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Common Solution for Fixed Points of a Finite Family of Nonexpansive Mappings and Variational Inclusion Problem

Prashant Patel

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Abstract The aim of this article is to present an iterative algorithm for finding a common solution to the variational inclusion problem with Lipschitz continuous single valued and maximal monotone multivalued mappings and the set of fixed points of a finite family of nonexpansive mappings. Under some conditions, we prove a strong convergence theorem which converges to this common solution.

1 Introduction and Preliminaries

Suppose Σ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$ and 2^{Σ} denotes the family of all the nonempty subsets of Σ. In 1976, Rockafellar [16] studied the inclusion problem of finding

$$\eta^{\dagger} \in H^{-1}(0), \tag{1.1}$$

where H is a maximal monotone set-valued mapping defined on a Hilbert space Σ. We focus on the following variational inclusion problem

$$\text{find } \eta^{\dagger} \in \Sigma \text{ such that } \eta^{\dagger} \in (G+H)^{-1}(0), \tag{1.2}$$

where $G : \Sigma \to \Sigma$ is a single valued and $H : \Sigma \to 2^{\Sigma}$ is a multivalued mapping. When $G = 0$ then the problem (1.2) reduces to the inclusion problem (1.1) which plays an important role in minimization problems and other fields of mathematics. Due to its applications in several fields of science, engineering, management, and social sciences in the last many years, the inclusion problem has been broadened and generalized in numerous ways; see, for example, [3, 6, 17, 11, 14, 13, 18, 7, 10, 15] and the references therein.

A point $\eta^{\dagger} \in \Sigma$ is said to be the fixed point of the mapping $S : \Sigma \to \Sigma$ if $S(\eta^{\dagger}) = \eta^{\dagger}$. We can easily verify that a point $\eta^{\dagger} \in \Sigma$ is a solution of the problem (1.2) if and only if $\eta^{\dagger}$ is a fixed point of the mapping $J_{\lambda}^{H}(I - \lambda G)$, that is $\eta^{\dagger} = J_{\lambda}^{H}(I - \lambda G)(\eta^{\dagger})$, where J_{λ}^{H} is the resolvent operator of H. This provides us numerous benefits for constructing new algorithms and proving the convergence of algorithms.

Mann [9] introduced an algorithm to approximate the fixed points of a nonexpansive mapping, which is as follows: $\eta_1 \in \Sigma$

$$\eta_{n+1} = \Upsilon_n \eta_n + (1 - \Upsilon_n) S(\eta_n)$$

for all $n \in \mathbb{N}$ and $\{\Upsilon_n\}$ is a sequence in $[0, 1]$. On the other hand, Halpern [4] introduced an algorithm as $\eta_1 = \eta \in \Sigma$ and

$$\eta_{n+1} = \Upsilon_n \eta + (1 - \Upsilon_n) S(\eta_n)$$

for all $n \in \mathbb{N}$ and $\{\Upsilon_n\}$ is a sequence in $[0, 1]$. In the recent years, there are many researchers who modified the Mann and Halpern type iteration and defined new algorithms [5, 22, 8]. In

2008, Zhang et. al. [23] presented an iterative algorithm for finding a common solution of the set of fixed points of a nonexpansive mapping and the set of solutions to the variational inclusion problem with inverse strongly monotone mapping and multivalued maximal monotone mapping in Hilbert spaces. Their algorithm is as follows:

$$\begin{cases} \varsigma_n = J_\lambda^H(\eta_n - \lambda G(\eta_n)), \\ \eta_{n+1} = \Upsilon_n \eta + (1-\Upsilon_n) S(\varsigma_n), \ \forall n \geq 0. \end{cases} \tag{1.3}$$

Here the mapping $G : \Sigma \to \Sigma$ is α-inverse strongly monotone mapping, $H : \Sigma \to 2^\Sigma$ a maximal monotone mapping, $S : \Sigma \to \Sigma$ a nonexpansive mapping and $\lambda \in (0, 2\alpha]$ and Υ_n is given sequence in the interval $[0, 1]$ having the given conditions:

(1) $\Upsilon_n \to 0, \sum\limits_{n=0}^{\infty} \Upsilon_n = \infty,$

(2) $\sum\limits_{n=0}^{\infty} |\Upsilon_{n+1} - \Upsilon_n| < \infty.$

Then the sequence $\{\eta_n\}$ converges strongly to a point of $F(S) \bigcap (G + H)^{-1}(0)$.

