**Palestine Journal of Mathematics**

**Vol 13(4)(2024)** , 353–381 © Palestine Polytechnic University-PPU 2024

# NUMERICAL ANALYSIS OF A SEMI-IMPLICIT FRACTIONAL FEM-SCHEME FOR A NEW MODEL OF FLUID FLOW WITH FRACTIONAL TIME-DERIVATIVE

S. Azoug, A. Ouardani and T. Aliziane

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 65M12; Secondary 65M22.

Keywords and phrases : Semi-implicit scheme, Non-Newtonian fluid, Kelvin-Voigt, Polymer, Fractional time-derivative.

**Abstract** The main aim of this paper is the numerical study of a generalized model of fractional polymer aqueous solutions and Navier-Stokes equations using a fully discrete fractional semi-implicit FEM scheme which is developed to study this type of equations. The studied model describes an incompressible fluid flow that takes into account the relaxation properties with a fractional time derivative of Caputo. The Existence and uniqueness results were obtained for the weak discrete solution and with the help of a newly introduced trilinear form. The convergence and stability of the developed numerical scheme are demonstrated for certain criteria and time step limits are obtained. Numerical simulations are developed and the effects of various parameters of the discrete system are studied. The results obtained are analysed and the application to the lid-driven cavity problem in a complex geometry is presented.

## 1 Introduction

### 1.1 Mathematical formulation & main results

Many real fluids, such as blood, cornstarch, paint, and aqueous polymer solutions, cannot be described by the Navier-Stokes equations and are known as non-Newtonian fluids. In this work, we numerically study an initial-boundary value problem that models the unsteady flow of a new type of fluid model that generalizes the fractional Navier-Stokes equations by using Caputo fractional temporal derivatives, defined as follows :
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with regular boundary $\partial\Omega$. We denote $\Omega_T$ the cylinder $\Omega \times (0,T)$ $(T > 0)$. The studied model is giving as following

$$(\mathbf{P}) \begin{cases} D_t^{\beta}(\mathbf{u} - \alpha_1 \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + \mathbf{u} \nabla (\mathbf{u} - \alpha_2 \Delta \mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega_T, \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega_T, \\ \mathbf{u} = 0 & \text{on } \partial\Omega_T, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

- $t \in (0,T)$ denotes time and $x \in \Omega$ denotes space.
- $\nu > 0$ is the viscosity coefficient, $\alpha_1 \geq 0$ is the relaxation times and $\alpha_2 \geq 0$ is the relaxation viscosity.
- $\mathbf{u} = (u_1, u_2)$ represents the velocity field.
- $p$ is the pressure.
- $\mathbf{f}$ is the external forces field.

$\nabla$ denotes the gradient with respect to the space variables $x$ such as $\nabla. = \left(\frac{\partial .}{\partial x_1}, \frac{\partial .}{\partial x_2}\right)$

- div is the divergence operator such as $\text{div}. = \frac{\partial .}{\partial x_1} + \frac{\partial .}{\partial x_2}$
- $\Delta$ is the laplacien operator such as $\Delta. = \frac{\partial^2 .}{\partial x_1^2} + \frac{\partial^2 .}{\partial x_2^2}$

• $D_t^{\beta}(.)$ denotes the VO (in time, t) fractional time-derivatives in Caputo's sense, which is defined by Coimbra in [1] by :

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\beta}} d\tau, \quad 0 < \beta < 1, \tag{1.2}$$

where $\Gamma(.)$ is the gamma function. Using the fractional derivative instead the ordinary one is to describe the rate of change of a function with respect to a non-integer power of time. It is particularly useful in modelling fluids dynamic with a complex structure such as viscoelasticity, it allows us to understand how a system evolves over time in more nuanced way than what traditional calculus can offer.
If $\beta = 1$ and $\alpha_1 = \alpha_2$, this CO(constant order) fractional derivatives is reduced to the ordinary time-derivative because $D_t^1 f = \frac{df}{dt}$. The problem (**P**) will be written as follow

$$(\mathbf{ASP}) \begin{cases} \frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathbf{u}\nabla(\mathbf{u} - \alpha\Delta\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \mathit{\Omega}_T, \\ \text{div}\,(\mathbf{u}) = 0 & \text{in } \Omega_T, \\ \mathbf{u} = 0 & \text{on } \partial\Omega_T, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \mathit{\Omega}. \end{cases} \tag{1.3}$$

. This model has been experimentally established by L.I Sedov in (1966)[2] and generously studied by A.P.Oskolkov(1977)[3], in a series of papers, see ([4],[5],[6],[7]), and by Azoug et al.(2018)[8]. In this articles, A.P.Oskolkov, has been interested to the uniqueness and global solvability of different models of aqueous solutions of polymers. The stationary model was studied by C. Amrouche and EH. Ouazar in(2008)[9] where they proved the existence of a week solution in two dimension. In (2022)[10], they studied the same model with a damping term in a bounded domain, they proved that the solution of the problem (NSV) converges to a solution of the steady-state damped Navier-Stokes system as $\alpha_2$ tends to zero.
When $\alpha_1 = \alpha_2 = 0$, the system (**P**) is reduced to the fractional Navier-Stokes equations.

