau} e^{\beta A_s}\left|\left(L_s - \Delta^{n,d}_s\right)^+\right|^2 ds\right]\right\}\\
&\leq TC_\beta\left(\mathbb{E}\left[\sup_{0\leq t\leq T}\left|e^{2\beta A_t}(L_t)^+\right|^2\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}\left|e^{2\beta A_t}(U_t)^-\right|^2\right]\right),
\end{aligned}$$

and from (4.2) and (4.5), we obtain

$$\sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left[e^{\beta A_{\delta_\tau}}\left(\left|\Delta^{n,d}_{\delta_\tau}\right|^2 + \left|Y^n_{\delta_\tau}\right|^2\right)\right]\right\} \leq C_\beta \mathbb{E}\left[\sup_{0\leq t\leq T}\left|e^{2\beta A_t}(U_t)^-\right|^2\right],$$

for some constant C_β independent of n.

Thus, from Theorem .1, for each $n \in \mathbb{N}$, there exists a unique triplet of processes $(\tilde{Y}^n, \tilde{Z}^n, \tilde{N}^n) \in \left(\mathcal{S}^2_\beta \cap \mathcal{S}^{2,\alpha}_\beta\right) \times \mathcal{H}^2_\beta \times \mathcal{M}^2_\beta$ solution of the following BSDE: $\forall t \in [\tau, \delta_\tau]$ a.s.

$$\tilde{Y}^n_t = \left(Y^n_{\delta_\tau} - \Delta^{n,d}_{\delta_\tau}\right) + \int_t^{\delta_\tau} g(s)d\langle M\rangle_s + \int_t^{\delta_\tau} n\left(\tilde{L}^n_s - \tilde{Y}^n_s\right)^+ ds - \int_t^{\delta_\tau}\tilde{Z}^n_s dM_s - \int_t^{\delta_\tau} d\tilde{N}^n_s. \tag{4.18}$$

Now, from the definition of the stopping times $\{\delta^n_\tau\}_{n\in\mathbb{N}}$, it is well known that $\Delta^{n,d}_t = 0$, $\forall t \in [\tau, \delta_\tau[$, this, with the continuity of the Lebesgue-measure, implies $K^{n,+}_t = n\int_\tau^t \left(\tilde{Y}^n_s - \tilde{L}^n_s\right)^- ds$, for $t \in [\tau, \delta_\tau]$. Finally writing the forward SDE (4.17) backwardly on $[\tau, \delta_\tau]$, yields

$$Y^n_t - \Delta^{n,d}_t = \left(Y^n_{\delta_\tau} - \Delta^{n,d}_{\delta_\tau}\right) + \int_t^{\delta_\tau} g(s)d\langle M\rangle_s + n\int_t^{\delta_\tau}\left(\tilde{L}^n_s - Y^n_s\right)^+ ds - \int_t^{\delta_\tau} Z^n_s dM_s - \int_t^{\delta_\tau} dN^n_s. \tag{4.19}$$

Henceforth, leveraging the uniqueness property of BSDEs (4.18) and (4.19) on $[\tau, \delta_\tau]$, we deduce that the newly introduced process $\tilde{Y}^n$ defined on $[\tau, \delta_\tau]$ can be written in terms of Y^n on $[\tau, \delta_\tau]$ as

$$\tilde{Y}^n_t = Y^n_t - \Delta^{n,d}_t = Y^n_t - \left(K^{n,d}_t - K^{n,d}_\tau\right), \quad t \in [\tau, \delta_\tau].$$

Moreover, the quadruplet $\left(\tilde{Y}^n, Z^n, K^n, N^n\right)$ represents the unique solution of a reflected BSDE on $[\tau, \delta_\tau]$ expressed by: $\forall t \in [\tau, \delta_\tau]$ a.s.

$$\begin{cases} \tilde{Y}^n_t = \left(Y^n_{\delta_\tau} - \Delta^{n,d}_{\delta_\tau}\right) + \int_t^{\delta_\tau} g(s) d\langle M\rangle_s + (K^{n,+}_{\delta_\tau} - K^{n,+}_t) - \int_t^{\delta_\tau} Z^n_s dM_s - \int_t^{\delta_\tau} dN^n_s. \\ \tilde{L}^n_t \wedge \tilde{Y}^n_t \leq \tilde{Y}^n_t, \ \forall t \in [\tau, \delta_\tau], \\ \int_\tau^{\delta_\tau} (\tilde{Y}^n_s - \tilde{L}^n_t \wedge \tilde{Y}^n_t) dK^{n,+}_s = 0, \ \mathbb{P}\text{-a.s. with } K^{n,+}_t = n \int_\tau^t \left(\tilde{Y}^n_s - \tilde{L}^n_s\right)^- ds, \ \forall t \in [\tau, \delta_\tau]. \end{cases} \tag{4.20}$$

Indeed, is is clear that $\tilde{L}^n_t \wedge \tilde{Y}^n_t \leq \tilde{Y}^n_t, \forall t \in [\tau, \delta_\tau]$ and

$$\begin{aligned} 0 \leq \int_\tau^{\delta_\tau} \left(\tilde{Y}^n_s - \tilde{L}^n_s \wedge \tilde{Y}^n_s\right) dK^{n,+}_s &= n \int_\tau^{\delta_\tau} \left(\tilde{Y}^n_s - \tilde{L}^n_s \wedge \tilde{Y}^n_s\right) \left(\tilde{Y}^n_s - \tilde{L}^n_s\right)^- ds \\ &= n \int_\tau^{\delta_\tau} \left(\tilde{Y}^n_s - \tilde{L}^n_s \wedge \tilde{Y}^n_s\right)^+ \left(\tilde{Y}^n_s - \tilde{L}^n_s\right)^- ds \\ &\leq n \int_\tau^{\delta_\tau} \left(\tilde{Y}^n_s - \tilde{L}^n_s\right)^+ \left(\tilde{Y}^n_s - \tilde{L}^n_s\right)^- ds \\ &= 0. \end{aligned}$$

Thus $\int_\tau^{\delta_\tau} \left(\tilde{Y}^n_s - \tilde{L}^n_s \wedge \tilde{Y}^n_s\right) dK^{n,+}_s = 0$. An interesting fact is that the process $\Delta^{n,d}$ constitutes the predictable jump part of the process Y^n, arising from the positive predictable jumps of the upper obstacle $(U_t)_{t \leq T}$ on $[\tau, \delta_\tau]$. So, from the given definition $\tilde{L}^n = L - \Delta^{n,d}$ and the form of RBSDE (4.20) satisfied by the process $\left(\tilde{Y}_t\right)_{t \leq \delta_\tau}$, we may say that the obstacle $\tilde{L}^n$ has the same jumps as the martingale part $\left(\int_\tau^{\cdot} Z^n_s dM_s + \int_\tau^{\cdot} dN^n_s\right)$ on $[\tau, \delta_\tau]$. This explains the jump structure of $\tilde{Y}$ which is described by the relation $\tilde{Y}^n_t = Y^n_t - \Delta^{n,d}_t$ and why the choice of the reflection continuous process $K^{n,+}$ on $[\tau, \delta_\tau]$ is convenient. This proves the minimality condition. Making us of the representation property for $\tilde{Y}^n_t$ for $t \in [\tau, \delta_\tau]$ in terms of the Snell envelope notion given in Theorem .3 and the fact that $\left(\tilde{L}^n_\sigma \wedge \tilde{Y}^n_\sigma\right) \mathbb{1}_{\{\sigma < \delta_\tau\}} = \left(L_\sigma \wedge Y^n_\sigma\right) \mathbb{1}_{\{\sigma < \delta_\tau\}}$ for any $\sigma \in \mathcal{T}^{\delta_\tau}_\tau$, we get: $\forall t \in [\tau, \delta_\tau]$,

$$\tilde{Y}^n_t = \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}^{\delta_\tau}_t} \mathbb{E}^{\mathcal{F}_t} \left[\left(Y^n_{\delta_\tau} - \Delta^{n,d}_{\delta_\tau}\right) \mathbb{1}_{\{\sigma = \delta_\tau\}} + \left(L_\sigma \wedge Y^n_\sigma\right) \mathbb{1}_{\{\sigma < \delta_\tau\}} + \int_t^\sigma g(s) d\langle M\rangle_s \right].$$

We point out that the reflected BSDE (4.20) can be viewed as a penalized version of another BSDE associated with terminal value $Y_{\delta_\tau} - K'^{d,-}_{\delta_\tau}$, driver g and lower reflecting barrier $\tilde{L} := L - K'^{d,-}$. Therefor, it is natural to consider the following reflected BSDE on the time interval $[\tau, \delta_\tau]$: $\forall t \in [\tau, \delta_\tau]$ a.s.

$$\begin{cases} \text{(i) } (\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{N}) \in \mathfrak{A}^2_\beta. \\ \text{(ii) } \tilde{Y}_t = Y_{\delta_\tau} - K'^{d,-}_{\delta_\tau} + \int_t^{\delta_\tau} g(s) d\langle M\rangle_s + \left(\tilde{K}^+_{\delta_\tau} - \tilde{K}^+_t\right) - \int_t^{\delta_\tau} \tilde{Z}_s dM_s - \int_t^{\delta_\tau} d\tilde{N}_s. \\ \text{(iii) } \tilde{Y}_t \geq \tilde{L}_t. \\ \text{(iv) The process } \tilde{K}^+_t = \tilde{K}^{+,c}_t + \tilde{K}^{+,d}_t \text{ satisfies the following minimality condition:} \\ \quad \bullet \int_\tau^{\delta_\tau} \left(\tilde{Y}_s - \tilde{L}_s\right) d\tilde{K}^{+,c}_s = 0, \text{where } \bar{K}^{+,c} \text{ is its continuous part.} \\ \quad \bullet \tilde{K}^{+,d}_t = \sum_{\tau < s \leq t} \left(\tilde{L}_{s-} - \tilde{Y}_s\right)^+ \mathbb{1}_{\{\tilde{Y}_{s-} = L_{s-}\}}, \text{ where } \tilde{K}^{+,d} \text{ is the predictable} \\ \quad \text{purely discontinuous part.} \end{cases} \tag{4.21}$$

The existence and uniqueness of the quadruplet $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{N})$, solution of the RBSDE (4.21) on $[\tau, \delta_\tau]$, is guaranteed through Theorem .3. Furthermore, we have the following description for

$\tilde{Y}_t$: $\forall t \in [\tau, \delta_\tau]$

$$\tilde{Y}_t = \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_t^{\delta_\tau}} \mathbb{E}^{\mathcal{F}_t} \left[\left(Y_{\delta_\tau} - K'^{d,-}_{\delta_\tau} \right) \mathbb{1}_{\{\sigma = \delta_\tau\}} + L_\sigma \mathbb{1}_{\{\sigma < \delta_\tau\}} + \int_t^\sigma g(s) d\langle M \rangle_s \right],$$

since $K'^{d,-}_t \mathbb{1}_{\{t<\delta_\tau\}} = 0$, for all $t \in [\tau, \delta_\tau]$ (see (4.16)-(d)).