Recently, in 2023 Younis et. al. [21] modified above algorithm (1.3) for all type of Lipschitz continuous mappings and they removed the restriction on $\lambda \in (0, 2\pi]$ to $\lambda \in \mathbb{R}^+$ and presented a strong convergence result for a new algorithm as follows:

Theorem 1.1. *Suppose $G : \Sigma \to \Sigma$ a single valued Lipschitz continuous, $H : \Sigma \to 2^\Sigma$ a multivalued maximal monotone, $S : \Sigma \to \Sigma$ a nonexpansive mapping. Suppose $\Theta = F(S) \bigcap (G+H)^{-1} \neq \emptyset$. Suppose $\eta = \eta_0 \in \Sigma$ and the sequence $\{\eta_n\}$ generated by*

$$\begin{cases} \varsigma_n = J_\lambda^{(G+H)}(\eta_n), \\ \eta_{n+1} = \Upsilon_n \eta + (1-\Upsilon_n) S(\varsigma_n), \ \forall n \geq 0. \end{cases} \tag{1.4}$$

where $\lambda \in \mathbb{R}^+$ and $\forall n \in \mathbb{N}$, $\{\Upsilon_n\}$ is the given sequence having the given conditions:

(1) $\Upsilon_n \to 0, \sum\limits_{n=0}^{\infty} \Upsilon_n = \infty,$

(2) $\sum\limits_{n=0}^{\infty} |\Upsilon_{n+1} - \Upsilon_n| < \infty.$

Then the sequence $\{\eta_n\}$ generated by (1.4) converges strongly to a point of $F(S) \bigcap (G+H)^{-1}(0)$.

In this article, motivated by Zhang et. al. [23], Younis et. al. [21] and others we consider a new algorithm and prove that the sequence generated by the algorithm converges strongly to the common solution of a finite family of nonexpansive mappings and variational inclusion problem i.e. to obtain $\eta^\dagger \in \Sigma$ such that

$$\eta^\dagger \in \bigcap_{i=1}^{m} F(S_i) \bigcap (G + H)^{-1}(0).$$

Now we present, some basic definitions and facts from the literature.

Lemma 1.2. [21]. *$\zeta = J_\lambda^{(G+H)}(\zeta)$ for all $\lambda \in \mathbb{R}^+$ iff $\zeta \in \Sigma$ is a solution of the variational inclusion problem (1.2).*

Definition 1.3. The single valued mapping $J_\lambda^H : \Sigma \to \Sigma$ defined by

$$J_\lambda^H(\eta) = [I + \lambda H]^{-1}(\eta), \ \ \forall \eta \in \Sigma,$$

is known as resolvent mapping for $H : \Sigma \to 2^\Sigma$.

Definition 1.4. A mapping $S : \Sigma \to \Sigma$ is said to be nonexpansive if for all $\eta, \zeta \in \Sigma$

$$\|S(\eta) - S(\zeta)\| \leq \|\eta - \zeta\|.$$

The resolvent mapping J_λ^H is also a nonexpansive, i.e. for all $\eta, \zeta \in \Sigma$

$$\|J_\lambda^H(\eta) - J_\lambda^H(\zeta)\| \leq \|\eta - \zeta\|.$$

Definition 1.5. A mapping $S : \Sigma \to \Sigma$ is said to be monotone if for each $\eta, \zeta \in \Sigma$

$$\langle \eta - \zeta, S(\eta) - S(\zeta) \rangle \geq 0.$$

α-inverse strongly monotone with constant $\alpha > 0$, if for each $\eta, \zeta \in \Sigma$

$$\langle \eta - \zeta, S(\eta) - S(\zeta) \rangle \geq \alpha \|S(\eta) - S(\zeta)\|^2.$$

