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PROPERTY (UW_E) FOR FUNCTIONS OF OPERATORS

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Abstract In this paper, we study property (UW_E) for functions of operators on an infinite dimensional complex separable Hilbert space $\mathcal{H}$. We show that if $T + K$ satisfies property (UW_E) for all $K \in K(\mathcal{H})$, then so does $f(T) + K$ where f is an injective function that is analytic in the neighborhood of $\sigma(T)$. In addition, we study functions of Toeplitz operators with nonconstant continuous symbols satisfying property (UW_E).

1 Introduction and Preliminaries

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on infinite dimensional complex separable Hilbert space $\mathcal{H}$ and $K(\mathcal{H})$ denotes the ideal of all compact operators on $\mathcal{H}$. For an operator $T \in B(\mathcal{H})$, we denote T^*, $N(T)$, $R(T)$, $\alpha(T)$, and $\beta(T)$ for the adjoint, the kernel, the range, the nullity, and the defect respectively. If $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$) and $R(T)$ is closed, then T is called an upper semi-Fredholm (resp., lower semi-Fredholm) operator. Let $\Phi_+(\mathcal{H})$ denote the class of all upper semi-Fredholm operators and $\Phi_-(\mathcal{H})$ denote the class of all lower semi-Fredholm operators. An operator T is called semi-Fredholm, $T \in SF$, if $T \in \Phi_+(\mathcal{H}) \cup \Phi_-(\mathcal{H})$. Let $\Phi(\mathcal{H}) = \Phi_-(\mathcal{H}) \cap \Phi_+(\mathcal{H})$ denote the collection of all Fredholm operators. For $T \in SF$, the index of T denoted by $i(T) \in \mathbb{Z} \cup \{-\infty, \infty\}$ is defined as $i(T) = \alpha(T) - \beta(T)$. An operator $T \in B(\mathcal{H})$ is Weyl if it is Fredholm of index zero. Let $SF_+(\mathcal{H}) = \{T \in SF : i(T) \leq 0\}$ and $SF_-(\mathcal{H}) = \{T \in SF : i(T) \geq 0\}$ be the collection of all upper semi-Weyl operators and the collection of all lower semi-Weyl operators respectively.
Ascent of an operator T is denoted by $p := p(T)$ is the smallest non-negative integer such that $p = \min\{n \in \mathbb{N} \cup \{\infty\} : N(T^n) = N(T^{n+1})\}$. The descent of an operator T is denoted by $q := q(T)$ and is defined by $q = \min\{n \in \mathbb{N} \cup \{\infty\} : R(T^n) = R(T^{n+1})\}$. It is admitted that if ascent and descent are both finite, then $p(T) = q(T)$ [1]. If $p(T) = q(T) < \infty$, then the operator $T \in B(\mathcal{H})$ is said to be Drazin invertible. Let

$$DI(\mathcal{H}) = \{T \in B(\mathcal{H}) : T \text{ is Drazin invertible}\}.$$

Drazin invertible operators can be written as the direct sum of an invertible and a nilpotent operator. The essential spectrum $\sigma_e(T)$, the Wolf spectrum $\sigma_{lre}(T)$, the Weyl spectrum $\sigma_w(T)$, the upper semi-Weyl spectrum $\sigma_{uw}(T)$, the lower semi-Weyl spectrum $\sigma_{lw}(T)$, and the Drazin invertible spectrum $\sigma_{dz}(T)$ of $T \in B(\mathcal{H})$ are given by

$$\begin{aligned}
\sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(\mathcal{H})\}, \\
\sigma_{lre}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin SF(\mathcal{H})\}, \\
\sigma_w(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\},
\end{aligned}$$

$$\begin{aligned}\sigma_{uw}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+(\mathcal{H})\},\\ \sigma_{lw}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_-(\mathcal{H})\},\\ \sigma_{dz}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin DI(\mathcal{H})\}.\end{aligned}$$