As we have already mentioned, the main idea is to show that the solutions of penalized reflected BSDEs (4.20) converges increasingly toward the solution of the BSDE (4.21). More precisely, we prove that $\mathbb{P}$-a.s. for any $t \in [\tau, \delta_\tau]$, $\tilde{Y}^n_t \nearrow \tilde{Y}_t$. First, note that the increasing property of $\{S^n\}_{n\in\mathbb{N}}$ studied before, we deduce that $\left(Y^n_{\delta_\tau} - \Delta^{n,d}_{\delta_\tau}\right) \mathbb{1}_{\{t=\delta_\tau\}} \nearrow \left(Y_{\delta_\tau} - K'^{d,-}\right) \mathbb{1}_{\{t=\delta_\tau\}}$. Additionally, since $Y \geq L$ on $[0,T]$ in particular on $[\tau, \delta_\tau]$, we get $(L_\sigma \wedge Y^n_\sigma) \mathbb{1}_{\{\sigma<\delta_\tau\}} \nearrow L_\sigma \mathbb{1}_{\{\sigma<\delta_\tau\}}$. Since all the processes $\{Y^n\}_{n\in\mathbb{N}}$, $\left\{\Delta^{n,d}\right\}_{n\in\mathbb{N}}$, Y, L and $K'^{d,-}$ are all in class $\mathcal{D}([\tau, \delta_\tau])$, applying Proposition (A1) in [23], we deduce that, $\mathbb{P}$-a.s., $\forall t \in [\tau, \delta_\tau]$, $\tilde{Y}^n_t \nearrow \tilde{Y}_t$, i.e. $Y = \tilde{Y} + K'^{d,-}$ on the time interval $[\tau, \delta_\tau]$. This, combined with the BSDE verified by the process $\tilde{Y}$ given in the second line of (4.21), gives

$$\begin{aligned} Y_t = Y_{\delta_\tau} - \left(K'^{d,-}_{\delta_\tau} - K'^{d,-}_t\right) + \int_t^{\delta_\tau} g(s) d\langle M \rangle_s + \left(\tilde{K}^+_{\delta_\tau} - \tilde{K}^+_t\right) \\ - \int_t^{\delta_\tau} \tilde{Z}_s dM_s - \int_t^{\delta_\tau} d\tilde{N}_s, \quad \forall t \in [\tau, \delta_\tau]. \end{aligned} \tag{4.22}$$

- *Construction of the processes Z' and N' on $[0,T]$:*

  For $t \leq T$, let us set: $Z'_t = \tilde{Z}_t \mathbb{1}_{[\tau,\delta_\tau]}(t)$ and $dN'_t = \mathbb{1}_{]\tau,\delta_\tau]}(t) d\tilde{N}_t$. Clearly, we have $(Z'_t, N'_t)_{t\leq T} \in \mathcal{H}^2_\beta \times \mathcal{M}^2_\beta$.

- *Construction of the processes $K'^{+,c}$ and $K'^{+,d}$ on $[0,T]$:*

  First, we consider the natural decomposition of the processes $\tilde{K}^+$ on $[\tau, \delta_\tau]$ into its continuous increasing part $\tilde{K}^{+,c}$ and its $\mathcal{F}_t$-predictable increasing purely-discontinuous part $\tilde{K}^{+,d}$. Then, we set $K'^{+,c}_t := \tilde{K}^{+,c}_{(t\vee\tau)\wedge\delta_\tau} - \tilde{K}^{+,c}_\tau$, for $t \in [0,T]$. Note that $\left(K'^{+,c}_t\right)_{t\leq T}$ is continuous $\mathbb{R}^+$-valued increasing process since $K'^{+,c}_\tau = 0$. Moreover, we have $dK'^{+,c}_t = d\tilde{K}^{+,c}_t$ for $t \in [\tau, \delta_\tau]$, thus $\int_\tau^{\delta_\tau} (Y_s - L_s) dK'^{+,c}_s = \int_\tau^{\delta_\tau} \left(\tilde{Y}_s - \tilde{L}_s\right) d\tilde{K}^{+,c}_s = 0$. Using the same principal, we define $K'^{+,d}_t := \tilde{K}^{+,d}_{(t\vee\tau)\wedge\delta_\tau} - \tilde{K}^{+,d}_\tau$, $t \leq T$. Obviously, the process $\left(K'^{+,d}_t\right)_{t\leq T}$ is $\mathcal{F}_t$-predictable increasing purely-discontinuous. Now, assume that $K'^{+,d}_t$ jumps at some $\mathbb{F}$-predictable stopping time $\theta \in]\tau, \delta_\tau[$, i.e. $\Delta K'^{+,d}_\theta > 0$. Therefore, $\Delta K'^{+,d}_\theta = \Delta\tilde{K}^{+,d}_\theta = (Y_\theta - L_{\theta-})^-$ since $K'^{d,-} = 0$ on $[\tau, \delta_\tau[$ (assertion (4.16)-(d)). It remains two cases, if the jump occurs at τ and at δ_τ. The first case does not need to be studied, since $K'^{+,d}_\tau = 0$. For this, let's discuss the latest. Assume now that $\theta = \delta_\tau$ and $\Delta K'^{+,d}_\theta > 0$. From the BSDE (4.22), we have $\Delta Y_\eta = K'^{d,-}_\eta - \Delta\tilde{K}^{+,d}_\eta$. Indeed, this is due to the quasi-left continuity of $\mathbb{F}$ and the fact that $K'^{d,-}_t = 0$ for $t \in [\tau, \delta_\tau[$. Thus, for $\eta = \theta = \delta_\tau$, and from that Skorokhod condition (last line in (4.21)) and the fact that $Y_- = \tilde{Y}_-$, $L_- = \tilde{L}_-$ on $[\tau, \delta_\tau]$, we get $\Delta K'^{+,d}_\theta = \Delta\tilde{K}^{+,d}_\theta = Y_{\theta-} - Y_\theta + K'^{d,-}_\theta = \tilde{Y}_{\theta-} - Y_\theta + K'^{d,-}_\theta = L_{\theta-} - Y_\theta + K'^{d,-}_\theta$. Thanks again to (4.16)-(d), if $K'^{d,-}_\theta > 0$, then $Y_{\theta-} = U_{\theta-}$, $Y_\theta > U_{\theta-}$, and $K'^{d,-}_\theta = Y_\theta - U_{\theta-}$, then $\Delta K'^{+,d}_\theta = L_{\theta-} - U_{\theta-} \leq 0$, which a contradiction. Therefore, necessary, we have $K'^{d,-}_\theta = 0$. As a result, $\Delta K'^{+,d}_\theta = (L_{\theta-} - Y_\theta)^+$. In conclusion, $\int_\tau^{\delta_\tau} (Y_{t-} - L_{t-}) dK'^+_t = 0$, where $K'^+ = K'^{+,c} + K'^{+,d}$. Finally, note that $\left(K'^{d,-}_t, K'^+_t\right)_{t\leq T} \in \mathcal{S}^2 \times \mathcal{S}^2$, and the proof of statements (4.16)-(a)-(b)-(c) is then completed.

□

4.2 Analysis of the decreasing penalization scheme

We now consider the following decreasing penalization scheme

$$\begin{cases} \text{(i)} \left(Y'^n, Z'^n, , K'^{n+}, N'^n\right) \in \mathfrak{A}^2_\beta. \\ \text{(ii)} \; Y'^n_t = \xi + \int_t^T g(s) d\langle M\rangle_s - \int_t^T n(Y'^n_s - U_s)^+ ds + (K'^n_T - K'^n_t) \\ \qquad - \int_t^T Z^n_s dM_s - \int_t^T dN'^n_s, \quad 0 \le t \le T. \\ \text{(iii)} \; Y'^n \ge L, \quad \int_0^T (Y^n_s - L_s) dK'^{n,c}_s = 0 \text{ and } K'^{n,d}_t = \sum_{0<s\le t} (Y_s - L_{s-})^-, \quad 0 \le t \le T. \end{cases} \tag{4.23}$$

For any $n \in \mathbb{N}$, the quintuplet $(Y'^n, Z'^n, V'^n, K'^n, M'^n)$ exists through Theorem .3. From the comparison Theorem .5, we have for any $n \ge 0$, $L \le Y'^{n+1} \le Y'^n$. Therefore, there exists an optional process $(Y'_t)_{t\le T}$ such that $\mathbb{P}$-a.s. $Y' \ge L$ and, for any $t \le T$, $Y'_t := \lim_{n\to\infty} Y'^n_t$. Furthermore, the Lebesgue dominate convergence theorem implies that the sequence $(Y'^n)_{n\in\mathbb{N}}$ converges to Y' in $\mathcal{S}^{2,\alpha}_\beta$, and the process Y' belongs to $\mathcal{S}^2_\beta \cap \mathcal{S}^{2,\alpha}_\beta$. Now, for any stopping time τ and $n \ge 0$, let us set

$$\begin{aligned} \lambda^n_\tau &:= \inf\{s \ge \tau : K'^n_s - K'^{n+}_\tau > 0\} \wedge T \\ &= \inf\{s \ge \tau : K'^{n,c}_s - K'^{n+,c}_\tau > 0\} \wedge \inf\{s \ge \tau : K'^{n,d}_s - K'^{n,d}_\tau > 0\} \wedge T. \end{aligned}$$

The same analysis reveals that $(\lambda^n_\tau)_{n\in\mathbb{N}}$ is a non-decreasing sequence of stopping times that converges to another stopping time $\lambda_\tau := \lim_{n\to\infty} \lambda^n_\tau$. Moreover, we have

$$\begin{aligned} Y'^n_t = Y'^n_{\lambda^n_\tau} + \int_t^{\lambda^n_\tau} g(s) d\langle M\rangle_s - n\int_t^{\lambda^n_\tau} (U_s - Y'^n_s)^- ds + (K'^{n,d}_{\lambda^n_\tau} - K'^{n,d}_t) \\ - \int_t^{\lambda^n_\tau} Z^n_s dM_s - \int_t^{\lambda^n_\tau} dN^n_s, \quad \forall t \in [\tau, \lambda^n_\tau], \end{aligned}$$

and

$$\forall t \in [\tau, \lambda^n_\tau], \quad K'^{n,d}_{\lambda^n_\tau} - K'^{n,d}_t \le \left(L_{\lambda^n_\tau -} - Y'^n_{\lambda^n_\tau}\right)^+ \mathbb{1}_{\{t<\delta^n_\tau\}\cap\{Y'^n_{\delta^n_\tau -} = U_{\delta^n_\tau -}\}}.$$

The following properties related to Y' which are analogous to the ones the Propositions 4.2 and 4.4 hold true.