β-strongly monotone if there exists a positive real number β such that

$$\langle \eta - \zeta, S(\eta) - S(\zeta) \rangle \geq \beta \|\eta - \zeta\|^2.$$

The set of zeros of the mapping S is defined by

$$S^{-1}(0) = \{\eta \in \Sigma : 0 \in S(\eta)\}.$$

Definition 1.6. [19] Suppose $\mathcal{E} \neq \emptyset$ be a closed convex subset of Σ. Then $\forall\ \eta \in \Sigma$, $\exists$ one and only one nearest point in $\mathcal{E}$, known as a metric projection of $\eta \in \mathcal{E}$ and denoted by $P_{\mathcal{E}}(\eta)$, that is

$$\|\eta - P_{\mathcal{E}}(\eta)\| \leq \|\eta - \zeta\| \text{ for all } \zeta \in \mathcal{E}.$$

Remark 1.7. A metric projection $P_{\mathcal{E}}$ has the following properties:

(i) $P_{\mathcal{E}} : \Sigma \to \mathcal{E}$ is nonexpansive

$$\|P_{\mathcal{E}}(\eta) - P_{\mathcal{E}}(\zeta)\| \leq \|\eta - \zeta\| \text{ for all } \eta, \zeta \in \Sigma.$$

(ii) $P_{\mathcal{E}}$ is called firmly nonexpansive if

$$\|P_{\mathcal{E}}(\eta) - P_{\mathcal{E}}(\zeta)\|^2 \leq \langle P_{\mathcal{E}}(\eta) - P_{\mathcal{E}}(\zeta), \eta - \zeta \rangle \text{ for all } \eta, \zeta \in \Sigma.$$

(iii) For all $\eta \in \Sigma$

$$\nu = P_{\mathcal{E}}(\eta) \Leftrightarrow \langle \eta - \nu, \nu - \zeta \rangle \geq 0 \text{ for all } \zeta \in \mathcal{E}.$$

Lemma 1.8. [1] *A mapping $G + H : \Sigma \to 2^{\Sigma}$ is maximal monotone, if $G : \Sigma \to \Sigma$ is Lipschitz continuous and $H : \Sigma \to 2^{\Sigma}$ is maximal monotone.*

Definition 1.9. Let $G : \Sigma \to \Sigma$ be a Lipschitz continuous mapping and $H : \Sigma \to 2^{\Sigma}$ a maximal monotone mapping. Then a new resolvent mapping of the maximal monotone mapping $G + H$ is defined as

$$J_{\lambda}^{(G+H)}(\eta) = [I + \lambda(G + H)]^{-1}(\eta), \text{ for all } \eta \in \Sigma. \tag{1.5}$$

Remark 1.10. The resolvent operator defined by (1.5) is nonexpansive and 1-inverse strongly monotone.

Lemma 1.11. [20, 12]. *Assume $\{\tau_n\}$ be a given sequence of non negative real numbers that satisfy*

$$\tau_{n+1} \leq (1 - \sigma_n)\tau_n + \xi_n + \delta_n,$$

$\forall\ n \geq 0$, *where $\{\xi_n\}$ and $\{\delta_n\}$ are sequences in $\mathbb{R}$ and $\{\sigma_n\}$ is a subsequence in $(0, 1)$. Let us consider that:*

(1) $\sum_{n=1}^{\infty} \sigma_n = \infty$,

(2) $\sum_{n=1}^{\infty} |\xi_n| < \infty$ or $\limsup_{n\to\infty} \frac{\xi_n}{\sigma_n} \leq 0$,

(3) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then, $\lim_{n\to\infty} \tau_n = 0$.