Let $\rho_{SF}(T)$ stand for the semi-Fredholm domain of T and is given by $\rho_{SF}(T) = \mathbb{C} \setminus \sigma_{lre}(T)$ and let

$$\begin{aligned}\rho_{SF}^{+}(T) &:= \{\lambda \in \rho_{SF}(T) : i(T - \lambda I) > 0\},\\ \rho_{SF}^{-}(T) &:= \{\lambda \in \rho_{SF}(T) : i(T - \lambda I) < 0\},\\ \rho_{SF}^{0}(T) &:= \{\lambda \in \rho_{SF}(T) : i(T - \lambda I) = 0\}.\end{aligned}$$

Obviously, $\sigma_w(T) = \mathbb{C} \setminus \rho_{SF}^0(T)$ and $\rho_{SF}(T) = \rho_{SF}^0(T) \cup \rho_{SF}^+(T) \cup \rho_{SF}^-(T)$. Let $\emptyset \neq \sigma \subseteq \sigma(T)$, where σ is a clopen subset. Then there exists an Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$, where Ω is an analytic Cauchy domain. Then the operator $E(\sigma; T)$ defined by

$$E(\sigma; T) = \tfrac{1}{2\pi i} \int_\Gamma (T - \lambda I)^{-1} d\lambda,$$

where $\Gamma = \partial\Omega$ positively oriented with respect to Ω. $E(\sigma; T)$ commutes with every operator that commutes with T often called Riesz idempotent operator. Let $H(\sigma; T) = R(E(\sigma; T))$. We simply write $H(\lambda; T)$ instead of $H(\{\lambda\}; T)$. An isolated point λ of $\sigma(T)$ is called a normal eigenvalue of T if $\dim H(\lambda; T) < \infty$. The set of all normal eigenvalues of T will be denoted as $\sigma_0(T)$. We denote $E(T)$, the set of all isolated eigenvalues and $E_0(T)$ the set of all points in $E(T)$ with finite multiplicity.
Hermann Weyl [14] studied that the points λ in the spectrum of self-adjoint operator T on $\mathcal{H}$ but not in $\cap\{\sigma(T + K) : K \in K(\mathcal{H})\}$ are precisely the points in $E_0(T)$. Coburn [7] studied that this result remains holds for hyponormal operators. This remarkable result is known as Weyl's theorem. According to Coburn [7], an operator $T \in B(\mathcal{H})$ is said to satisfy Weyl's theorem, $T \in (W)$, if

$$\sigma(T) \setminus \sigma_w(T) = E_0(T).$$

Weyl-type theorems and its variations are studied for many classes of operators on Hilbert spaces and Banach spaces [2, 3, 4, 5, 6, 10, 11, 12, 13, 15, 16, 17]. Li, Zhu, and Feng [11] proved for any bounded linear operator T on $\mathcal{H}$ there is an arbitrary small compact perturbation of T satisfying Weyl's theorem. In [6], Berkani and Kachad introduced and studied several variants of Weyl's theorem such as property (UW_E), (W_π), (UW_{π_a}), and property (UW_π).
Operator $T \in B(\mathcal{H})$ is said to satisfy property (UW_E) if

$$\sigma_a(T) \setminus E(T) = \sigma_{uw}(T).$$

Berkani and Kachad [6] studied the connection between property (UW_E) and other variations of Weyl-type theorems. In [13] the authors studied stability of property (UW_E) under compact perturbation and proved that $T + K$ satisfies property (UW_E) for all $K \in K(\mathcal{H})$ if and only if isolated point of $\sigma_{lre}(T)$ is empty and $\sigma_{uw}(T)$ is simply connected. An operator $T \in B(\mathcal{H})$ is said to satisfy property (UW_π) [6] if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_{dz}(T).$$

Stablity of property (UW_π) under quasi-nilpotent and Riesz perturbations and the passage of property (UW_π) from T to $f(T)$ where f is an analytic function defined on the neighbourhood of $\sigma(T)$ is studied by Aiena and Kachad [3, 4].