Proposition 4.5. *(i) The limiting process Y' is an RCLL semi-martingale on $[\tau, \lambda_\tau]$ such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_s} |Y'_s|^2 + \int_0^T |Y'_s|^2 d\langle M\rangle_s\right] < \infty. \tag{4.24}$$

(ii) $\mathbb{P}$-a.s., $\mathbb{1}_{\{\lambda_\tau<T\}} Y'_{\lambda_\tau} \le \mathbb{1}_{\{\lambda_\tau<T\}} (L_{\lambda_\tau} + \mathbb{1}_{\{\lambda_\tau>\tau\}} (\Delta L_{\lambda_\tau})^-)$.

(iii) There exists a quintuplet of processes $(Z'', V'', K''^-, K''^{d,+}, M'')$ which in association with

Y' satisfies the following reflected BSDE:

$$\begin{cases} (a)\ Z'' \in \mathcal{H}^2_\beta, \quad K''^{,-} \in \mathcal{S}^2, \quad K''^{d,+} \in \mathcal{S}^2 \text{ and } N'' \in \mathcal{M}^2_\beta. \\ (b)\ Y'_t = Y'_{\lambda_\tau} + \int_t^{\lambda_\tau} g(s)\mathrm{d}\langle M\rangle_s + (K''^{d,+}_{\lambda_\tau} - K''^{d,+}_t) - (K''^{-}_{\lambda_\tau} - K''^{-}_t) \\ \qquad - \int_t^{\lambda_\tau} Z''_s dM_s - \int_t^{\lambda_\tau} dN''_s, \quad t \in [\tau, \lambda_\tau]. \\ (c)\ \forall t \in [0,T],\ L_t \leq Y'_t \leq U_t. \\ (d)\ K''^{-}_\tau = 0,\ \int_\tau^{\lambda_\tau} (Y'_s - U_s) dK''^{-,c}_s = 0 \text{ and } K''^{-,d}_t = \sum_{\tau < s \leq t} (Y'_s - U_{s-})^+,\ \forall t \in [\tau, \lambda_\tau], \\ \quad \text{where } K''^{-,c} \text{ is the continuous part of } K''^{-} \text{ and } K''^{-,d} \text{ it's predictable jump part.} \\ (e)\ K''^{d,+} \text{ is predictable purely-discontinuous part of } K''^{+}\ : \\ \quad K''^{d,+}_\tau = 0,\ K''^{d,+}_t = 0,\ \forall t \in [\tau, \lambda_\tau[, \\ \quad \text{and if } K''^{d,+}_{\lambda_\tau} > 0 \text{ then } Y'_{\lambda_\tau -} = L_{\lambda_\tau -} \text{ and } K''^{d,+}_{\lambda_\tau} = (Y'_{\lambda_\tau} - L_{\lambda_\tau -})^-. \end{cases} \tag{4.25}$$

Proof. We follow the same scheme of the proof of Propositions 4.2 and 4.4 combined with Remark .7. Namely, if $(Y^n, Z^n, V^n, K^{n-}, M^n)$ is a solution of the RBSDE defined as in (4.1) but associated with $(-\xi, -g, -L, -U)$, then, by uniqueness, we get $(Y^n, Z^n, V^n, K^n, M^n) = (-Y'^n, -Z'^n, -V'^n, K'^n, -M'^n)$ which can lead to the desired results. □

Now we move to the construction of a local solution for the reflected BSDE (2.1) associated with (ξ, g, L, U) using the limits of the increasing and decreasing approximating schemes, i.e. Y and Y', respectively.

4.3 Existence of the local solution

Proposition 4.6. *$\mathbb{P}$-a.s., for any $t \leq T$, $Y_t = Y'_t$. Additionally the process Y is RCLL on $[0,T]$ and verifies: $\mathbb{P}$-a.s.,*

- $Y_{\delta_\tau} \mathbb{1}_{\{\delta_\tau < T\}} \geq \left(U_{\delta_\tau} - \mathbb{1}_{\{\delta_\tau > \tau\}} (\Delta U_{\delta_\tau})^+\right) \mathbb{1}_{\{\delta_\tau < T\}},$
- $\mathbb{1}_{\{\lambda_\tau < T\}} Y_{\lambda_\tau} \leq \mathbb{1}_{\{\lambda_\tau < T\}} (L_{\lambda_\tau} + \mathbb{1}_{\{\lambda_\tau > \tau\}} (\Delta L_{\lambda_\tau})^-).$

Proof. The proof will be divided into two parts, where we employ the proof of Proposition 3.4 from [24]. By utilizing their proof as a basis in both parts, we adapt their arguments and reasoning to our more general problem. This allows us to extend their results and demonstrate their applicability in our specific context.
Part 1: Y and Y' are indistinguishable on $[0,T]$.
First, we apply Meyer-Ito's formula to the semi-martingale $Y^n - Y'^m$ with the convex func-

tion $\psi(x)=(x^+)^2$ between t and T, we obtain

$$\begin{aligned}
&\left((Y_t^n-Y_t'^m)^+\right)^2+\int_t^T \mathbb{1}_{\{Y_{s-}^n>Y_{s-}'^m\}}\left|Z_s^n-Z_s'^m\right|^2 d\langle M^c\rangle_s+\int_t^T \mathbb{1}_{\{Y_{s-}^n>Y_{s-}'^m\}}d\langle (N^n-N'^m)^c\rangle_s\\
&\quad+2\int_t^T \mathbb{1}_{\{Y_{s-}^n>Y_{s-}'^m\}}d\langle (N^n-N'^m)^c,M_s^c\rangle_s\\
&\quad+\sum_{t<s\le T}\left\{\psi(Y_s^n-Y_s'^m)-\psi(Y_{s-}^n-Y_{s-}'^m)-\psi'(Y_{s-}^n-Y_{s-}'^m)\Delta\left(Y^n-Y'^m\right)_s\right\}\\
&=2\int_t^T \mathbb{1}_{\{Y_{s-}^n>Y_{s-}'^m\}}\left(Y_{s-}^n-Y_{s-}'^m\right)\left(n(Y_s^n-L_s)^- - m(Y_s'^m-U_s)^+\right)ds\\
&\quad-2\int_t^T \mathbb{1}_{\{Y_{s-}^n>Y_{s-}'^m\}}(Y_{s-}^n-Y_{s-}'^m)(dK_s'^{m+}+dK_s^{n-})\\
&\quad-2\int_t^T \mathbb{1}_{\{Y_{s-}^n>Y_{s-}'^m\}}(Y_{s-}^n-Y_{s-}'^m)\left(Z_s^n-Z_s'^m\right)dM_s\\
&\quad-2\int_t^T \mathbb{1}_{\{Y_{s-}^n>Y_{s-}'^m\}}(Y_{s-}^n-Y_{s-}'^m)\left(dN_s^n-dN_s'^m\right).
\end{aligned}\tag{4.26}$$

On the set $\{Y^n>Y'^m\}$, we have $U\ge Y^n>Y'^m\ge L$, then

$$\int_t^T \mathbb{1}_{\{Y_s^n>Y_s'^m\}}\left(Y_s^n-Y_s'^m\right)\left(n(Y_s^n-L_s)^- - m(Y_s'^m-U_s)^+\right)ds=0.$$

Also

$$-\int_t^T \mathbb{1}_{\{Y_{s-}^n>Y_{s-}'^m\}}(Y_{s-}^n-Y_{s-}'^m)(dK_s'^{m+}+dK_s^{n-})=-\int_t^T (Y_{s-}^n-Y_{s-}'^m)^+(dK_s'^{m+}+dK_s^{n-})\le 0.$$

Going back to (4.26) and taking expectation on both sides gives

$$\mathbb{E}\left[(Y_t^n-Y_t'^m)^{+2}\right]\le 0,\quad \forall t\le T.$$

Then $Y^n\le Y'^m$ since Y^n and Y'^m are RCLL on $[0,T]$. Therefore, passing to the limit in n and m, we deduce that $Y_t\le Y_t'$ $0\le t\le T$, $\mathbb{P}$-a.s.