$$(\mathbf{FNSE}) \begin{cases} D_t^{\beta}(\mathbf{u}) - \nu\Delta\mathbf{u} + \mathbf{u}\nabla(\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \mathit{\Omega} \times (0,T), \\ \text{div}\,(\mathbf{u}) = 0 & \text{in } \mathit{\Omega} \times (0,T), \\ \mathbf{u} = 0 & \text{on } \partial\mathit{\Omega} \times (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \mathit{\Omega}. \end{cases} \tag{1.4}$$

Lions(1959), in [11] has obtained priori estimates for fractional order derivative in time. Also, for the numerical study we can cite the article [12](2018) by J. Zhang and J Wang, where they used a finite difference approach in fractional derivative and Legendre-spectral method approximations in space and they have concluded that the scheme is unconditionally stable. In (2014)[13]

they studied the Navier-Stokes equations with a time-fractional derivative in a tube, using the coupling of Adomian decomposition method and Laplace transform method. using the technique of a domain decomposition and the Laplace transform method, for some related study see [14],[15],[16],[17], [18].
In the case where $\alpha_2 = 0$, the problem is reduced to a system of partial differential equations introduced by W. Voigt (1892)[19] which is called the Kelvin-Voigt equations describing the motion of certain viscoelastic incompressible fluids. This model has been studied by many authors, in (2016)[20] they studied the well-posedness of the system on $L^q(\Omega)$ were $\Omega$ is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. Also, Baranovskii (2020)[21] has been studied the existence of the strong solution by using Faedo-Galerkin method with a special basis, where $\alpha_1 > 0$ a length scale parameter characterizing the elasticity of the fluid.

$$(\mathbf{FKV})\begin{cases} D_t^{\beta}(\mathbf{u} - \alpha_1\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathbf{u}\nabla(\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega \times (0,T), \\ \text{div}\,(\mathbf{u}) = 0 & \text{in } \Omega \times (0,T), \\ \mathbf{u} = 0 & \text{on } \partial\Omega \times (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

The system reduces to the aqueous solution of polymer when $\alpha_1 = \alpha_2$, we get the following system

$$(\mathbf{FASP})\begin{cases} D_t^{\beta}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathbf{u}\nabla(\mathbf{u} - \alpha\Delta\mathbf{u}) + \nabla p = \mathbf{f} & \Omega \times (0,T), \\ \text{div}\,(\mathbf{u}) = 0 & \Omega \times (0,T), \\ \mathbf{u} = 0 & \partial\Omega \times (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0 \in \Omega. \end{cases} \tag{1.6}$$

The model (**FASP**) was investigated by Azoug et al(2020)[22]. In this paper they presented a fully discrete fractional semi-implicit FEM-scheme to study numerically the model which describes the non-Newtonian fluid flow of polymer aqueous solutions.
Let the final time $T > 0$ be fixed. Let us introduce the following spaces used in the paper :
We use the standard notation for the Lebesgue spaces $L^p(\Omega)$ with the norm $\|.\|_{L^p(\Omega)}$, if $p = 2$ we denote its usual norm by $|.|_0$.
We introduce also the Sobolev spaces

$$H^m(\Omega) \underset{=}{\text{def}} W^{m,2}(\Omega), \quad m \in \mathbb{N}.$$

equipped by the semi-norms and norms $|.|_m$ $\|.\|_m$.
Set

$$X = (H^3(\Omega))^2 \cap (H_0^1(\Omega))^2 \text{ and } Y = L_0^2(\Omega)$$

Let $V$ an Hilbert space defined by

$$V = \left\{ \mathbf{v} \in (H_0^1(\Omega))^2,\ \Delta\mathbf{v} \in (L^2(\Omega))^2,\ div(\mathbf{v}) = 0 \text{ p.p in } \Omega,\ \mathbf{v} = 0 \text{ on } \partial\Omega \right\} \tag{1.7}$$

equipped with the scalar product

$$((u|v))_V = (\nabla\mathbf{u}, \nabla\mathbf{v})_{L^2(\Omega)} + (\Delta\mathbf{u}, \Delta\mathbf{v})_{L^2(\Omega)}$$

The bilinear forms a(.,.) , b(.,.) in X are defined by :

$$a(\mathbf{u},\mathbf{v}) = \nu \int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}dx;\ \mathbf{u},\mathbf{v} \in X, \tag{1.8}$$

$$b(\mathbf{u},q) = -\int_\Omega q \text{ div}(\mathbf{u})dx;\ \mathbf{u} \in X,\ q \in Y, \tag{1.9}$$

We also introduce the following trilinear form in X by :

$$C(\mathbf{w},\mathbf{z},\mathbf{v}) = \sum_{i,j=1}^{N} \int_\Omega w_i \frac{\partial z_j}{\partial x_i} v_j dx;\ \mathbf{u},\mathbf{v},\mathbf{w} \in X. \tag{1.10}$$

For the sake of simplicity, let us assume that **f** is independent on time and **u** is zero at the edges, the domain $\Omega \in \mathbb{R}^2$ is assumed polygonal and convex. We denote by $X_h \subset X$ and $Y_h \subset Y$ sub finite dimensional spaces and are such that the form b satisfies the condition inf-sup discrete. In the next steps a semi implicit scheme in time will be presented and the stability and convergence will be studied. We denote $\Delta t > 0$ the time step discretization with $\Delta t = T/n, n \in \mathbb{N}^*$ and we consider $\mathbf{u}_h^k(x)$ the approximation of $\mathbf{u}$, $\mathbf{u}_h^k(x) \in X_h$ with $t^k = k.\Delta t$, $k \geqslant 0$ and $p_h^k(x) \in Y_h$ is the approximation of $p(x, t^k)$.