2 Results

Aiena and Kachad [3] studied class of operators satisfying property (UW_π). In a similar manner, we study it for property (UW_E). If T is algebraic, then its spectrum is a finite set of poles and every pole is an eigenvalue. Then $\sigma_a(T) = \sigma(T) = E(T)$ and $\sigma_{uw}(T) \neq \emptyset$. Thus, every algebraic operator fails to satisfy property (UW_E). It is evident that every finite-dimensional operator does not satisfy property (UW_E). Also, quasi-nilpotent operators with empty point spectrum satisfies property (UW_E). Let T be a non-injective operator and N be a nilpotent operator with $TN = NT$. If property (UW_E) is satisfied for T, $T+N$ satisfies property (UW_E). In [3], Aiena and Kachad proved that $\sigma(T)$ and $\sigma_a(T)$ are invariant under commuting quasi-nilpotent perturbations and $\sigma_p(T+N) = \sigma_p(T)$. Hence $E(T+N) = E(T)$. Therefore, $T+N$ satisfies property (UW_E). In [16], Zhu, Li, and Zhou are given the necessary and sufficient condition for functions of operators satisfying property (w) and a-Weyl's theorem. Now we study stability of property (UW_E) for functions of operators. We start with the following result from [13].

Lemma 2.1 ([13], Theorem 2.9). *Let $T \in B(\mathcal{H})$. Then, Property (UW_E) hold for T if and only if*
(i) $\sigma(T)$ is the union of $\sigma_w(T)$ and $\sigma_0(T)$,
(ii) all isolated eigenvalues are normal eigenvalues, and
(iii) there exist no $\lambda \in \sigma_p(T)$ such that $T - \lambda I$ is semi-Fredholm with $i(T) < 0$.

Let iso σ denotes the collection of points in σ that are not accumulation points.

Lemma 2.2 ([8], Chapter XI, Proposition 6.9). *Let $T \in B(\mathcal{H})$ and $\lambda \in$ iso $\sigma(T)$. Then, the following statements are equivalent.*
(i) $\lambda \in \rho_{SF}(T)$.
(ii) $\lambda \in \rho^0_{SF}(T)$.
(iii) $\lambda \in \sigma_0(T)$.

The upper semi-Weyl spectrum does not hold the spectral mapping theorem. However, if f is injective, we have the following result due to Aiena [5].

Lemma 2.3 ([5], Lemma 2.5). *Let $T \in B(\mathcal{H})$ and suppose that f is injective on $\sigma(T)$. Then,*

$$\sigma_{uw}(f(T)) = f(\sigma_{uw}(T)) \text{ and } \sigma_{lw}(f(T)) = f(\sigma_{lw}(T)).$$

Let $Hol'(\sigma(T))$ denotes the collection of all analytic functions that are injective on any connected neighborhood of $\sigma(T)$. An operator $T \in B(\mathcal{H})$ is said to be isoloid if the isolated points of the spectrum are all eigenvalues.

Theorem 2.4. *Let $T \in B(\mathcal{H})$, Then $f(T) \in (UW_E)$ for all $f \in Hol'(\sigma(T))$ if and only if the following conditions hold*
(i) $T \in (UW_E)$;
(ii) If $\rho^-_{SF}(T) \neq \emptyset$, then $\sigma_0(T) = \emptyset$ and there exist no $\lambda \in \rho_{SF}(T)$ such that $0 < i(T-\lambda I) < \infty$;
(iii) If $E(T) \neq \emptyset$, then T is isoloid.