Now, we want to show the reverse inequality. To this end, let τ be a $[0,T]$-valued stopping time, and let us define another stopping time α_τ^p by

$$\alpha_\tau^p:=\inf\{s\ge\tau: Y_s\ge U_s-\frac{1}{p}\text{ or } Y_s'\le L_s+\frac{1}{p}\}\wedge T,\quad p\ge 1.$$

The sequel monotonicity of $(Y^n)_n$ and $(Y'^n)_n$ and the fact that $L\le Y,Y'\le U$, implies that, for each n, $Y_{s-}^n<U_{s-}$ and $Y_{s-}'^n>L_{s-}$, $\mathbb{P}$-a.s. for each $s\in[\tau,\alpha_\tau^p]\cap\{\alpha_\tau^p>\tau\}$. Therefore, from the minimality conditions, the predictable reflection processes (K^{n-}) and (K'^{n+}) do not increase on $[\tau,\alpha_\tau^p]$, i.e. $(dK_s^{n-}+dK_s'^{n+})=0$, $\forall s\in[\tau,\alpha_\tau^p]$, and in particular

$$2\int_\tau^{\alpha_\tau^p}(Y_{s-}^n-Y_{s-}'^n)(dK_s^{n-}+dK_s'^{n+})=0.\tag{4.27}$$

Clearly, equality (4.27) is given in order to apply Itô's formula to the process $(Y'^n-Y^n)^2$ on the interval $[\tau,\alpha_\tau^p]$, which yields after reducing inequalities by the help of the monotonicity condition on the generators and taking the expectation in both side to

$$\mathbb{E}\left[(Y_\tau'^n-Y_\tau^n)^2\right]\le\mathbb{E}\left[(Y_{\alpha_\tau^p}'^n-Y_{\alpha_\tau^p}^n)^2\right].$$

Next, by applying the Lebesgue dominated convergence theorem and letting $n\to\infty$, we can show that

$$\mathbb{E}\left[(Y_\tau'-Y_\tau)^2\right]\le\mathbb{E}\left[(Y_{\alpha_\tau^p}'-Y_{\alpha_\tau^p})^2\right].\tag{4.28}$$

Note that we cannot apply Ito's formula with $Y'-Y$ since we lack knowledge on whether $Y'-Y$ is a semimartingale over $[\tau,\alpha^p_\tau]$. However, by utilizing **Part 1** along with the right continuity of Y^n and Y'^n and the fact that $L\le U$, we can obtain

$$\begin{aligned}0\le\left(Y'^n_{\alpha^p_\tau}-Y^n_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau<\alpha^p_\tau\}}&=\left(Y'^n_{\alpha^p_\tau}-Y^n_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau<\alpha^p_\tau\}\cap\{\alpha^p_\tau<T\}}+\left(Y'^n_{\alpha^p_\tau}-Y^n_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau<\alpha^p_\tau\}\cap\{\alpha^p_\tau=T\}}\\&\le\left(Y'^n_{\alpha^p_\tau}-L_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau<\alpha^p_\tau\}\cap\{\alpha^p_\tau<T\}}+\left(U_{\alpha^p_\tau}-Y^n_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau<\alpha^p_\tau\}\cap\{\alpha^p_\tau<T\}}\\&\le\frac{1}{p}.\end{aligned}$$

Based on the previous analysis, we take the limit as $n\to+\infty$ and conclude that

$$0\le\left(Y'_{\alpha^p_\tau}-Y_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau<\alpha^p_\tau\}}\le\frac{1}{p},\quad\mathbb{P}\text{-a.s.}$$

Conversely, by utilizing Proposition 4.2, Proposition 4.4-(ii), Proposition 4.5-(i)-(ii) and (4.25)-c, we can derive the following:

$$\begin{aligned}0&\le\left(Y'_{\alpha^p_\tau}-Y_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau=\alpha^p_\tau\}}\\&=\left(Y'_{\alpha^p_\tau}-Y_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau=\alpha^p_\tau\}\cap\{\tau<\delta_\tau\wedge\lambda_\tau\}}+\left(Y'_{\alpha^p_\tau}-Y_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau=\alpha^p_\tau\}\cap\{\tau=\delta_\tau\wedge\lambda_\tau\}}\\&\le\frac{1}{p}+\left\{\left(Y'_{\alpha^p_\tau}-U_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau=\delta_\tau\}\cap\{\delta_\tau\le\lambda_\tau\}}+\left(Y'_{\alpha^p_\tau}-L_{\alpha^p_\tau}\right)\mathbb{1}_{\{\tau=\lambda_\tau\}\cap\{\delta_\tau>\lambda_\tau\}}\right\}\mathbb{1}_{\{\tau=\alpha^p_\tau\}\cap\{\tau<T\}}\\&\le\frac{1}{p}\end{aligned}$$

Thus,

$$0\le\left(Y'_{\alpha^p_\tau}-Y_{\alpha^p_\tau}\right)^2\le\frac{1}{p^2},\quad\mathbb{P}\text{-a.s.},$$

and from the integrability properties (4.3) and (4.24), we may infer, using (4.28) and the dominated convergence theorem, that

$$\mathbb{E}\left[(Y'_\tau-Y_\tau)^2\right]\le\lim_{p\to\infty}\mathbb{E}\left[\left(Y'_{\alpha^p_\tau}-Y_{\alpha^p_\tau}\right)^2\right]=0$$

Henceforth, $Y_\tau=Y'_\tau$ for any stopping time $\tau\in\mathcal{T}_0^T$. An application of the section theorem yields, $\mathbb{P}$-a.s., $\forall t\le T$, $Y_t=Y'_t$.

Now, let's move to the second property. It is clear that, $\forall t\in[0,T)$, $\forall n\in\mathbb{N}$,

$$Y^n_t=\liminf_{s\downarrow t}Y^n_s\le\liminf_{s\downarrow t}Y_s=\liminf_{s\downarrow t}Y'_s\le\limsup_{s\downarrow t}Y_s=\limsup_{s\downarrow t}Y'_s\le\limsup_{s\downarrow t}Y'^n_s=Y'^n_t.$$

As $n\to\infty$, we can achieve the right continuity of Y and Y', as $Y=Y'$. To accomplish the proof of the proposition, we need to demonstrate that Y and Y' has left-sided limits. To this end, let $\left(\bar{S}_t\right)_{t\in(0,T]}$ and $(\underline{S}_t)_{t\in(0,t]}$ be two $\mathcal{F}_t$-predictable processes defined as:

$$\bar{S}_t:=\limsup_{s\uparrow t}Y_s,\qquad\text{and}\qquad\underline{S}_t:=\liminf_{s\uparrow t}Y_s$$

Clearly, the claim can be proven by showing that $\bar{S}_\sigma$ and $\underline{S}_\sigma$ are equal for every $\mathbb{F}$-predictable stopping time σ. So, let σ be any predictable $\mathbb{F}$-stopping time. From Theorem 2.15 in [26], pp. 19, there exists a sequence of $\mathbb{F}$-stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$ such that $\sigma_n<\sigma$ on $\{\sigma>0\}$ and $\sigma_n\uparrow\sigma$ as $n\to+\infty$, $\mathbb{P}$-a.s. Therefore, we have,

$$\bar{S}_\sigma=\limsup_{n\to\infty}Y_{\sigma_n}=\limsup_{n\to\infty}Y'_{\sigma_n}\le Y'^n_{\sigma-}=Y'^n_\sigma+\left(Y'^n_\sigma-L_{\sigma-}\right)^-\xrightarrow[n\to\infty]{}Y_\sigma-\left(Y_\sigma-L_{\sigma-}\right)^-,$$

since $-\Delta Y_\sigma=\Delta K'^{n,d}_\sigma=(Y_\sigma-L_{\sigma-})^-$ (see (4.23)-(iii)). Similarly, we may show that

$$\underline{S}_\sigma=\liminf_{n\to+\infty}Y_{\sigma_n}\ge Y^n_{\sigma-}=Y^n_\sigma-\left(Y^n_\sigma-U_{\sigma-}\right)^+.$$

Letting $n \to \infty$, we get

$$\bar{S}_\sigma \leq Y_\sigma + (Y_\sigma - L_{\sigma-})^-, \text{ and } \underline{S}_\sigma \geq Y_\sigma - (Y_\sigma - U_{\sigma-})^+.$$

But from Proposition 4.2-(ii) and (4.25)-(c), we know that $L_{\sigma-} \leq \underline{S}_\sigma \leq \bar{S}_\sigma \leq U_{\sigma-}$. Therefore,

$$L_{\sigma-} \vee \left(Y_\sigma - (Y_\sigma - U_{\sigma-})^+\right) \leq \underline{S}_\sigma \leq \bar{S}_\sigma \leq U_{\sigma-} \wedge \left(Y_\sigma - (Y_\sigma - L_{\sigma-})^-\right)$$

It is evident that the right-hand side and the left-hand side are the same and equal to the expression $L_{\tau-}\mathbb{1}_{\{Y_\tau < L_{\tau-}\}} + Y_\tau \mathbb{1}_{\{L_{\tau-} \leq Y_\tau \leq U_{\tau-}\}} + U_{\tau-}\mathbb{1}_{\{Y_\tau > U_{\tau-}\}}$. By applying the predictable section theorem (Theorem 2.14 in [26], pp. 19), we obtain that $\bar{S}_t = \underline{S}_t$, $\mathbb{P}$-a.s., which gives the desired result, i.e., Y has left limits. The lest two properties follows from Propositions 4.2 and 4.5. □

Construction of the local solution $(Y, Z^\tau, K^{\tau+}, K^{\tau-}, N^\tau)$ for the doubly reflected BSDE (2.1) associated with (ξ, g, L, U):

The aim of the current section is to show that the DRBSDE (2.1) with parameters (ξ, g, L, U) has a local solution. Namely, we will prove that for any stopping time $\tau \in \mathcal{T}_0^T$, there exists another stopping time $\theta_\tau \in \mathcal{T}_\tau^T$ and a quintuplet of process $\left(Y_t, Z_t^\tau, K_t^{\tau+}, K_t^{\tau-}, N_t^\tau\right)_{t \leq T}$ that solves the DRBSDE(ξ, g, L, U) on $[\tau, \theta_\tau]$.

Definition of $(Y, Z^\tau, K^{\tau+}, K^{\tau-}, N^\tau)$ by concatenation and verification of the BSDE. Let $(Y_t)_{t \leq T}$ be the adapted process, defined as the limit of the increasing (or decreasing) scheme, which has right-continuous with left limits paths on $[0, T]$ and satisfies $L_t \leq Y_t \leq U_t$ and $Y_T = \xi$, $\mathbb{P}$-a.s. On the other hand, let τ be a stopping time and let δ_τ be the stopping time defined in the preceding section. Finally, let us set $\theta_\tau = \lambda_{\delta_\tau}$.