### 1.2 Modified trilinear form

For stability reasons, the modified trilinear form $\tilde{C}(.,.,.)$ is introduced as follow, see Azoug et al. [8], [22]

**Definition 1.1.**

$$\tilde{C}(\mathbf{w},\mathbf{z},\mathbf{v}) = \frac{1}{2}[C(\mathbf{w},\mathbf{z},\mathbf{v})-C(\mathbf{w},\mathbf{v},\mathbf{z})]-\frac{\alpha_2}{2}[C(\mathbf{w},\Delta\mathbf{z},\mathbf{v})-C(\mathbf{w},\mathbf{v},\Delta\mathbf{z})-C(\mathbf{w},\Delta\mathbf{z},\mathbf{z})+C(\mathbf{w},\mathbf{z},\Delta\mathbf{z})] \tag{1.11}$$

The trilinear forms C and $\tilde{C}(.,.,.)$ have some properties, hereafter some ones that will be used in the next sections.

**Properties 1.1.** *For all* $\boldsymbol{w}$, $\boldsymbol{z}$ *in X and* $\boldsymbol{v}$ *in V we have*

1. $\tilde{C}(\boldsymbol{w},\boldsymbol{w},\boldsymbol{v}) = C(\boldsymbol{w},\boldsymbol{w},\boldsymbol{v})$,
2. $\tilde{C}(\boldsymbol{w},\boldsymbol{z},\boldsymbol{z}) = 0$,
3. $\tilde{C}(\boldsymbol{w},\boldsymbol{z},\boldsymbol{v}) = \frac{1}{2}[C(\boldsymbol{w},\boldsymbol{z},\boldsymbol{v}) - C(\boldsymbol{w},\boldsymbol{v},\boldsymbol{z})]$, *for* $\alpha_2 = 0$.

PROOF. see Azoug et al. [8], [22]

### Estimation of C, $\tilde{\mathbf{C}}$

In this subsection, the obtained and proved results given in Azoug et al.(2020)[22] for the estimation of C(., ., .) are recalled. The first and the second estimations of $\tilde{C}(.,.,.)$ are gathered in the following lemmas :

**Lemma 1.2.** *The form C(., ., .) is trilinear continuous on* $[H^1(\Omega)]^3$ *and*

$$C(\boldsymbol{w},\boldsymbol{z},\boldsymbol{v}) \leqslant c\|\boldsymbol{w}\|_0^{1/2}\|\boldsymbol{w}\|_1^{1/2} \mid \boldsymbol{z}\mid_1 \|\boldsymbol{v}\|_0^{1/2}\|\boldsymbol{v}\|_1^{1/2}. \tag{1.12}$$

**Lemma 1.3.** $\tilde{C}(.,.,.)$ *verifies :*

$$\tilde{C}(\boldsymbol{w},\boldsymbol{z},\boldsymbol{v}) \leqslant c \mid \boldsymbol{w}\mid_1 [\mid \boldsymbol{v}\mid_1\mid \boldsymbol{z}\mid_1 + \alpha_2 \mid \Delta\boldsymbol{z}\mid_1 (\mid \boldsymbol{v}\mid_1 + \mid \boldsymbol{z}\mid_1)], \tag{1.13}$$

*for all* $\boldsymbol{w},\boldsymbol{v} \in [H^1(\Omega)]^2$ *and* $\boldsymbol{z} \in [H^3(\Omega)]^2$.
*Particularly, for* $\alpha = 0$, *we have*

$$\tilde{C}(\boldsymbol{w},\boldsymbol{z},\boldsymbol{v}) \leqslant c \mid \boldsymbol{w}\mid_1\mid \boldsymbol{v}\mid_1\mid \boldsymbol{z}\mid_1, \tag{1.14}$$

*for all* $\boldsymbol{w}$, $\boldsymbol{z}$, $\boldsymbol{v} \in [H^1(\Omega)]^2$.

**Lemma 1.4.** $\tilde{C}(.,.,.)$ *verifies :*

$$\tilde{C}(\boldsymbol{w},\boldsymbol{z},\boldsymbol{v}) \leqslant c\|\boldsymbol{w}\|_0 \left[\mid \boldsymbol{v}\mid_1 (\|\boldsymbol{z}\|_{W^{1,\infty}(\Omega)} + \alpha_2\|\Delta\boldsymbol{z}\|_{W^{1,\infty}(\Omega)}) + \alpha_2\|\boldsymbol{z}\|_{W^{1,\infty}}\|\Delta\boldsymbol{z}\|_{W^{1,\infty}(\Omega)}\right], \tag{1.15}$$

*for all* $\boldsymbol{z}$, $\Delta\boldsymbol{z} \in [W^{1,\infty}(\Omega)]^2$, $\boldsymbol{v} \in [H^1(\Omega)]^2, \boldsymbol{w} \in [L^2(\Omega)]^2$.
*Particularly, for* $\alpha_2 = 0$, *we have*

$$\tilde{C}(\boldsymbol{w},\boldsymbol{z},\boldsymbol{v}) \leqslant c\|\boldsymbol{w}\|_0 \mid \boldsymbol{v}\mid_1 \|\boldsymbol{z}\|_{W^{1,\infty}(\Omega)}, \tag{1.16}$$

*for all* $\boldsymbol{z} \in [W^{1,\infty}(\Omega)]^2$, $\boldsymbol{v} \in [H^1(\Omega)]^2, \boldsymbol{w} \in [L^2(\Omega)]^2$.