Lemma 1.12. [2] *If Σ is a real Hilbert space, then*

$$\|\eta_1 + \eta_2\|^2 \leq \|\eta_1\|^2 + 2\langle \eta_2, \eta_1 + \eta_2 \rangle \text{ for all } \eta_1, \eta_2 \in \Sigma.$$

2 Main Results

Theorem 2.1. *Suppose $G : \Sigma \to \Sigma$ a single valued Lipschitz continuous mapping, $H : \Sigma \to 2^{\Sigma}$ a multivalued maximal monotone mapping, and $S_i : \Sigma \to \Sigma$, $i = 1, 2, \cdots m$ be a finite family of nonexpansive mappings. Suppose $\Theta = \bigcap_{i=1}^{m} F(S_i) \bigcap (G + H)^{-1} \neq \emptyset$. Suppose $\eta = \eta_0 \in \Sigma$ and the sequence $\{\eta_n\}$ generated by*

$$\begin{cases} \varsigma_n = J_\lambda^{(G+H)}(\eta_n), \\ \eta_{n+1} = \Upsilon_n \eta + (1 - \Upsilon_n) S_i(\varsigma_n), \ \forall n \geq 0. \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}^+$ and $\forall n \in \mathbb{N}$, $\{\Upsilon_n\}$ is a given sequence having these conditions:

(1) $\Upsilon_n \to 0$, $\sum_{n=0}^{\infty} \Upsilon_n = \infty$,

(2) $\sum_{n=0}^{\infty} |\Upsilon_{n+1} - \Upsilon_n| < \infty$.

Then the sequence $\{\eta_n\}$ converges strongly to a point of $\bigcap_{i=1}^{m} F(S_i) \bigcap (G + H)^{-1}(0)$.

Proof. First we proof that the sequences $\{\eta_n\}$ and $\{\varsigma_n\}$ are bounded. Suppose $\zeta \in \Theta$ using Lemma 1.2, we have

$$\zeta = J_\lambda^{(G+H)}(\zeta).$$

Now,

$$\begin{aligned} \|\varsigma_n - \zeta\| &= \|J_\lambda^{(G+H)}(\eta_n) - J_\lambda^{(G+H)}(\zeta)\| \\ &\leq \|\eta_n - \zeta\|. \end{aligned} \tag{2.2}$$

$$\begin{aligned} \|\eta_{n+1} - \zeta\| &= \|\Upsilon_n \eta + (1 - \Upsilon_n) S_i(\varsigma_n) - \zeta\| \\ &= \|\Upsilon_n(\eta - \zeta) + (1 - \Upsilon_n)(S_i(\varsigma_n) - \zeta)\| \\ &\leq \Upsilon_n \|\eta - \zeta\| + (1 - \Upsilon_n)\|S_i(\varsigma_n) - \zeta\| \\ &\leq \Upsilon_n \|\eta - \zeta\| + (1 - \Upsilon_n)\|\varsigma_n - \zeta\| \end{aligned}$$

Using inequality (2.2) in the above equation we get

$$\begin{aligned} \|\eta_{n+1} - \zeta\| &\leq \Upsilon_n \|\eta - \zeta\| + (1 - \Upsilon_n)\|\eta_n - \zeta\| \\ &\leq \max\{\|\eta - \zeta\|, \|\eta_n - \zeta\|\} \\ &\cdots \\ &\leq \max\{\|\eta - \zeta\|, \|\eta_0 - \zeta\|\} \\ &= \|\eta - \zeta\|. \end{aligned}$$

Thus we can say the sequences $\{\eta_n\}$ and $\{\varsigma_n\}$ are bounded. Since S_i is the family of nonexpansive mappings and G is Lipschitz continuous, the sequences $\{G(\eta_n)\}$ and $\{S_i(\varsigma_n)\}$ are also bounded in Σ.

Now we prove that $\|\eta_{n+1} - \eta_n\| \to 0$ and $\|\varsigma_{n+1} - \varsigma_n\| \to 0$ as $n \to \infty$.

Now, we have

$$\begin{aligned} \|\varsigma_{n+1} - \varsigma_n\| &= \left\| J_\lambda^{(G+H)}(\eta_{n+1}) - J_\lambda^{(G+H)}(\eta_n) \right\| \\ &\leq \|\eta_{n+1} - \eta_n\|, \end{aligned}$$