From Proposition 4.5, there exists a quadruplet of processes $(Z'', K''^-, K''^{d,+}, M'')$ which in association with Y satisfies the following reflected BSDE:

$$\begin{cases} \text{(a) } Z'' \in \mathcal{H}_\beta^2, \quad K''^{,-} \in \mathcal{S}^2, \quad K''^{d,+} \in \mathcal{S}^2 \text{ and } N'' \in \mathcal{M}_\beta^2. \\ \text{(b) } Y_t = Y_{\theta_\tau} + \displaystyle\int_t^{\theta_\tau} g(s) \mathrm{d}\langle M \rangle_s + (K''^{d,+}_{\theta_\tau} - K''^{d,+}_t) - (K''^-_{\theta_\tau} - K''^-_t) \\ \qquad - \displaystyle\int_t^{\theta_\tau} Z''_s dM_s - \int_t^{\theta_\tau} dN''_s, \quad t \in [\delta_\tau, \theta_\tau]. \\ \text{(c) } \forall t \in [0, T], \ L_t \leq Y_t \leq U_t. \\ \text{(d) } K''^-_{\delta_\tau} = 0, \quad \displaystyle\int_{\delta_\tau}^{\theta_\tau} (Y_s - U_s) dK''^{-,c}_s = 0 \text{ and } K''^{-,d}_t = \sum_{\delta_\tau < s \leq t} (Y_s - U_{s-})^+, \ \forall t \in [\delta_\tau, \theta_\tau] \\ \quad \text{where } K''^{-,c} \text{ is the continuous part of } K''^- \text{ and } K''^{-,d} \text{ it's predictable jump part.} \\ \text{(e) } K''^{d,+} \text{ is predictable purely-discontinuous part of } K''^+ : \\ \quad K''^{d,+}_{\delta_\tau} = 0, \ K''^{d,+}_t = 0, \ \forall t \in [\delta_\tau, \theta_\tau[, \\ \quad \text{and if } K''^{d,+}_{\theta_\tau} > 0, \text{ then } Y_{\theta_\tau -} = L_{\theta_\tau -} \text{ and } K''^{d,+}_{\theta_\tau} = (Y_{\theta_\tau} - L_{\theta_\tau -})^-. \end{cases} \tag{4.29}$$

Now using the result of Proposition 4.4, we get the existence of a quadruplet $(Z', K'^+, K'^{d,-}, N')$ defined on $[\tau, \delta_\tau]$ satisfying (4.16).

Our local solution will be built by merging different components using the process of concatenation. More precisely, for any $t \leq T$, we set

- $Z_t^\tau := Z' \mathbb{1}_{[\![\tau, \delta_\tau]\!]}(t) + Z''_t \mathbb{1}_{]\!]\delta_\tau, \theta_\tau]\!]}(t)$,
- $K_t^{\tau c,+} := K'^{+,c}_{(t \vee \tau) \wedge \delta_\tau}$; $K_t^{\tau d,+} := K'^{+,d}_{(t \vee \tau) \wedge \delta_\tau} + K''^{d,+}_{(t \vee \delta_\tau) \wedge \theta_\tau}$ and $K_t^{\tau+} := K_t^{\tau c,+} + K_t^{\tau d,+}$,
- $K_t^{\tau c,-} := K''^{-,c}_{(t \vee \delta_\tau) \wedge \theta_\tau}$; $K_t^{\tau d,-} := K'^{d,-}_{(t \vee \tau) \wedge \delta_\tau} + K''^{-,d}_{(t \vee \delta_\tau) \wedge \theta_\tau}$ and $K_t^{\tau-} := K_t^{\tau c,-} + K_t^{\tau d,-}$,
- $dN_t^\tau := \mathbb{1}_{[\![\tau, \delta_\tau]\!]}(t) dN'_t + \mathbb{1}_{]\!]\delta_\tau, \theta_\tau]\!]}(t) dN''_t$,

It is evident that the quadruplet $(Z^\tau, K^{\tau+}, K^{\tau-}, N^\tau)$ belongs to $\mathcal{H}^2_\beta \times \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{M}^2_\beta$, and $K^\pm$ is a predictable process with $K^{\tau\pm}_\tau = 0$. We will now demonstrate that the processes Y, Z, N, and $K^\pm$ satisfy the relations in equation (2.1) associated with the driver $g(\omega, t)$ on the interval $[\tau, \theta_\tau]$.

Let t be any point in the interval $[\tau, \theta_\tau]$. Firstly, let us assume that t belongs to the sub-interval $[\delta_\tau, \theta_\tau]$. Using equation (4.29)-(b), the given definitions, and the construction of stochastic integration with respect to semi-martingales, we can write

$$\begin{aligned}
&Y_{\theta_\tau} + \int_t^{\theta_\tau} g(s)d\langle M\rangle_s + \int_t^{\theta_\tau} dK^{\tau+}_s - \int_t^{\theta_\tau} dK^{\tau-}_s - \int_t^{\theta_\tau} Z^\tau_s dM_s - \int_t^{\theta_\tau} dN^\tau_s \\
&= Y_{\theta_\tau} + \int_t^{\theta_\tau} g(s)d\langle M\rangle_s + \int_t^{\theta_\tau} dK''^{d,+}_s - \int_t^{\theta_\tau} dK''^{-}_s - \int_t^{\theta_\tau} Z''_s dM_s - \int_t^{\theta_\tau} dN''_s \\
&= Y_t.
\end{aligned}$$

Suppose that t belongs to the interval $[\tau, \delta_\tau)$. In this case, using Proposition 4.4, we can deduce that

$$\begin{aligned}
&Y_{\theta_\tau} + \int_t^{\theta_\tau} g(s)d\langle M\rangle_s + \int_t^{\theta_\tau} dK^{\tau+}_s - \int_t^{\theta_\tau} dK^{\tau-}_s - \int_t^{\theta_\tau} Z^\tau_s dM_s - \int_t^{\theta_\tau} dN^\tau_s \\
&= Y_{\theta_\tau} + \int_{\delta_\tau}^{\theta_\tau} g(s)d\langle M\rangle_s + \int_{\delta_\tau}^{\theta_\tau} dK^{\tau+}_s - \int_{\delta_\tau}^{\theta_\tau} dK^{\tau-}_s - \int_t^{\theta_\tau} Z^\tau_s dM_s - \int_{\delta_\tau}^{\theta_\tau} dN^\tau_s \\
&\quad + \int_t^{\delta_\tau} g(s)d\langle M\rangle_s + \int_t^{\delta_\tau} dK^{\tau+}_s - \int_t^{\delta_\tau} dK^{\tau-}_s - \int_t^{\delta_\tau} Z^\tau_s dM_s - \int_t^{\delta_\tau} dN^\tau_s \\
&= Y_{\delta_\tau} + \int_t^{\delta_\tau} g(s)d\langle M\rangle_s + \int_t^{\delta_\tau} dK'^{+}_s - \int_t^{\delta_\tau} dK'^{d,-}_s - \int_t^{\delta_\tau} Z'_s dM_s - \int_t^{\delta_\tau} dN'_s \\
&= Y_t.
\end{aligned}$$

Henceforth, it is shown that the quadruplet $\left(Y_t, Z_t, K^+_t, K^-_t, N_t\right)_{t\le T}$ satisfies the following BSDE:

$$Y_t = Y_{\theta_\tau} + \int_t^{\theta_\tau} g(s)d\langle M\rangle_s + \int_t^{\theta_\tau} dK^{\tau+}_s - \int_t^{\theta_\tau} dK^{\tau-}_s - \int_t^{\theta_\tau} Z^\tau_s dM_s - \int_t^{\theta_\tau} dN^\tau_s,\ t \in [\tau, \theta_\tau]. \tag{4.30}$$

Skorokhod condition.

Continuous reflection property: From the definition of the process $K^{c,+}$ and (4.16)-(c), we have $\int_\tau^{\theta_\tau}(Y_s - L_s)dK^{\tau c,+}_s = \int_\tau^{\delta_\tau}(Y_s - L_s)dK'^{+,c}_s = 0$. Similarly, the definition of $K^{\tau c,-}$ with (4.25)-(d) yields to $\int_\tau^{\theta_\tau}(Y_s - L_s)dK^{\tau c,-}_s = \int_{\delta_\tau}^{\theta_\tau}(Y_s - L_s)dK''^{-,c}_s = 0$.

Jumps reflection property: Let η be a predictable stopping time such that $\tau \le \eta \le \theta_\tau$, $\mathbb{P}$-a.s. From the BSDE (4.30) satisfied by Y on $[\tau, \delta_\eta]$, we have $\Delta Y_\eta = \Delta K^{\tau d,-}_\eta - \Delta K^{\tau d,+}_\eta$. Consider $(\omega, t) \in \{(\omega, t) \in \Omega \times [0, T] : \Delta K^{\tau d,-}_t(\omega) > 0\}$. Thus, from (4.16)-(d) and (4.25)-(d), either $\Delta K''^{-,d}_t(\omega) > 0$, for $t \in [\delta_\tau(\omega), \theta_\tau(\omega)]$ or $K'^{d,-}_{\delta_\tau(\omega)}(\omega) > 0$ if $t = \delta_\tau(\omega)$. In both cases, we we necessarily have $Y_t(\omega) > U_{t-}(\omega)$, hence $\{\Delta Y_\eta > 0\} \subset \{\Delta K^{\tau d,-} > 0\} \subset \{Y > U_-\}$. Similarly (using (4.16)-(c) and (4.25)-(e)), we obtain $\{\Delta Y_\eta < 0\} \subset \{\Delta K^{\tau d,+} > 0\} \subset \{Y < L_-\}$.