PROOF. see Azoug et al. [22]

### 1.3 Energy estimate

For the energy estimate, let us take **v** = **u** in the weak formulation and thanks to the $\tilde{C}(.,.,.)$ properties one gets the following :

$$\frac{1}{2}D_t^{\beta}\|\mathbf{u(t)}\|_0^2 + \frac{\alpha_1}{2}D_t^{\beta}\|\nabla\mathbf{u(t)}\|_0^2 + \nu\|\mathbf{u(t)}\|_0^2 \leqslant \|\mathbf{f}\|_0\|\mathbf{u(t)}\|_0. \tag{1.17}$$

Multiplying by 2 and using the propriety of Caputo's fractional derivative defined by (see Coimbra (2003))[1]

$$D_t^{-\beta}D_t^{\beta}f(t) = f(t) - f(0), \tag{1.18}$$

one obtains the following energy estimate.

**Lemma 1.5.** *For all* $\boldsymbol{u}$ *in* $[L^2(\Omega)]^2$ *the following energy estimate is obtained :*

$$\|\boldsymbol{u(t)}\|_0^2 - \|\boldsymbol{u(0)}\|_0^2 + \alpha_1(\|\nabla\boldsymbol{u(t)}\|_0^2 - \|\nabla\boldsymbol{u(0)}\|_0^2) + 2\nu D_t^{-\beta}\|\nabla\boldsymbol{u(t)}\|_0^2 \leqslant \|\boldsymbol{f}\|_0 D_t^{-\beta}\|\boldsymbol{u(t)}\|_0. \tag{1.19}$$

PROOF. see Azoug et al. [22]

## 2 Discritization scheme with Fractional derivative in time

For numerical solution of the nonlinear steady state problem (1.1) a combination of the finite element method in space and a fractional derivative scheme in time is used. More precisely, let $\mathbf{u}_h^0 \in X_h$ be given. We seek for $(\mathbf{u}_h^k, p_h^k),\ k = 1,..,m$ such as :

$$(P_h)\begin{cases} D_t^{\beta}(\mathbf{u}_h^k - \alpha_1\Delta\mathbf{u}_h^k - \mathbf{u}_h^{k-1} + \alpha_1\Delta\mathbf{u}_h^{k-1}, \mathbf{v}_h) + a(\mathbf{u}_h^k, \mathbf{v}_h) + \tilde{C}(\mathbf{u}_h^{k-1}, \mathbf{u}_h^k, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^k) = \mathbf{f}^k; \\ \hfill \forall \mathbf{v}_h \in X_h, \\ \hfill b(\mathbf{u}_h^k, q_h) = 0; \\ \hfill \forall q_h \in Y_h. \end{cases} \tag{2.1}$$

Using the following formula for the fractional derivative discretisation (see Zhuang et al. (2009))[23]

$$D_t^{\beta}(w(t^{n+1})) = \frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}\sum_{k=0}^{n} c_k(w_i^{n+1-k} - w_i^{n-k}), \tag{2.2}$$

where $c_k = (k+1)^{1-\beta} - (k)^{1-\beta}$. Using the proposed semi-implicit scheme, the fully discrete problem (2.1) leads to a linear problem to be solved in the variables $\mathbf{u}_h^{k+1}$ and $p_h^{k+1}$ only for each time step $t^{k+1}$. In the following sections the proposed numerical scheme will be analyzed and the existence and the uniqueness of the time-discrete solution will be demonstrated.

## 3 Existence and uniqueness results

The existence and uniqueness results for the time-discrete problem $(P_h)$ can be resumed from the following theorem.

**Theorem 1.** Let $X_h$ and $Y_h$ satisfy the discrete "Inf-Sup" condition, then for $u_h^k$ given, the problem $(P_h)$ has a unique solution $(u_h^{k+1}, p_h^{k+1}) \in (X_h \times Y_h)$.

PROOF. For a given $\mathbf{u}_h^k$ and for all u, v in X, we define the following bilinear form $\tilde{a}(.,.)$

$$\tilde{a}(\mathbf{u},\mathbf{v}) = \Big(\frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}(\mathbf{u} - \alpha_1\Delta\mathbf{u}, \mathbf{v})\Big) + a(\mathbf{u},\mathbf{v}) + \tilde{C}(\mathbf{u}_h^{k-1}, \mathbf{u}, \mathbf{v})$$

Thus, we have

$$\tilde{a}(\mathbf{u},\mathbf{u}) = \Big(\frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}(\mathbf{u} - \alpha_1\Delta\mathbf{u}, \mathbf{u})\Big) + a(\mathbf{u},\mathbf{u}) + \underbrace{\tilde{C}(\mathbf{u}_h^{k-1}, \mathbf{u}, \mathbf{u})}_{=0}$$

By integrating by parts one obtain :

$$\tilde{a}(\mathbf{u},\mathbf{u}) = \Big(\frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}\Big)\|\mathbf{u}\|_0^2 + \Big(\frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}\alpha_1 + \nu\Big) \mid \mathbf{u} \mid_1^2 . \tag{3.1}$$

Hence, the form ã(., .) is continuous and coercive on $X \times X$ According to the Lax-Milgram's lemma, the existence and uniqueness results are obtained in X. In the case of $\alpha_1 = 0$, the existence and uniqueness results are then obtained in $[H_0^1(\Omega)]^2$

$$\tilde{a}(\mathbf{u},\mathbf{u}) = \Big(\frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}\Big)\|\mathbf{u}\|_0^2 + \nu \mid \mathbf{u} \mid_1^2 . \tag{3.2}$$

But with $\tilde{C}(\mathbf{u},\mathbf{v},\mathbf{w}) = \tilde{C}$ and the result in $[H_0^1(\Omega)]^2$ is then obtained.