and

$$\begin{aligned}\|\eta_{n+1}-\eta_n\| &= \|\Upsilon_n\eta+(1-\Upsilon_n)S_i(\varsigma_n)-\Upsilon_{n-1}\eta-(1-\Upsilon_{n-1})S_i(\varsigma_{n-1})\| \\ &= \|(\Upsilon_n-\Upsilon_{n-1})(\eta-S_i(\varsigma_{n-1}))+(1-\Upsilon_n)(S_i(\varsigma_n)-S_i(\varsigma_{n-1}))\| \\ &\leq |\Upsilon_n-\Upsilon_{n-1}|\|\eta-S_i(\varsigma_{n-1}\|+(1-\Upsilon_n)\|S_i(\varsigma_n)-S_i(\varsigma_{n-1})\| \\ &\leq D|\Upsilon_n-\Upsilon_{n-1}|+(1-\Upsilon_n)\|\varsigma_n-\varsigma_{n-1}\| \\ &\leq D|\Upsilon_n-\Upsilon_{n-1}|+(1-\Upsilon_n)\|\eta_n-\eta_{n-1}\|,\end{aligned}$$

where $D = \sup_{n\geq 1}\|\eta - S_i(\varsigma_{n-1})\|$. If we take $\tau_n = \|\eta_n - \eta_{n-1}\|$, $\xi_n = D|\Upsilon_n - \Upsilon_{n-1}|$, and $\delta_n = 0$ then all the conditions of Lemma 1.11 are satisfied, and hence $\lim_{n\to\infty}\|\eta_{n+1}-\eta_n\| = 0$, $\lim_{n\to\infty}\|\varsigma_{n+1}-\varsigma_n\| = 0$.

Now we prove that $\lim_{n\to\infty}\|\eta_n - S_i(\varsigma_n)\| \to 0$.
Now,

$$\begin{aligned}\|\eta_n-S_i(\varsigma_n)\| &\leq \|\eta_n-S_i(\varsigma_{n-1})\|+\|S_i(\varsigma_{n-1})-S_i(\varsigma_n)\| \\ &\leq \Upsilon_{n-1}\|\eta-S_i(\varsigma_{n-1})\|+\|\varsigma_{n-1}-\varsigma_n\|.\end{aligned}$$

Since $\Upsilon_n \to 0$ and $\|\varsigma_{n-1}-\varsigma_n\| \to 0$ as $n\to\infty$, we get $\|\eta_n - S_i(\varsigma_n)\| \to 0$.

Now we prove that $\|\eta_n-\varsigma_n\| \to 0$ and $\|S_i(\varsigma_n)-\varsigma_n\| \to 0$. For $\zeta \in \bigcap_{i=1}^{n} F(S_i)\bigcap(G+H)^{-1}(0)$ and using remark 1.10 we get

$$\begin{aligned}\|\varsigma_n-\zeta\|^2 &= \left\|J_\lambda^{(G+H)}(\eta_n)-J_\lambda^{(G+H)}(\zeta)\right\|^2 \\ &= \left\langle \eta_n-\zeta, J_\lambda^{(G+H)}(\eta_n)-J_\lambda^{(G+H)}(\zeta)\right\rangle \\ &= \langle \eta_n-\zeta, \varsigma_n-\zeta\rangle \\ &= \frac{1}{2}\left\{\|\eta_n-\zeta\|^2+\|\varsigma_n-\zeta\|^2-\|\eta_n-\zeta-(\varsigma_n-\zeta)\|^2\right\} \\ &\leq \frac{1}{2}\left\{\|\eta_n-\zeta\|^2+\|\eta_n-\zeta\|^2-\|\eta_n-\varsigma_n\|^2\right\}.\end{aligned}$$

We have

$$\|\varsigma_n-\zeta\|^2 \leq \|\eta_n-\zeta\|^2-\frac{1}{2}\|\eta_n-\varsigma_n\|^2.$$

Now,

$$\begin{aligned}\|\eta_{n+1}-\zeta\|^2 &= \|\Upsilon_n\eta+(1-\Upsilon_n)S_i(\varsigma_n)-\zeta\|^2 \\ &= \|\Upsilon_n(\eta-\zeta)+(1-\Upsilon_n)(S_i(\varsigma_n)-\zeta)\|^2 \\ &\leq \Upsilon_n\|(\eta-\zeta)\|^2+(1-\Upsilon_n)\|(S_i(\varsigma_n)-\zeta)\|^2 \\ &\leq \Upsilon_n\|(\eta-\zeta)\|^2+(1-\Upsilon_n)\|(\varsigma_n-\zeta)\|^2 \\ &\leq \Upsilon_n\|(\eta-\zeta)\|^2+(1-\Upsilon_n)\left\{\|(\eta_n-\zeta)\|^2-\frac{1}{2}\|\eta_n-\varsigma_n\|^2\right\}.\end{aligned}$$