Proof. Assume that $f(T) \in (UW_E)$ for all $f \in Hol'(\sigma(T))$. If $f(\lambda) = \lambda$, then (i) follows. If (ii) does not hold, then we can choose an $\alpha \in \rho^-_{SF}(T)$ and $\beta \in \rho_{SF}(T) \cap \sigma_p(T)$ such that $0 \leq i(T - \beta I) < \infty$. We can choose $k \in \mathbb{N}$ such that $k.i(T - \alpha I) + i(T - \beta I) < 0$. Set $f_2(\lambda) = (\lambda - \alpha)^k(\lambda - \beta)$. Then, $f_2(T) = (T - \alpha I)^k(T - \beta I) \in SF$ and $i(f_2(T)) < 0$. Clearly, $N(f_2(T)) \geq N(T - \beta I) > 0$. Thus, $0 \in \rho^-_{SF}(f_2(T)) \cap \sigma_P(f_2(T))$. By Lemma 2.1, $f_2(T) \notin (UW_E)$, a contradiction. Thus (ii) holds. Suppose (iii) does not hold, then there exist a $\lambda_1 \in E(T)$ and $\lambda_2 \in$ iso $\sigma(T)$ such that $\lambda_2 \notin \sigma_p(T)$. Then $\lambda_2 \in \sigma_{lre}(T)$, by Lemma 2.2. Put $f_3(T) = (T - \lambda_1 I)(T - \lambda_2 I)$ and so $0 \in \sigma_{lre}(f_3(T))$ and $N(f_3(T)) = N(T - \lambda_1 I) > 0$. Since $\lambda_1, \lambda_2 \in$ iso $\sigma(T)$, $0 \in$ iso $\sigma(f_3(T))$. Hence $0 \in E(f_3(T))$. Then by Lemma 2.1, it follows that $f_3(T)$ does not satisfy property (UW_E), a contradiction. Conversely, assume that (i),(ii) and (iii) holds. By a similar argument as in Theorem 1.2 [16], we get $\sigma_a(f(T)) \setminus \sigma_{uw}(f(T)) \subseteq E(f(T))$. Now we prove $E(f(T)) \subseteq \sigma_a(f(T)) \setminus \sigma_{uw}(f(T))$. Let $\lambda \in E(f(T))$. Then

$$f(T) - \lambda I = (T - \lambda_1)^{k_1}(T - \lambda_2)^{k_2}....(T - \lambda_n)^{k_n} g(T),$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$ and $g(T)$ is invertible. Since $\lambda \in E(f(T))$, $\lambda \in \sigma(f(T))$. Also, $\lambda_i \in \sigma(T)$ for all i and there exist an i_0 such that $\lambda_{i_0} \in \sigma_p(T)$. Hence $\lambda_{i_0} \in E(T)$. By (iii), $\lambda_i \in E(T)$ for all i. Since $T \in (UW_E)$, we have $\lambda_i \notin \sigma_{uw}(T)$ for all i, which implies $f(\lambda_i) \neq \lambda$ for all i. Since $f \in Hol'(\sigma(T))$, $\lambda \notin f(\sigma_{uw}(T))$. Hence $\lambda \notin \sigma_{uw}(f(T))$. This gives $\lambda \in \sigma_a(f(T)) \setminus \sigma_{uw}(f(T))$.

□

In [13], authors studied the stability of property (UW_E) under compact perturbation.

Theorem 2.5. *If $T \in B(\mathcal{H})$ and $T + K \in (UW_E)$ for all $K \in K(\mathcal{H})$, then $f(T) + K \in (UW_E)$ for all $K \in K(\mathcal{H})$ and for every $f \in Hol'(\sigma(T))$.*

Proof. By [13, Theorem 2.6], it is enough to prove $\mathbb{C}\backslash\sigma_{uw}(f(T))$ is connected and iso $\sigma_{lre}(f(T)) = \emptyset$. Suppose that $\mu_0 \in$ iso $\sigma_{lre}(f(T))$ and

$$f(T) - \mu_0 I = (T - \lambda_1)^{n_1}(T - \lambda_2)^{n_2}.....(T - \lambda_k)^{n_k} g(T),$$

where $g(T)$ is invertible and $\lambda_i \neq \lambda_j$ for $i \neq j$. If $\lambda_1^{(n)} \to \lambda_1$, then $f(\lambda_1^{(n)}) \to \mu_0$. Then,