Note also that $\{\Delta K^{\tau d,+} > 0\} \cap \{\Delta K^{\tau d,-} > 0\} = \emptyset$ since $L_- < U_-$ (assumption **(H3)**), meaning that $K^{\tau d,+}$ and $K^{\tau d,-}$ have no common predictable jump times, and Y does not have any negative (resp. positive) predictable jumps on $]\tau, \delta_\tau[$ (resp. on $]\delta_\tau, \theta_\tau[$). Hence, if $\Delta Y_\eta > 0$ (resp. $\Delta Y_\eta < 0$), then $\Delta Y_\eta = \Delta K^{\tau d,-}_\eta$ (resp. $\Delta Y_\eta = -\Delta K^{\tau d,+}_\eta$). Additionally, within the time frame of $[\delta_\tau, \theta_\tau]$, it can be noted that

$$\begin{aligned}
&\Delta Y_\eta \mathbb{1}_{\{\Delta Y_\eta>0\}} \\
&= \Delta K_\eta^{\tau d,-} \mathbb{1}_{\{\Delta K_\eta^{\tau d,-}>0\}} = \Delta K_\eta^{\tau d,-} \mathbb{1}_{\{\Delta K_\eta^{\tau d,-}>0\}\cap\{\delta_\tau<\eta\leq\theta_\tau\}} + \Delta K_\eta^{\tau d,-} \mathbb{1}_{\{\Delta K_\eta^{\tau d,-}>0\}\cap\{\eta=\delta_\tau\}} \\
&= \Delta K_\eta^{\prime\prime-,d} \mathbb{1}_{\{\Delta K_\eta^{\prime\prime-,d}>0\}\cap\{\delta_\tau<\beta\leq\theta_\tau\}} + K_\eta^{\prime d,-} \mathbb{1}_{\{K_\eta^{\prime d,-}>0\}\cap\{\eta=\delta_\tau\}} \\
&= (Y_\eta - U_{\eta-})^+ \mathbb{1}_{\{\delta_\tau<\eta\leq\theta_\tau\}} + (Y_\eta - U_{\eta-})^+ \mathbb{1}_{\{\eta=\delta_\tau\}} = (Y_\eta - U_{\eta-})^+ .
\end{aligned}$$

Employing a similar argument, we can reach

$$\begin{aligned}
\Delta Y_\eta \mathbb{1}_{\{\Delta Y_\eta<0\}} &= -\Delta K_\eta^{\tau d,+} \mathbb{1}_{\{\Delta K_\eta^{\tau d,+}>0\}} \\
&= -(Y_\beta - L_{\eta-})^- \mathbb{1}_{\{\tau<\eta\leq\delta_\tau\}} - (Y_\eta - U_{\eta-})^- \mathbb{1}_{\{\beta=\theta_\tau\}} = -(Y_\eta - U_{\eta-})^+ .
\end{aligned}$$

Finally, applying Proposition 4.6, we deduce the existence of two stopping times δ, $\theta \in \mathcal{T}_0^T$ such that $\tau \leq \delta \leq \theta \leq \theta_\tau$ and

$$\begin{cases} Y_\delta \mathbb{1}_{\{\delta<T\}} \geq \left(U_\delta - \mathbb{1}_{\{\delta>\tau\}} (\Delta U_\delta)^+\right) \mathbb{1}_{\{\delta<T\}}, \\ Y_\theta \mathbb{1}_{\{\theta<T\}} \leq \mathbb{1}_{\{\theta<T\}} (L_\theta + \mathbb{1}_{\{\theta>\tau\}} (\Delta L_\theta)^-), \end{cases} \tag{4.31}$$

with $\delta = \delta_\tau$ and $\theta = \lambda_\tau$.

5 Global solution for the DRBSDE (2.1) with two completely separated RCLL barriers

In the following section, we assume that the right continuous points and the left limits of the two reflecting barriers $(L_t)_{t\leq T}$ and $(U_t)_{t\leq T}$ are completely separated. Specifically, the obstacles meet the following assumption:

$$\textbf{[CS]}: \ \mathbb{P}\text{-a.s.},\ \forall t \leq T, \quad L_t < U_t \quad \text{and} \quad L_{t-} < U_{t-}.$$

5.1 Case when the driver f is independent of (y, z)

In the next section, we will demonstrate that the DRBSDE (2.1) associated with (ξ, g, L, U) has a solution when the condition **[CS]** is satisfied, and the coefficient f does not depend on y or z. This result is presented as follows:

Theorem 5.1. *Under Assumptions (H1)-(H2) and [CS], the DRBSDE (2.1) associated with $(\xi, g(t), L, U)$ has a unique solution.*

Proof.
Uniqueness: Please refer to Proposition 3.2.
Existence: The proof will be based on the result of the Section 4.3, which allows to construct a local solution for our DRBSDE for every fixed stopping time. Let $Y := (Y_t)_{0\leq t\leq T}$ be the process defined as the limit of the increasing (or decreasing) scheme, which is RCLL and satisfies $Y_T = \xi$. Now, we will construct a sequence of stopping time $(\eta_k)_{k\in\mathbb{N}}$ recursively as follows: $\eta_0 = 0$, $\eta_1 = \theta_0$, $\eta_k = \theta_{\eta_{k-1}}$ for $k \geq 1$. Between η_{k-1} and η_k, there exists a quadruplet $(Z^k, K^{k,+}, K^{k,-}, M^k)$ belonging to $\mathcal{H}_\beta^2 \times \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{M}_\beta^2$, and such that $(Y, Z^k, V^k, K^{k,+}, K^{k,-}, M^k)$ is a solution of the BSDE (4.30) with $\tau = \eta_k$ and $\theta_\tau = \eta_{k+1}$. Note that the sequence of stopping times $(\eta_k)_{k\geq1}$ is increasing. Then, it converges to another stopping time $\rho := \lim_{k\to\infty} \eta_k$.
First, let us show that for any $k \geq 1$, $\mathbb{P}(\{\eta_{k-1} = \eta_k\} \cap \{\eta_k < T\}) = 0$.
Let $\omega \in \{\eta_{k-1} = \eta_k\} \cap \{\eta_k < T\}$. From (4.31), we have $U_{\eta_k} \leq Y_{\eta_k} \leq L_{\eta_k}$. This implies $U_{\eta_k} = Y_{\eta_k} = L_{\eta_k}$. However, $\mathbb{P}$-a.s., $L < U$, thus $\mathbb{P}(\{\eta_{k-1} = \eta_k\} \cap \{\eta_k < T\}) = 0$.
We will now prove that the sequence $(\eta_k)_{k\geq1}$ is of stationary type, which means that $\mathbb{P}\big(\cap_{k\geq1} \{\eta_k < T\}\big) = 0$. In other word, for a fixed $\omega \in \Omega$, there exists some integer $k_0(\omega)$ such that $\eta_{k-1}(\omega) = \eta_k(\omega) = T$ for all $k \geq k_0(\omega)$.

Let $A = \cap_{k\geq 1}\{\eta_k < T\}$, and let's show that $\mathbb{P}(A) = 0$. For $\omega \in A$, again from (4.31), for every $k \geq 1$, there exist two real numbers $t_k(\omega)$ and $t'_k(\omega)$ such that $Y_{t_k} \geq U_{t_k} \wedge U_{t_k-}$, $Y_{t'_k} \leq L_{t'_k} \wedge L_{t'_k-}$, and $t_k(\omega)$, $t'_k(\omega) \in [\eta_{k-1}(\omega), \eta_k(\omega)]$. Let $(t_k(\omega))_{k\geq 1}$ and $(t'_k(\omega))_{k\geq 1}$ be the two sequences constructed in this way. Since $\eta_k < \eta_{k+1}$, the sequences $(t_k(\omega))_{k\geq 1}$ and $(t'_k(\omega))_{k\geq 1}$ are not of stationary type. Taking the limit as $k \to +\infty$, we obtain $Y_{\rho-}(\omega) \leq L_{\rho-}(\omega) \leq U_{\rho-}(\omega) \leq Y_{\rho-}(\omega)$. It means that $L_{\rho-}(\omega) = U_{\rho-}(\omega)$. However, this is impossible under **[CS]** because $L_- < U_-$, and then the sequence $(\eta_k)_{k\geq 1}$ is of stationary type.

Let us introduce the next concatenated processes: $\mathbb{P}$-a.s., $\forall t \leq T$

(i) $Z_t := Z^1_t \mathbb{1}_{[0,\eta_1]}(t) + \sum_{k\geq 1} Z^k_t \mathbb{1}_{]\eta_k,\eta_{k+1}]}(t)$,

(ii) $dN_t := \mathbb{1}_{[0,\eta_1]}(t) dN^1_t + \sum_{k\geq 1} \mathbb{1}_{]\eta_k,\eta_{k+1}]}(t) dN^k_t$,

(iii) $K^{c,\pm}_t := K^{1,\pm,c}_t \mathbb{1}_{[0,\eta_1]}(t) + \sum_{k\geq 1} (K^{k,+,c}_{\eta_k} + K^{k+1,+,c}_t) \mathbb{1}_{]\eta_k,\eta_{k+1}]}(t)$,

(iv) $K^{d,\pm}_t := K^{1,\pm,d}_t \mathbb{1}_{[0,\eta_1]}(t) + \sum_{k\geq 1} (K^{k,+,d}_{\eta_k} + K^{k+1,+,d}_t) \mathbb{1}_{]\eta_k,\eta_{k+1}]}(t)$, and $K^{\pm}_t := K^{c,\pm}_t + K^{d,\pm}_t$.

The analysis in Section 4.3 leads us to the conclusion that the quintuplet $(Y_t, Z_t, K^+_t, K^-_t, N_t)_{t\leq T}$ is the unique solution to the DRBSDE (2.1) associated with (ξ, g, L, U) . □

Remark 5.2.

- The constructed stationary sequence $(\eta_k)_{k\in\mathbb{N}}$ above (i.e., in the proof of Theorem 5.1) ensures the local integrability of the processes Z, $K^{\pm}$, and M. Specifically, we have

$$\mathbb{E}\left[\int_0^{\eta_k} e^{\beta A_s}\left\{|Z_s|^2 \, d\langle M\rangle_s + \int_0^{\eta_k} d\,[N]_s\right\} + (K^+_{\eta_k})^2 + (K^-_{\eta_k})^2\right] < \infty. \tag{5.1}$$

- Under **[CS]**, there exists a sequence of stopping times $(\eta_k)_{k\in\mathbb{N}}$ such that: $\eta_0 = 0$, for any $k \geq 0$, $\eta_k \leq \eta_{k+1}$, and the sequence $(\eta_k)_{k\in\mathbb{N}}$ is of stationary type, i.e., $\mathbb{P}\left(\bigcap_{k\geq 0}\{\eta_k < T\}\right) = 0$.

  For further information on the two aforementioned points, please consult Section 4 in [17] and Lemma 4.1 in [24].