It implies

$$\frac{(1-\Upsilon_n)}{2}\|\eta_n-\varsigma_n\|^2 \leq \Upsilon_n\|(\eta-\zeta)\|^2+(\|\eta_n-\zeta\|^2-\|\eta_{n+1}-\zeta\|^2). \tag{2.3}$$

Since $\Upsilon_n \to 0$ and

$$\left|\|\eta_n-\zeta\|^2-\|\eta_{n+1}-\zeta\|^2\right| \leq \|\eta_{n+1}-\eta_n\|(\|\eta_n\|+\|\eta_{n+1}\|) \to 0.$$

Using the above condition in the equation (2.3) we get $\|\eta_n - \varsigma_n\| \to 0$.

Hence
$$\|S_i(\varsigma_n) - \varsigma_n\| \le \|S_i(\varsigma_n) - \eta_n\| + \|\eta_n - \varsigma_n\| \to 0.$$

Now we prove that
$$\limsup_{n\to\infty} \langle \eta - \varpi, S_i(\varsigma_n) - \varpi\rangle \le 0,$$

where $\varpi = P_{\bigcap_{i=1}^{n} F(S_i) \bigcap (G+H)^{-1}(0)}(\eta)$. Since the sequence $\{\varsigma_n\}$ is bounded in Σ, there exists a subsequence $\{\varsigma_{n_i}\}$ of $\{\varsigma_n\}$ such that $\varsigma_{n_i} \rightharpoonup \varsigma \in \Sigma$ and
$$\limsup_{n\to\infty} \langle \eta - \varpi, S_i(\varsigma_n) - \varpi\rangle = \lim_{n_i\to\infty} \langle \eta - \varpi, S_i(\varsigma_{n_i}) - \varpi\rangle.$$

Since $\|S_i(\varsigma_n) - \varsigma_n\| \to 0$, $\|S_i(\varsigma_{n_i}) - \varsigma_{n_i}\| \to 0$ and the family of mappings S_i is nonexpansive so $I - S_i : \Sigma \to \Sigma$ is the family demiclosed mappings, so we have $S_i(\varsigma) = \varsigma$, that is $\varsigma \in \bigcap_{i=1}^{n} F(S_i)$. Now we prove that
$$\varsigma \in (G+H)^{-1}(0).$$

Since the mapping $G : \Sigma \to \Sigma$ is Lipschitz continuous and $H : \Sigma \to 2^{\Sigma}$ is maximal monotone, using Lemma 1.8 the mapping $G+H$ is also maximal monotone. Now suppose $(p,q) \in Graph(G+H)$, that is $q \in (G+H)(p)$. Since $\varsigma_{n_i} = J_{\lambda}^{G+H}(\eta_{n_i})$, we have $\eta_{n_i} \in [I + (G+H)](\varsigma_{n_i})$, that is
$$\frac{1}{\lambda}(\eta_{n_i} - \varsigma_{n_i}) \in (G+H)(\varsigma_{n_i}).$$

Using the maximal monotonicity of mapping $(G+H)$, we have
$$\left\langle p - \varsigma_{n_i}, q - \frac{1}{\lambda}(\eta_{n_i} - \varsigma_{n_i}) \right\rangle \ge 0.$$

Hence
$$\langle p - \varsigma_{n_i}, q\rangle \ge \left\langle p - \varsigma_{n_i}, \frac{1}{\lambda}(\eta_{n_i} - \varsigma_{n_i}) \right\rangle.$$

Since, $\|\eta_{n_i} - \varsigma_{n_i}\| \to 0$ and $\varsigma_{n_i} \rightharpoonup \varsigma$, we get
$$\lim_{n_i\to\infty} \langle p - \varsigma_{n_i}, q\rangle = \langle p - \varsigma, q\rangle \ge 0.$$

Since the mapping $G+H$ is maximal monotone, this implies that $0 \in (G+H)(\varsigma)$, that is $\varsigma \in (G+H)^{-1}(0)$. So $\varsigma \in \bigcap_{i=1}^{n} F(S_i) \bigcap (G+H)^{-1}(0)$.