## 4 Stability

It is assumed that the triangulation $\tau_h$ is uniformly regular and the initial data $\mathbf{u}_0^h$ is such that $\|\mathbf{u}_h^0\|_{[H^1(\Omega)]^2}$ is bounded independently of h and $\Delta t$ and f is bounded in $L^2(\Omega)$.

**Theorem 4.1.** *(Stability) The semi discrete scheme $(P_h)$ is unconditionally stable, for all n with $1 \leq n \leq M-1$*

$$\|\boldsymbol{u}_h^{n+1}\|_0^2 + (2\nu c_\beta + \alpha_1)\|\nabla \boldsymbol{u}_h^{n+1}\|_0^2 \leqslant \|\boldsymbol{u}_h^0\|_0^2 + \alpha_1\|\nabla \boldsymbol{u}_h^0\|_0^2 + cc_\beta \sum_{j=1}^{n+1} \|\boldsymbol{f}^j\|_0^2$$

*with $c_\beta = \Delta t^\beta \Gamma(2-\beta)$*

PROOF. Taking $\mathbf{v}_h = \mathbf{u}_h^k$ in equation $(P_h)$, one gets :

$$\frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}\{(\mathbf{u}_h^k - \alpha_1\Delta\mathbf{u}_h^k - \mathbf{u}_h^{k-1} + \alpha_1\Delta\mathbf{u}_h^{k-1}, \mathbf{u}_h^k) + \sum_{s=1}^{k-1} c_s(\mathbf{u}_h^{k-s} - \alpha_1\Delta\mathbf{u}_h^{k-s} - \mathbf{u}_h^{k-s-1} + \alpha_1\Delta\mathbf{u}_h^{k-s-1}, \mathbf{u}_h^k)\}$$
$$+ a(\mathbf{u}_h^k,\mathbf{u}_h^k) + \tilde{C}(\mathbf{u}_h^{k-1},\mathbf{u}_h^k,\mathbf{u}_h^k) + b(\mathbf{u}_h^k,p_h^k) = (\mathbf{f},\mathbf{u}_h^k)$$
$$\forall\, \mathbf{u}_h^k \in X_h$$

with $c_s = (s+1)^{1-\beta} - (s)^{1-\beta}$
We multiply by $c_\beta$

$$(\mathbf{u}_h^k - \alpha_1\Delta\mathbf{u}_h^k - \mathbf{u}_h^{k-1} + \alpha_1\Delta\mathbf{u}_h^{k-1}, \mathbf{u}_h^k) + \sum_{s=1}^{k-1} c_s(\mathbf{u}_h^{k-s} - \alpha_1\Delta\mathbf{u}_h^{k-s} - \mathbf{u}_h^{k-s-1} + \alpha_1\Delta\mathbf{u}_h^{k-s-1}, \mathbf{u}_h^k)$$
$$+ c_\beta a(\mathbf{u}_h^k,\mathbf{u}_h^k) + c_\beta \underbrace{\tilde{C}(\mathbf{u}_h^{k-1},\mathbf{u}_h^k,\mathbf{u}_h^k)}_{0} + c_\beta \underbrace{b(\mathbf{u}_h^k,p_h^k)}_{0} = c_\beta(\mathbf{f},\mathbf{u}_h^k)$$
$$\forall \mathbf{u}_h^k \in X_h$$

$$(\mathbf{u}_h^k - \alpha_1\Delta\mathbf{u}_h^k - \mathbf{u}_h^{k-1} + \alpha_1\Delta\mathbf{u}_h^{k-1}, \mathbf{u}_h^k) + \sum_{s=1}^{k-1} (c_s)(\mathbf{u}_h^{k-s} - \alpha_1\Delta\mathbf{u}_h^{k-s} - \mathbf{u}_h^{k-s-1} + \alpha_1\Delta\mathbf{u}_h^{k-s-1}, \mathbf{u}_h^k)$$
$$+ c_\beta a(\mathbf{u}_h^k,\mathbf{u}_h^k) = c_\beta(\mathbf{f},\mathbf{u}_h^k)$$
$$\forall \mathbf{u}_h^k \in X_h$$

$$(\mathbf{u}_h^k - \alpha_1 \Delta \mathbf{u}_h^k, \mathbf{u}_h^k) - (\mathbf{u}_h^{k-1} - \alpha_1 \Delta \mathbf{u}_h^{k-1}, \mathbf{u}_h^k) + \sum_{s=1}^{k-1} c_s (\mathbf{u}_h^{k-s} - \alpha_1 \Delta \mathbf{u}_h^{k-s}, \mathbf{u}_h^k)$$
$$- \sum_{s=1}^{k-1} c_s (\mathbf{u}_h^{k-s-1} - \alpha_1 \Delta \mathbf{u}_h^{k-s-1}, \mathbf{u}_h^k) + c_\beta a(\mathbf{u}_h^k, \mathbf{u}_h^k) = c_\beta (\mathbf{f}, \mathbf{u}_h^k)$$
$$\forall \mathbf{u}_h^k \in X_h$$