- The hypothesis labeled as **[CS]** is less stringent than Mokobodski's condition, denoted as **[Mk]**, which reads as follows:

$$\textbf{[Mk]:} \begin{cases} \text{there exists two supermartingales } (H_t)_{t\leq T} \textit{ and } (G_t)_{t\leq T} \text{ such that} \\ \bullet\ \mathbb{E}\big[\sup_{0\leq t\leq T}\{\,|H^k_t|^2 + |G^k_t|^2\,\}\big] < \infty, \\ \bullet\ \mathbb{P}\text{-}a.s.,\ \forall t \leq T,\ H_t \geq 0, G_t \geq 0 \text{ and } L_t \leq H_t - G_t \leq U_t. \end{cases}$$

  This condition can be challenging to confirm in practical situations. For instance, consider the solution $\left(y_t, z_t, k^+_t, k^-_t, n_t\right)_{t\leq T}$ of the DRBSDE (2.1) with a null generator, i.e., $g = 0$. Furthermore, let $(\eta_k)_{k\in\mathbb{N}}$ be the sequence of stationary stopping times provided in the proof of Theorem 5.1.

  For $k \geq 1$, set

$$H^k_{t\wedge\eta_k} := \mathbb{E}^{\mathcal{F}_{t\wedge\eta_k}}\left[y^+_{\eta_k} + \left(k^+_{\eta_k} - k^+_{t\wedge\eta_k}\right)\right], \quad \text{and} \quad G_{t\wedge\eta_k} := \mathbb{E}^{\mathcal{F}_{t\wedge\eta_k}}\left[y^-_{\eta_k} + \left(k^-_{\eta_k} - k^-_{t\wedge\eta_k}\right)\right],$$

  for $t \in [0,T]$. We may then deduce that $\left(H^k_t\right)_{t\leq T}$ and $\left(G^k_t\right)_{t\leq T}$ are two non-negative $\mathbb{F}$-supermartingale such that $\mathbb{E}\big[\sup_{0\leq t\leq\eta_k}\{\,|H^k_t|^2 + |G^k_t|^2\,\}\big] < \infty$ (from (5.1)) and $L_t \leq H^k_t - G^k_t \leq U_t$ for any $t \in [0,\eta_k]$, $\mathbb{P}$-a.s. In other words, Mokobodski's condition **[Mk]** is locally satisfied when assumption **[CS]** is fulfilled.

In the following section, we will state the main result of the current paper. Namely, the existence and uniqueness of the DRBSDE (2.1) in the case of a general stochastic Lipschitz generator f.

5.2 Main result

To demonstrate the existence of a solution for the DRBSDE (2.1) when the coefficient f depends on variables (y,z), we will initially establish a solution in the scenario where the driver depends solely on the y-variable. This will be achieved through a contraction argument, leveraging the outcome outlined in Theorem 5.1. Subsequently, building upon this result, we will extend our findings to the general form of the driver f.

Case of a generator depending only on y

In this part, we initially consider the scenario where f takes the form $f=f(t,\omega,y)$, and we present the following result:

Theorem 5.3. *Under conditions **(H1)-(H3)** and **[CS]**, the DRBSDE (2.1) associated with data $(\xi,f(t,y),L,U)$ has a unique solution.*

Proof.
Uniqueness: Please refer to Proposition 3.2.
Existence: As mentioned earlier, the existence will be established through a fixed-point argument. Specifically, we'll consider the space $\mathcal{S}_\beta^{2,\alpha}$ equipped with the norm

$$\| Y \|^2_{\mathcal{S}_\beta^{2,\alpha}}=\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\alpha_s Y_s\right|^2 d\langle M\rangle_s\right]=\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|Y_s\right|^2 dA_s\right].$$

Consider the map Ψ from $\mathcal{S}_\beta^{2,\alpha}$ to itself. This map takes a component Y from $\mathcal{S}_\beta^{2,\alpha}$ and associates it with the corresponding component $\hat{Y}$. In other words, $\Psi(Y)=\hat{Y}$, where $\left(\hat{Y},\hat{Z},\hat{K}^+,\hat{K}^-,\hat{N}\right)$ is the solution of the DRBSDE associated with $\left(\xi,f\left(t,Y_t\right),L,U\right)$. This implies that, for every $t\in[0,T]$ a.s.

$$\begin{cases}\text{(i) } \hat{Y}_t=\xi+\displaystyle\int_t^T f(s,Y_s)d\langle M\rangle_s+(\hat{K}_T^+-\hat{K}_t^+)-(\hat{K}_T^--\hat{K}_t^-)-\int_t^T \hat{Z}_s dM_s-\int_t^T d\hat{N}_s.\\ \text{(ii) } L_t\leq \hat{Y}_t\leq U_t.\\ \text{(iii) Skorokhod condition: } \displaystyle\int_0^T\left(\hat{Y}_{s-}-L_{s-}\right)d\hat{K}_s^+=\int_0^T\left(U_{s-}-\hat{Y}_{s-}\right)d\hat{K}_s^-=0.\end{cases}$$

Now, let's consider another component Y' from $\mathcal{S}_\beta^{2,\alpha}$. Applying Ψ to Y', we obtain $\hat{Y}'=\Psi\left(Y'\right)$, and $\left(\hat{Y}',\hat{Z}',\hat{K}'^{,+},\hat{K}'^{,-},\hat{N}\right)$ is the solution of the DRBSDE associated with $\left(\xi,f\left(t,Y'_t\right),L,U\right)$.

Once again, the lack of integrability of the processes $\left(\hat{Z},\hat{Z}',\hat{N},\hat{N}'\right)$ requires us to use the sequence of stopping times $\{\sigma_k\}_{k\geq 1}$ defined in the proof of Proposition 3.2. Applying Itô's formula and using a similar argument as in the proof of Proposition 3.2, we obtain

$$\begin{aligned}&\beta\mathbb{E}\left[\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left|\hat{Y}_s-\hat{Y}'_s\right|^2 d\langle M\rangle_s\right]\\&\leq \mathbb{E}\left[e^{\beta A_{\sigma_k}}\left|\hat{Y}_{\sigma_k}-\hat{Y}'_{\sigma_k}\right|^2\right]+2\mathbb{E}\left[\int_{t\wedge\sigma_k}^{\sigma_k} e^{\beta A_s}\left(\hat{Y}_s-\hat{Y}'_s\right)\left(f\left(s,Y_s\right)-f\left(s,Y'_s\right)\right)d\langle M\rangle_s\right]\end{aligned}\tag{5.2}$$

On the other hand, since $\sigma_k\nearrow T$ a.s. as $k\to+\infty$ and $e^{\beta A_{\sigma_k}}\left|\hat{Y}_{\sigma k}-\hat{Y}'_{\sigma k}\right|^2\to 0$ as $k\to+\infty$, then by applying the monotonic convergence theorem and the Lebesgue dominated convergence theorem to (5.2) as $k\to+\infty$, we derive

$$\begin{aligned}&\beta\mathbb{E}\left[\int_t^T e^{\beta A_s}\left|\hat{Y}_s-\hat{Y}'_s\right|^2 d\langle M\rangle_s\right]\\&\leq 2\mathbb{E}\left[\int_t^T e^{\beta A_s}\left(\hat{Y}_s-\hat{Y}'_s\right)\left(f\left(s,Y_s\right)-f\left(s,Y'_s\right)\right)d\langle M\rangle_s\right]\end{aligned}$$

Next, using assumption **(H2)**-(ii) and the inequality $2ab \leq 2a^2 + \frac{1}{2}b^2$ (in the sense of signed measures), we get

$$\begin{aligned}
2(\hat{Y}_s - \hat{Y}'_s)(f(s,Y_s) - f(s,Y'_s))d\langle M\rangle_s &\leq 2\kappa_s \left|\hat{Y}_s - \hat{Y}'_s\right|^2 |Y_s - Y'_s| \, d\langle M\rangle_s \\
&\leq 2\left|\hat{Y}_s - \hat{Y}'_s\right|^2 |Y_s - Y'_s| \, dA_s \\
&\leq 2\left|\hat{Y}_s - \hat{Y}'_s\right|^2 dA_s + \frac{1}{2}|Y_s - Y'_s|^2 \, dA_s.
\end{aligned}$$

By choosing $\beta = 3$, we deduce that

$$\|\Psi(Y) - \Psi(Y')\|^2_{\mathcal{S}^{2,\alpha}_\beta} \leq \frac{1}{2}\|Y - Y'\|^2_{\mathcal{S}^{2,\alpha}_\beta}.$$

Then, the random functional Ψ becomes a strict contraction mapping on the Banach space $\mathcal{S}^{2,\alpha}_\beta$. As a result, there exists a process Y that serves as a fixed point of Ψ, meaning that $\Psi(Y) = Y$. This process, along with $\hat{Z}$, $\hat{K}^+$, $\hat{K}^-$, and $\hat{N}$, represents the unique solution to the DRBSDE (2.1) associated with (ξ, f, L, U). □

It is now time to present the main result of this paper.

General Case

Building upon the insights gained from Theorem 5.3 and adopting a similar approach, we are now prepared to unveil the central outcome of this paper. Specifically, we establish the existence and uniqueness of a solution to the DRBSDE (2.1) in the more general scenario where the coefficient f incorporates dependence on both y and z, i.e., $f = f(t,\omega,y,z)$. The uniqueness result has already been established in Proposition 3.2. For the existence part, the proof closely follows the one presented in ([18], Theorem 4.2, Step 2), despite the distinction that the obstacles in this paper may exhibit inaccessible jumps. As this difference doesn't affect the argument, we choose to omit it.

Theorem 5.4. *Under conditions **(H1)-(H3)** and **[CS]**, the DRBSDE (2.1) associated with data (ξ, f, L, U) has a unique solution.*

6 Conclusion

This paper aims to establish existence and uniqueness results for a class of doubly reflected BSDEs under mild conditions on the data. The approach involves initially exploring local solutions by analyzing the convergence of the increasing and decreasing penalization schemes. Subsequently, we extend our findings to global solutions, covering the entire time horizon, leveraging the complete separation of the barriers and their left limits.