$$f(T) - f(\lambda_1^{(n)})I = (T - \lambda_1^{(n)})^m (T - \lambda_1')^{m_1}(T - \lambda_2')^{m_2}.....(T - \lambda_n')^{m_n} h(T),$$

where $h(T)$ is invertible and $\lambda_i' \neq \lambda_j'$ for $i \neq j$. Since $f(T) - f(\lambda_1^{(n)})I \in SF$, $\lambda_1 \in$ iso $\sigma_{lre}(T) \cup \rho_{SF}(T)$. Since iso $\sigma_{lre}(T) = \emptyset$, $\lambda_1 \in \rho_{SF}(T)$. In a similar manner, we can show that $\lambda_i \in \rho_{SF}(T)$ for all i. This implies that $f(T) - \mu_0 I$ is semi-Fredholm, which is a contradiction. Therefore, iso $\sigma_{lre}(f(T)) = \emptyset$. Since f is injective, we have $\rho_{uw}(f(T))$ is connected by [5, Theorem 3.3]. □

The following example shows that Theorem 2.5 may not hold for non-injective functions.

Example 2.6. Let $T : l^2 \to l^2$ given by $T(a_1, a_2, a_3, ...) := (a_2, a_3, a_4, ...)$. We have

$$\sigma_a(T) = \sigma_{uw}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \text{ and } \sigma_{lre}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Hence, $T \in (UW_E)$ for all $K \in K(\mathcal{H})$ by [13, Theorem 2.6]. But if we take f as the zero function on the closed unit disc, then $f(T)$ is the zero operator. This implies that $\sigma_a(f(T)) \setminus \sigma_{uw}(f(T)) = \emptyset$ and $E(f(T)) = \{0\}$. Consequently $f(T)$ does not satisfy property (UW_E).

Yang and Cao [15] studied property (UW_π) for functions of operators. Now we prove that if $T + K$ satisfies property (UW_π) for all $K \in K(\mathcal{H})$, then $f(T) + K$ satisfies property (UW_π) for all $K \in K(\mathcal{H})$ and for every $f \in Hol'(\sigma(T))$.

Lemma 2.7 ([15], Theorem 1.5). *Let $T \in B(\mathcal{H})$. Then, $T + K$ satisfies property (UW_π) and is isoloid for all $K \in K(\mathcal{H})$ if and only if*
(i) iso $\sigma_w(T) = \emptyset$
(ii) $\sigma_{uw}(T)$ is simply connected.

Theorem 2.8. *Let $T \in B(\mathcal{H})$. If $T + K \in (UW_\pi)$ and is isoloid for all $K \in K(\mathcal{H})$, then so does $f(T) + K \in (UW_\pi)$ and is isoloid for all $K \in K(\mathcal{H})$ and for every $f \in Hol'(\sigma(T))$.*

Proof. Assume that $T + K \in (UW_\pi)$ and is isoloid for all $K \in K(\mathcal{H})$. By Lemma 2.7, it is enough to prove $\mathbb{C} \setminus \sigma_{uw}(f(T))$ is connected and iso $\sigma_w(f(T)) = \emptyset$. If iso $\sigma_w(f(T)) \neq \emptyset$, then there exist a $\mu_0 \in$ iso $\sigma_w(f(T))$ such that

$$f(T) - \mu_0 I = (T - \lambda_1)^{n_1}(T - \lambda_2)^{n_2}.....(T - \lambda_k)^{n_k} g(T) \ ,$$

where $g(T)$ is invertible and $\lambda_i \neq \lambda_j$ for $i \neq j$. If $\lambda \to \lambda_1$, then $f(\lambda) \to \mu_0$. Then,

$$f(T) - f(\lambda)I = (T - \lambda)^m (T - \lambda_1')^{m_1}(T - \lambda_2')^{m_2}.....(T - \lambda_n')^{m_n} h(T),$$