By integrating by parts and multiply by $(2)$ to obtain :

$$2(\mathbf{u}_h^k - \mathbf{u}_h^{k-1}, \mathbf{u}_h^k) + 2\alpha_1 (\nabla \mathbf{u}_h^k - \nabla \mathbf{u}_h^{k-1}, \nabla \mathbf{u}_h^k) + 2 \sum_{s=1}^{k-1} c_s (\mathbf{u}_h^{k-s} - \mathbf{u}_h^{k-s-1}, \mathbf{u}_h^k)$$
$$+ 2\alpha_1 \sum_{s=1}^{k-1} c_s (\nabla \mathbf{u}_h^{k-s} - \nabla \mathbf{u}_h^{k-s-1}, \nabla \mathbf{u}_h^k) + 2c_\beta a(\mathbf{u}_h^k, \mathbf{u}_h^k) = 2c_\beta (\mathbf{f}, \mathbf{u}_h^k)$$
$$\forall \mathbf{u}_h^k \in X_h$$

Cauchy Schwarz inequality :

$$2(\mathbf{u}_h^k - \mathbf{u}_h^{k-1}, \mathbf{u}_h^k) + 2.\alpha_1 \mid \mathbf{u}_h^k \mid_1^2 + 2 \sum_{s=1}^{k-1} c_s (\mathbf{u}_h^{k-s} - \mathbf{u}_h^{k-s-1}, \mathbf{u}_h^k)$$
$$+ 2\alpha_1 \sum_{s=1}^{k-1} c_s (\nabla \mathbf{u}_h^{k-s} - \nabla \mathbf{u}_h^{k-s-1}, \nabla \mathbf{u}_h^k) + 2c_\beta \nu \mid \mathbf{u}_h^k \mid_1^2 \leqslant 2c_\beta \mid \mathbf{f} \mid_0 \mid \mathbf{u}_h^k \mid_0 + 2\alpha_1 \mid \mathbf{u}_h^{k-1} \mid_1 \mid \mathbf{u}_h^k \mid_1$$
$$\forall \mathbf{u}_h^k \in X_h$$

for n=1,

$$2(\mathbf{u}^1 - \mathbf{u}_h^0, \mathbf{u}^1) + 2\alpha_1 (\nabla \mathbf{u}^0 - \nabla \mathbf{u}^0, \nabla \mathbf{u}^1) + 2c_\beta \nu \mid \mathbf{u}_h^k \mid_1^2 = 2c_\beta (\mathbf{f}^1, \mathbf{u}^1)$$

Using the following identity :

$$2(\mathbf{u} - \mathbf{v}, \mathbf{u}) = \|\mathbf{u}\|_0^2 - \|\mathbf{v}\|_0^2 + \|\mathbf{u} - \mathbf{v}\|_0^2$$

$$2(\mathbf{u}_h^k - \mathbf{u}_h^{k-1}, \mathbf{u}_h^k) = \|\mathbf{u}_h^k\|_0^2 - \|\mathbf{u}_h^{k-1}\|_0^2 + \|\mathbf{u}_h^k - \mathbf{u}_h^{k-1}\|_0^2$$

$$\|\mathbf{u}1\|_0^2 + \|\mathbf{u}^1 - \mathbf{u}^0\|_0^2 + \|\mathbf{u}^0\|_0^2 + \alpha_1 (\mid \mathbf{u}^1 \mid_1^2 - \mid \mathbf{u}^0 \mid_1^2 + \mid \mathbf{u}^1 - \mathbf{u}^0 \mid_1^2) + 2c_\beta \nu \mid \mathbf{u}^1 \mid_1^2 \leq c_\beta \nu \mid \mathbf{u}^1 \mid_1^2 + cc_\beta \mid \mathbf{f}^1 \mid^2$$

That is

$$\|\mathbf{u}1\|_0^2 + \alpha_1 \mid \mathbf{u}^1 \mid_1^2 + (c_\beta \nu + \alpha_1) \mid \mathbf{u}^1 \mid_1^2 \leq \|\mathbf{u}^0\|_0^2 + \alpha_1 \mid \mathbf{u}^0 \mid_1^2 + cc_\beta \mid \mathbf{f}^1 \mid^2$$

We assume :

$$\|\mathbf{u}_h^k\|_0^2 + (2\nu c_\beta + \alpha_1) \|\nabla \mathbf{u}_h^k\|_0^2 \leqslant \|\mathbf{u}_h^0\|_0^2 + \alpha_1 \|\nabla \mathbf{u}_h^0\|_0^2 + cc_\beta \sum_{j=1}^{k} \|\mathbf{f}^j\|_0^2, \ k = 1, ..., n.$$