Appendix: Special generalized BSDEs and generalized reflected BSDEs driven by an RCLL martingale

In this section, we present a special case of the existence and uniqueness result for a specific type of generalized BSDEs and generalized reflected BSDEs with jumps. In this case, the continuous finite variation process is related to another coefficient that depends only on y and satisfies a standard square integrable condition with respect to the Lebesgue measure. Throughout this section, we introduce the auxiliary space $\mathcal{C}^2_\beta$ as follows:

- $\mathcal{C}^2_\beta$: the space of one-dimensional $\mathcal{F}_t$-progressively measurable processes $(F_t)_{t\leq T}$ such that

$$\| F \|^2_{\mathcal{C}^2_\beta} = \mathbb{E}\left[\int_0^T e^{\beta A_s} |F_s|^2 \, ds\right] < \infty.$$

Special generalized BSDE without reflection driven by an RCLL martingale

Existence and uniqueness result: We provide a specific case of existence and uniqueness for BSDEs driven by the RCLL martingale M in this section when the coefficient depends only on y. Consider the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s)ds + \int_t^T g(s)d\langle M\rangle_s - \int_t^T Z_s dM_s - \int_t^T dN_s, \quad 0 \le t \le T. \tag{.1}$$

Theorem .1. *Assume that:*

(i) $\xi \in \mathbb{L}^2_\beta$,

(ii) $\dfrac{g(\cdot)}{\alpha.} \in \mathcal{H}^2_\beta$ *and* $f(\cdot, 0) \in \mathcal{C}^2_\beta$,

(iii) The driver f is uniformly Lipschitz continuous with respect to y, i.e. there exists a positive constant κ such that, almost every (ω, t), for all y, $y' \in \mathbb{R}$,

$$|f(t,y) - f(t,y')| \le \kappa |y - y'|$$

Then, the BSDE (.1) admit a unique solution $(Y, Z, N) \in \left(\mathcal{S}^2_\beta \cap \mathcal{S}^{2,\alpha}_\beta \cap \mathcal{C}^2_\beta\right) \times \mathcal{H}^2_\beta \times \mathcal{M}^2_\beta$.

Comparison theorem:

Theorem .2. *Met* (Y^1, Z^1, N^1), (Y^2, Z^2, N^2) *be solutions of BSDE (.1) associated with parameters* (ξ^1, f^1, g) *and* (ξ^2, f^2, g)*, respectively. Assume that* $\xi^1 \le \xi^2$ *and for any* $t \ge 0$, $f^1(t, y) \le f^2(t, y)$*, for all* $y \in \mathbb{R}$*,* $\mathbb{P}$*-a.s. Then* $Y^1 \le Y^2$ *a.s.*

Special case for generalized BSDEs with one reflecting RCLL barrier: Lower barrier case

Existence and uniqueness result: In this paragraph, we study the existence and uniqueness problem for a special case of one reflected BSDE described by:

$$\begin{cases} Y_t = \xi + \displaystyle\int_t^T g(s)d\langle M\rangle_s + \int_t^T f(s, Y_s)ds + K_T - K_t - \int_t^T Z_s dM_s - \int_t^T dN_s, \; 0 \le t \le T. \\ L_t \le Y_t, \; \forall t \le T, \text{ and } \displaystyle\int_0^T (Y_{s-} - L_{s-})dK_s = 0, \; \mathbb{P}\text{-a.s.} \end{cases} \tag{.2}$$

where the parameters (ξ, f, L) are given such that:

- The terminal variable ξ belongs to $\mathbb{L}^2_\beta$,
- $g(\cdot)$ is an $\mathcal{F}_t$-progressively measurable process, such that $\dfrac{g(\cdot)}{\alpha.} \in \mathcal{H}^2_\beta$,
- $f(\cdot, 0)$ is an $\mathcal{F}_t$-progressively measurable process, such that $f(\cdot, 0) \in \mathcal{C}^2_\beta$.
- There exists a positive constant κ such that, almost every (ω, t), for all y, $y' \in \mathbb{R}$,

$$|f(t,y) - f(t,y')| \le \kappa |y - y'|,$$

- The lower reflecting barrier $(L_t)_{t \le T}$ is a real-valued $\mathcal{F}_t$-progressively measurable RCLL processes satisfying:
  (i) $L_T \le \xi$, $\mathbb{P}$-a.s.
  (ii) $\mathbb{E}\left[\sup_{0 \le t \le T} \left|e^{\beta A_t} L_t^+\right|^2\right] < \infty$.

Theorem .3. *The reflected BSDE (.2) associated with (ξ, g, f, L) has a unique solution such that*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_t}|Y_t|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\alpha_s^2|Y_s|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}|Y_s|^2 ds\right]\\
&\quad+\mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[|K_T|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}d[N]_s\right]\\
&\le C_\beta\left(\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\right.\\
&\quad\left.+\mathbb{E}\left[\int_0^T e^{\beta A_s}|f(s,0)|^2 ds\right]+\mathbb{E}\left[\sup_{0\le t\le T}\left|e^{\beta A_s}(L_s)^+\right|^2\right]\right)
\end{aligned}$$

and the state process $(Y_t)_{t\le T}$ can be characterized using the Snell envelope of processes as follows:

$$Y_\sigma=\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\sigma^T}\mathbb{E}^{\mathcal{F}_\sigma}\left[\int_\sigma^\tau g(s)d\langle M\rangle_s+\int_\sigma^\tau f(s,Y_s)ds++S_\tau\mathbb{1}_{\{\tau<T\}}+\xi\mathbb{1}_{\{\tau=T\}}\right].$$

Remark .4. In the proof of Theorem .3 (also see the proof of Theorem 6 in [32]), it is important to note that we rely on a uniform estimation involving the sequence of penalized versions. These versions are defined as follows:

$$Y_t^n=\xi+\int_t^T g(s)d\langle M\rangle_s+\int_t^T\left\{f(s,y)+n(Y_s^n-L_s)^-\right\}ds-\int_t^T Z_s^n dM_s-\int_t^T dN_s^n,\quad n\in\mathbb{N}.$$

For the purpose of our argument, it is necessary to establish a uniform estimate for this sequence, given by:

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left[\sup_{0\le t\le T} e^{\beta A_t}|Y_t^n|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\alpha_s^2|Y_s^n|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}|Y_s^n|^2 ds\right]\right.\\
&\quad\left.+\mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s^n|^2 d\langle M\rangle_s\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}d[N^n]_s\right]+\mathbb{E}\left[|K_T^n|^2\right]\right\}\\
&\le C_{\beta,\kappa,T}\left(\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\right.\\
&\quad\left.+\mathbb{E}\left[\int_0^T e^{\beta A_s}|f(s,0)|^2 ds\right]+\mathbb{E}\left[\sup_{0\le t\le T}\left|e^{\beta A_s}(L_s)^+\right|^2\right]\right).
\end{aligned}$$

where $K_t^n := n\int_0^t (Y_s^n-L_s)^- ds$, $t\in[0,T]$ and $C_{\beta,\kappa,T}$ is a positive constant depending only on β, κ and T.

Comparison theorem:

Theorem .5. *Let $(Y_t^1, Z_t^1, K_t^1, N_t^1)_{t\le T}$ and $(Y_t^2, Z_t^2, K_t^2, N_t^2)_{t\le T}$ be the solution of the RBSDE (.2) associated with (ξ^1, g, f^1, L) and (ξ^2, g, f^2, L) respectively. Also assume that $\xi^1\le\xi^2$, and for any $t\ge 0$, $f^1(t,y)\le f^2(t,y)$, for all $y\in\mathbb{R}$, $\mathbb{P}$-a.s. Then, $Y_t^1\le Y^2$ and $K_t^1\ge K_t^2$ for $t\in[0,T]$, almost surely, and, for $0\le s\le t\le T$, $K_t^1-K_s^1\ge K_t^2-K_s^2$-almost surely.*

Special case for generalized BSDEs with one reflecting RCLL barrier: Upper barrier case

Building on the results presented in the previous section, we now delve into specific case of generalized BSDEs with a single upper RCLL reflecting barrier.

Corollary .6. *The BSDE with one upper reflecting barrier $(U_t)_{t\leq T}$, that is:*

$$\begin{cases} Y_t = \xi + \int_t^T g(s)d\langle M\rangle_s + \int_t^T f(s,Y_s)ds - (K_T - K_t) - \int_t^T Z_s dM_s - \int_t^T dN_s. \\ Y_t \leq U_t, \ \forall t \leq T, \\ \int_0^T (Y_s - U_s)dK_s^c = 0, \ \mathbb{P}\text{-a.s. and } K_t^d = \sum_{0<s\leq t} (Y_s - U_{s-})^+ . \end{cases}$$

where $(U_t)_{t\leq T}$ is an $\mathcal{F}_t$-progressively measurable RCLL real-valued process satisfying:

- $\xi \leq U_T$, $\mathbb{P}$-*a.s.*
- $\mathbb{E}\left[\sup_{0\leq t\leq T}\left|e^{\beta A_t}U_t^-\right|^2\right] < \infty$.

admit a unique solution $(Y,Z,K,N) \in \mathfrak{A}_\beta^2$ such that

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A_t}|Y_t|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s}\alpha_s^2|Y_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s}|Y_s|^2 ds\right] \\ &\quad + \mathbb{E}\left[\int_0^T e^{\beta A_s}|Z_s|^2 d\langle M\rangle_s\right] + \mathbb{E}\left[|K_T|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s} d[N]_s\right] \\ &\leq C_\beta\left(\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta A_s}\left|\frac{g(s)}{\alpha_s}\right|^2 d\langle M\rangle_s\right]\right. \\ &\quad \left. + \mathbb{E}\left[\int_0^T e^{\beta A_s}|f(s,0)|^2 ds\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}\left|e^{\beta A_s}(U_s)^-\right|^2\right]\right). \end{aligned}$$

and the state process $(Y_t)_{t\leq T}$ can be characterized using the Snell envelope of processes as follows:

$$Y_\sigma = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_\sigma^T} \mathbb{E}^{\mathcal{F}_\sigma}\left[\int_\sigma^\tau g(s)d\langle M\rangle_s + \int_\sigma^\tau f(s,Y_s)ds + U_\tau \mathbb{1}_{\{\tau<T\}} + \xi\mathbb{1}_{\{\tau=T\}}\right].$$

Remark .7. Note that, the quadruplet a triple (Y,Z,K,N) is a solution for the BSDE with a lower reflecting RCLL barrier L, drivers g, $f(t,y)$ and a terminal value ξ if and only if $(-Y,-Z,K,-N)$ is a solution for the BSDE with a lower reflecting RCLL barrier $-L$, drivers g, $-f(t,-y)$ and a terminal value $-\xi$.

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