where $h(T)$ is invertible and $\lambda_i' \neq \lambda_j'$ for $i \neq j$. Since $f(T) - f(\lambda)I$ is Weyl, $\lambda_1 \in$ iso $\sigma_w(T) \cup \rho_{SF}(T)$. Since iso $\sigma_w(T) = \emptyset$, $\lambda_1 \in \rho_{SF}(T)$. By a similar argument, we get $\lambda_i \in \rho_{SF}(T)$ for all i. Hence $f(T) - \mu_0 I$ is Weyl, a contradiction. Therefore, iso $\sigma_w(f(T)) = \emptyset$. By the same argument as in Theorem 2.5, we get $\rho_{uw}(f(T))$ is connected. This completes the proof. □

Hardy space of the unit circle $\mathbb{T}$ of the complex plane $\mathbb{C}$ is denoted by $H^2(\mathbb{T})$. Let $L^\infty(\mathbb{T})$ denote the set of all measurable functions that are essentially bounded on $\mathbb{T}$. Toeplitz operator with symbol $\phi \in L^\infty(\mathbb{T})$ on $H^2(\mathbb{T})$ is given by $T_\phi(f) = P(\phi f)$, where $f \in H^2(\mathbb{T})$ and P is the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. We denote $\mathbb{C}(\mathbb{T})$, the algebra of all continuous complex valued functions on $\mathbb{T}$. Aiena [5] studied some properties of spectra of Toeplitz operators. For $K \in K(\mathcal{H})$, $T_\phi + K$ satisfies a-Weyl's theorem if and only if ϕ is not constant and

the winding number of ϕ with respect to each hole of Γ is negative, where $\Gamma = \phi(\mathbb{T})$ [9]. If the winding number of ϕ with respect to each hole of Γ is negative and ϕ is not constant, then we have iso $\sigma_{lre}(T_\phi) = \emptyset$ and $\mathbb{C} \setminus \sigma_{uw}(T_\phi)$ is connected. Then, by [13, Theorem 2.6] property (UW_E) is invariant under compact perturbation of Toeplitz operators with nonconstant continuous symbol ϕ of winding number with respect to each hole of Γ is negative.
The following result shows that for continuous nonconstant symbol ϕ, the wolf spectrum of T_ϕ has no isolated points.

Lemma 2.9. *If $\phi \in C(\mathbb{T})$. Then the following are equivalent.*
(i) ϕ is nonconstant,
(ii) iso $\sigma_{uw}(T_\phi) = \emptyset$,
(iii) iso $\sigma_w(T_\phi) = \emptyset$, *and*
(iv) iso $\sigma_{lre}(T_\phi) = \emptyset$.

Proof. If ϕ is constant, then $\sigma_{lre}(T_\phi) = \{\mu\}$. Therefore, iso $\sigma_{lre}(T_\phi) = \emptyset$. Thus, ϕ is nonconstant. If iso $\sigma_{lre}(T_\phi) \neq \emptyset$, then Γ is singleton since the essential spectrum is connected. Therefore, ϕ is constant. Equivalency of (i), (ii), (iii) follows from [5, Theorem 3.5]. □

Theorem 2.10. *Let $\phi \in C(\mathbb{T})$ be nonconstant. Then both T_ϕ and T_ϕ^* satisfy property (UW_E).*

Proof. From [5, Theorem 3.3] and Lemma 2.9, we have iso $\sigma_a(T_\phi) = \emptyset$. Thus, $E(T_\phi) = \emptyset$. Since $\sigma_a(T_\phi) = \sigma_{uw}(T_\phi)$, T_ϕ satisfies property (UW_E). We have iso $\sigma_a(T_\phi^*) =$ iso $\sigma_{lw}(T_\phi) = \emptyset$ and so $E(T_\phi^*) = \emptyset$, $\sigma_a(T_\phi^*) = \sigma_{uw}(T_\phi^*)$. Thus, T_ϕ^* satisfies property (UW_E). □

Let $H_{nc}(\sigma(T_\phi))$ denotes the collection of analytic functions that are nonconstant on connected components of $\sigma(T)$.