For k = n + 1, we have

$$2\|\mathbf{u}_h^{n+1}\|_0^2 + 2(\nu c_\beta + \alpha_1) \|\nabla \mathbf{u}_h^{n+1}\|_0^2 = 2 \left( \sum_{j=0}^{n-1} (c_j - c_{j+1}) \mathbf{u}_h^{n-j} + c_n \mathbf{u}_h^0, \mathbf{u}_h^{n+1} \right)$$
$$+ 2\alpha_1 \left( \sum_{j=0}^{n-1} (c_j - c_{j+1}) \nabla \mathbf{u}_h^{n-j} + c_n \nabla \mathbf{u}_h^0, \nabla \mathbf{u}_h^{n+1} \right) + 2c_\beta (\mathbf{f}^{n+1}, \mathbf{u}_h^{n+1})$$

$$2\|\mathbf{u}_h^{n+1}\|_0^2+2(\nu c_\beta+\alpha_1)\|\nabla\mathbf{u}_h^{n+1}\|_0^2 \leq \sum_{j=0}^{n-1}(c_j-c_{j+1})(\|\mathbf{u}_h^{n-j}\|_0^2+\|\mathbf{u}_h^{n+1}\|_0^2)+c_n(\|\mathbf{u}_h^0\|_0^2+\|\mathbf{u}_h^{n+1}\|_0^2)$$

$$+\alpha_1\sum_{j=0}^{n-1}(c_j-c_{j+1})(|\ \mathbf{u}_h^{n-j}\ |_1^2+|\ \mathbf{u}_h^{n+1}\ |_1^2)+\alpha_1 c_n(|\ \mathbf{u}_h^1\ |_1^2+|\ \mathbf{u}_h^{n+1}\ |_1^2)+cc_\beta\|\mathbf{f}^{n+1}\|_0^2+\nu c_\beta\|\nabla\mathbf{u}_h^{n+1}\|_0^2$$

$$2\|\mathbf{u}_h^{n+1}\|_0^2+(\nu c_\beta+\alpha_1)\|\nabla\mathbf{u}_h^{n+1}\|_0^2 \leq (\sum_{j=0}^{n-1}(c_j-c_{j+1})+c_n)\left(\|\mathbf{u}_h^0\|_0^2+\alpha_1\|\nabla\mathbf{u}_h^0\|_0^2+cc_\beta\sum_{j=1}^{k}\|\mathbf{f}^j\|_0^2\right)$$

$$+\alpha_1(\sum_{j=0}^{n-1}(c_j-c_{j+1})+c_n)\left(\|\mathbf{u}_h^0\|_0^2)+\alpha_1\|\nabla\mathbf{u}_h^0\|_0^2+cc_\beta\sum_{j=1}^{k}\|\mathbf{f}^j\|_0^2\right)+cc_\beta\|\mathbf{f}^{n+1}\|_0^2$$

Note that $\sum_{j=0}^{n-1}(c_j-c_{j+1})+c_n=1$

$$2\|\mathbf{u}_h^{n+1}\|_0^2+(\nu c_\beta+\alpha_1)\|\nabla\mathbf{u}_h^{n+1}\|_0^2 \leq \|\mathbf{u}_h^0\|_0^2+\alpha_1\|\nabla\mathbf{u}_h^0\|_0^2+cc_\beta\sum_{j=1}^{n+1}\|\mathbf{f}^j\|_0^2.$$

For the case $\alpha_1=\alpha_2=0$, we obtain the stability results for the fractional Navier-stokes model **(FNSE)** in Zhang and Wang (2018)[12], and for the case $\alpha_1=\alpha_2$, we obtain the stability results for the fractional aqueuse solution of polymer(**FASP**) model in Azoug et al. (2020)[22].

**Theorem 4.2.** *(FNSE) The semi discrete scheme $(P_h)$ is uncondition stable, for all n with* $1\leq n\leq M-1$

$$\|\boldsymbol{u}_h^{n+1}\|_0^2+(2\nu c_\beta)\|\nabla\boldsymbol{u}_h^{n+1}\|_0^2 \leqslant \|\boldsymbol{u}_h^0\|_0^2+cc_\beta\sum_{j=1}^{n+1}\|\boldsymbol{f}^j\|_0^2,$$

*with* $c_\beta=\Delta t^\beta\Gamma(2-\beta)$.

**Theorem 4.3.** *(FASP) The semi discrete scheme $(P_h)$ is uncondition stable, for all n with* $1\leq n\leq M-1$

$$\|\boldsymbol{u}_h^{n+1}\|_0^2+(2\nu c_\beta+\alpha)\|\nabla\boldsymbol{u}_h^{n+1}\|_0^2 \leqslant \|\boldsymbol{u}_h^0\|_0^2+\alpha\|\nabla\boldsymbol{u}_h^0\|_0^2+cc_\beta\sum_{j=1}^{n+1}\|\boldsymbol{f}^j\|_0^2,$$

*with* $c_\beta=\Delta t^\beta\Gamma(2-\beta)$.

## 5 Convergence

We assume that the exact solution $(\mathbf{u},p)$ of system equations (FM) is regular, such that

$$\mathbf{u}\in C^0([0,T],[W^{3,\infty}(\Omega)]^2\cap X)\cap C^2([0,T],[H^3(\Omega)]^2)\cap H^2([0,T];[L^2(\Omega)]^2)$$

and

$$p\in H^1([0,T],H^1(\Omega)\cap Y)$$

**Theorem 5.1.** *(Convergence)*
*If :*

$$\Delta t^{\beta_0}\leq Min\left(\frac{\nu}{8\Gamma(2-\beta)};\frac{\nu}{2+\alpha_1+\alpha_2};\frac{\nu(2\alpha_1-\alpha_2)}{12\alpha_2 c_p^2\Gamma(2-\beta)}\right)$$

*and*

$$h \leq c\sqrt{\frac{(1-\frac{2\alpha_2+\alpha_1}{3})\nu^2-\alpha_2}{\alpha_2\nu}};\ \alpha_1 > \frac{\alpha_2}{2};\ \nu \geq \sqrt{\frac{4\alpha_2}{\alpha_2^2+4(1-\frac{2\alpha_2+\alpha_1}{3})}}$$