Since $\|S_i(\varsigma_n) - \varsigma_n\| \to 0$ and $\varsigma_{n_i} \rightharpoonup \varsigma \in \bigcap_{i=1}^{n} F(S_i) \bigcap (G+H)^{-1}(0)$. Now using Remark 1.7 we get
$$\begin{aligned}
\limsup_{n\to\infty} \langle \eta - \varpi, S_i(\varsigma_n) - \varpi\rangle &= \lim_{n_i\to\infty} \langle \eta - \varpi, S_i(\varsigma_{n_i}) - \varpi\rangle \\
&= \lim_{n_i\to\infty} \langle \eta - \varpi, S_i(\varsigma_{n_i}) - \varsigma_{n_i} + \varsigma_{n_i} - \varpi\rangle \\
&= \lim_{n_i\to\infty} \langle \eta - \varpi, \varsigma - \varpi\rangle \\
&\le 0.
\end{aligned}$$

Hence
$$\limsup_{n\to\infty} \langle \eta - \varpi, S_i(\varsigma_n) - \varpi\rangle \le 0. \tag{2.4}$$

Now finally we prove
$$\eta_n \to \varpi = P_{\bigcap_{i=1}^{n} F(S_i) \bigcap (G+H)^{-1}(0)}(\eta_0).$$

Now using Lemma 1.12 we get

$$\begin{aligned}\|\eta_{n+1}-\varpi\|^2 &= \|\Upsilon_n(\eta-\varpi)+(1-\Upsilon_n)(S_i(\varsigma_n)-\varpi)\|^2\\ &\le (1-\Upsilon_n)^2\|S_i(\varsigma_n)-\varpi\|^2+2\Upsilon_n\langle \eta-\varpi,\eta_{n+1}-\varpi\rangle\\ &\le (1-\Upsilon_n)^2\|\varsigma_n-\varpi\|^2+2\Upsilon_n\langle \eta-\varpi,\eta_{n+1}-\varpi\rangle\\ &\le (1-\Upsilon_n)^2\|\eta_n-\varpi\|^2+2\Upsilon_n\langle \eta-\varpi,\eta_{n+1}-\varpi\rangle.\end{aligned}$$

$$\|\eta_{n+1}-\varpi\|^2 \le (1-\Upsilon_n)^2\|\eta_n-\varpi\|^2+2\Upsilon_n\langle \eta-\varpi,\eta_{n+1}-\varpi\rangle. \tag{2.5}$$

Let

$$\Omega_n = \max\{0,\langle \eta-\varpi,\eta_{n+1}-\varpi\rangle\}.$$

Then $\Omega_n \ge 0$. Now we prove that $\lim_{n\to\infty} \Omega_n \to 0$. It follows from the equation (2.4) that for any given $\delta > 0$, there exists n_0 such that

$$\langle \eta-\varpi,\eta_{n+1}-\varpi\rangle < \delta.$$

So, we have

$$0 \le \Omega_n < \delta, \text{ for all } n \ge n_0.$$

Since $\delta > 0$ is arbitrary, we get $\Omega_n \to 0$. So we can write (2.5) as follows

$$\|\eta_{n+1}-\varpi\|^2 \le (1-\Upsilon_n)^2\|\eta_n-\varpi\|^2+2\Upsilon_n\Omega_n.$$

If we take $\tau_n = \|\eta_n-\varpi\|^2$, $\xi_n = 2\Upsilon_n\Omega_n$, and $\delta_n = 0$ then all the conditions of Lemma 1.11 are fulfilled. Hence $\eta_n \to \varpi$ when $n\to\infty$. Thus the proof is now completed. □