Theorem 2.11. *Let $\phi \in C(\mathbb{T})$ and $f \in H_{nc}(\sigma(T_\phi))$. Then the following are equivalent.*
(i) $\sigma_a(T_{f\circ\phi}) = f(\sigma_a(T_\phi))$.
(ii) $\sigma_{uw}(T_{f\circ\phi}) = f(\sigma_{uw}(T_\phi))$, and
(iii) $f(T_\phi)$ has property (UW_E).

Proof. We see that (i) and (ii) are equivalent from [5, Theorem 4.5]. Assume (i) hold

$$\text{iso } \sigma_a(f(T_\phi)) = f(\text{iso } \sigma_a(T_\phi)) = \emptyset.$$

This shows that $E(f(T_\phi)) = \emptyset$. We have,

$$\sigma_a(T_{f\circ\phi}) = f(\sigma_a(T_\phi)) = \sigma_a(f(T_\phi)) = \sigma_{uw}(f(T_\phi)).$$

Therefore, $f(T_\phi)$ satisfies property (UW_E). That is, (iii) holds.
Now assume (iii). Clearly $E(f(T_\phi)) = \emptyset$. This shows that

$$\sigma_a(f(T_\phi)) = \sigma_{uw}(f(T_\phi)) = \sigma_a(T_{f\circ\phi}).$$

This implies $\sigma_a(T_{f\circ\phi}) = f(\sigma_a(T_\phi))$ and so (i) holds. □

Corollary 2.12. *Let $\phi \in C(\mathbb{T})$ be nonconstant and $f \in H_{nc}(\sigma(T_\phi))$. If $\sigma_a(T_{f\circ\phi})$ has no holes, then $f(T_\phi)$ satisfies property (UW_E).*

Proof. We have $f(T_\phi) = T_{f\circ\phi} + K$, where K is a compact operator. Since $f \circ \phi$ is nonconstant, we have

$$\emptyset = \text{iso } \sigma_{lre}(T_{f\circ\phi}) = \text{iso } \sigma_{uw}(T_{f\circ\phi}) = \text{iso } \sigma_a(T_{f\circ\phi}).$$

Then, $f(T_\phi)$ satisfies property (UW_E). □

Let $\{b_1, b_2, b_3, ...\}$ be an orthonormal basis of $\mathcal{H}$. Then the unilateral weighted shift operator T on $\mathcal{H}$ is given by

$$Tb_i = w_i b_{i+1},$$

where $\{w_i\}_{i=1}^{\infty}$ is a sequence of complex numbers.

Corollary 2.13. *Let T be a unilateral weighted shift operator with weights $\{w_i\}_{i=1}^{\infty}$. Then $T+K$ satisfies property (UW_E) for all $K \in K(\mathcal{H})$ if and only if*

$$\liminf_i |w_i| = 0 < r(T),$$

where $r(T)$ denotes the spectral radius of T.

Proof. Suppose that T is a unilateral weighted shift operator with weights $\{w_i\}_{i=1}^{\infty}$. Then by [10], we have

$$\sigma(T) = \sigma_w(T) = \{z \in \mathbb{C} : |z| \leq r(T)\}.$$

By triangulability of T^*, we have $\rho_{SF}^-(T^*) = \emptyset$. Since $\rho_{SF}^-(T^*) = \rho_{SF}^+(T)$, $\rho_{SF}^+(T) = \emptyset$. We have $\rho_{SF}^-(T) = \emptyset$ if and only if $\liminf_i |w_i| = 0$. Therefore, $\sigma_{uw}(T) = \sigma_{lre}(T) = \sigma_w(T)$. Thus iso $\sigma_{lre}(T) = \emptyset$ and $\sigma_{uw}(T)$ is simply connected if and only if $\liminf_i |w_i| = 0 < r(T)$. □

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