Corollary 2.2. *Suppose $G : \Sigma \to \Sigma$ a single valued Lipschitz continuous mapping, $H : \Sigma \to 2^\Sigma$ a multivalued maximal monotone mapping, and $S : \Sigma \to \Sigma$ be a nonexpansive mapping. Suppose $\Theta = F(S)\bigcap(G+H)^{-1} \neq \emptyset$. Suppose $\eta = \eta_0 \in \Sigma$ and the sequence $\{\eta_n\}$ generated by*

$$\begin{cases}\varsigma_n = J_\lambda^{(G+H)}(\eta_n),\\ \eta_{n+1} = \Upsilon_n\eta+(1-\Upsilon_n)S(\varsigma_n),\ \forall n\ge 0.\end{cases} \tag{2.6}$$

where $\lambda \in \mathbb{R}^+$ and $\forall n \in \mathbb{N}$, $\{\Upsilon_n\}$ is a given sequence having these conditions:

(1) $\Upsilon_n \to 0$, $\sum_{n=0}^{\infty}\Upsilon_n = \infty$,

(2) $\sum_{n=0}^{\infty}|\Upsilon_{n+1}-\Upsilon_n| < \infty$.

Then the sequence $\{\eta_n\}$ converges strongly to a point of $F(S)\bigcap(G+H)^{-1}(0)$.

3 Examples

Example 3.1. Let $\Sigma = \mathbb{R}^n$ and $S_i : \Sigma \to \Sigma$ is the family of nonexpansive mappings for $i = 1, 2$ defined by

$$S_1(\eta) = (\eta_0,-\eta_1,-\eta_2,\cdots,\eta_n)$$
$$S_2(\eta) = (\eta_0,0,0\cdots,0).$$

Here $\bigcap_{i=1}^{2} F(S_i) = (1,0,\cdots,0)$.

Now define $H : \Sigma \to 2^\Sigma$ as

$$H(\eta_0,\eta_1,\cdots,\eta_n) = \{(-\eta_0,\eta_1,\eta_2,\cdots,\eta_n)\},$$

and $G : \Sigma \to \Sigma$ as

$$G(\eta_0,\eta_1,\cdots,\eta_n) = (\eta_0(1+\ln(\eta_0)),\eta_1,\eta_2,\cdots,\eta_n).$$

Here H is a maximal monotone, multivalued mapping and G is a Lipschitz continuous single valued mapping and $(G+H)^{-1}(0) = (1, 0, \cdots, 0) \neq \emptyset$. If $\{\eta_n\}$ is the sequence generated by (2.1), then the sequence $\{\eta_n\}$ converges to $\bigcap_{i=1}^{2} F(S_i) \bigcap (G+H)^{-1}(0) = (1, 0, \cdots, 0)$.

Example 3.2. Suppose $\Sigma = \mathbb{R}$ and $H : \Sigma \to 2^{\Sigma}$ a multivalued maximal monotone mapping defined as

$$H(\eta) = \{2\eta\},$$

and $G : \Sigma \to \Sigma$ a single valued Lipschitz continuous mapping defined as

$$G(\eta) = \tanh(\eta).$$

Here $(G+H)^{-1}(0) = 0$. Suppose $S_i : \Sigma \to \Sigma$ is the family of nonexpansive mappings for $i = 1, 2, 3, \ldots n$ defined as

$$S_i(\eta) = \frac{\eta}{2^i},$$

then $\bigcap_{i=1}^{n} F(S_i) = 0$. If $\{\eta_n\}$ is the sequence generated by (2.1), then the sequence $\{\eta_n\}$ converges to $\bigcap_{i=1}^{n} F(S_i) \bigcap (G+H)^{-1}(0) = 0$.

4 Conclusion remarks

In this paper, we have presented an iterative algorithm for finding a common solution to the variational inclusion problem with Lipschitz continuous single valued and maximal monotone multivalued mappings and the set of fixed points of a finite family of nonexpansive mappings and proved a strong convergence theorem which converges to this common solution. We have also presented an example to verify our results.

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Author information

Prashant Patel, Department of Mathematics, School of Advanced Sciences, VIT-AP University
Amaravati, 522237, Andhra Pradesh, India.
E-mail: prashant.patel9999@gmail.com, prashant.p@vitap.ac.in (P. Patel)

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