*then we have for all $k \geqslant 1$ :*

$$\|\boldsymbol{u}_h^k - \boldsymbol{u}(t^k)\|_0 \leqslant c"(|\ \boldsymbol{u}_h^0 - \boldsymbol{u}(t^0)\ |_1 + h^3 + \Delta t^{\beta} + \Delta t^{\frac{\beta'}{2}} + (2^{1-\beta}-1)\Delta t^{\frac{1}{4}} + \sqrt{\alpha_2}(h^{\frac{3}{2}} + \Delta t^{\frac{\beta}{2}} + \Delta t^{\frac{1}{4}} + \Delta t^{\frac{\beta}{2}+\frac{1}{4}}) + \alpha_2 h + \sqrt{\alpha_1}\Delta t^{\frac{\beta'}{2}}) \quad (5.1)$$

*with c, c" are independant of h and $\Delta t$ and $\beta_0 = Min(\beta, \beta'),\ \beta+\beta' = 1$.*

PROOF.
We denote :

$$\mathbf{w}_1^k = \mathbf{u}^k - \Pi_h^1\mathbf{u}(t^k),\ \eta_1^k = \Pi_h^1\mathbf{u}(t^k) - \mathbf{u}(t^k)$$
$$\mathbf{w}_2^k = p^k - \Pi_h^2 p(t^k),\ \eta_2^k = \Pi_h^2 p(t^k) - p(t^k)$$

To simply establish an estimate for $\mathbf{w}_1^k$, we can write the following :

$$(\mathbf{w}_1^k - \mathbf{w}_1^{k-1}, \mathbf{v}_h) + \sum_{s=1}^{k-1} c_s(\mathbf{w}_1^s - \mathbf{w}_1^{s-1}, \mathbf{v}_h) - \alpha_1(\Delta\mathbf{w}_1^k - \Delta\mathbf{w}_1^{k-1}, \mathbf{v}_h) - \alpha_1\sum_{s=1}^{k-1} c_s(\Delta\mathbf{w}_1^s - \Delta\mathbf{w}_1^{s-1}, \mathbf{v}_h) \quad (5.2)$$

$$+ c_\beta a(\mathbf{w}_1^k, \mathbf{v}_h) + c_\beta b(\mathbf{v}_h, w_2^k) = (\mathbf{u}^k - \mathbf{u}^{k-1}, \mathbf{v}_h) - (\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}), \mathbf{v}_h) + \sum_{s=1}^{k-1} c_s(\mathbf{u}_1^s - \mathbf{u}_1^{s-1}, \mathbf{v}_h)$$
$$- \sum_{s=1}^{k-1} c_s(\Pi\mathbf{u}(t^s) - \Pi\mathbf{u}(t^{s-1}), \mathbf{v}_h) - \alpha_1\sum_{s=1}^{k-1} c_s(\Delta\mathbf{u}_1^s - \Delta\mathbf{u}_1^{s-1}, \mathbf{v}_h) + \alpha_1\sum_{s=1}^{k-1} c_s(\Delta\Pi\mathbf{u}(t^s) - \Delta\Pi\mathbf{u}(t^{s-1}), \mathbf{v}_h)$$
$$+ c_\beta a(\Pi\mathbf{u}(t^k), \mathbf{v}_h) + c_\beta a(\mathbf{u}_h^k, \mathbf{v}_h) + c_\beta b(\mathbf{v}_h, p^k) - c_\beta b(\mathbf{v}_h, \Pi p(t^k)) - \alpha_1(\Delta\mathbf{u}_h^k - \Delta\mathbf{u}_h^{k-1}, \mathbf{v}_h)$$
$$+ \alpha_1(\Delta\Pi\mathbf{u}(t^k) - \Delta\Pi\mathbf{u}(t^{k-1}), \mathbf{v}_h)$$

We have :

$$(\mathbf{u}_1^k - \mathbf{u}_1^{k-1}, \mathbf{v}_h) - \alpha_1(\Delta\mathbf{u}_h^k - \Delta\mathbf{u}_h^{k-1}, \mathbf{v}_h) + \sum_{s=1}^{k-1} c_s(\mathbf{u}_1^s - \mathbf{u}_1^{s-1}, \mathbf{v}_h) - \alpha_1\sum_{s=1}^{k-1} c_s(\Delta\mathbf{u}_1^s - \Delta\mathbf{u}_1^{s-1}, \mathbf{v}_h) + c_\beta a(\mathbf{u}_h^k, \mathbf{v}_h) + c_\beta b(\mathbf{v}_h, p^k) = c_\beta(\mathbf{f}, \mathbf{v}_h) - c_\beta\tilde{C}(\mathbf{u}_h^{k-1}, \mathbf{u}_h^k, \mathbf{v}_h) \quad (5.3)$$

Eq.(5.3) is inserted in (5.2) to obtain :

$$= c_\beta(\mathbf{f}, \mathbf{v}_h) - c_\beta\tilde{C}(\mathbf{u}_h^{k-1}, \mathbf{u}_h^k, v_h) - (\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}), \mathbf{v}_h) - \sum_{s=1}^{k-1} c_s(\Pi\mathbf{u}(t^s) - \Pi\mathbf{u}(t^{s-1}), \mathbf{v}_h)$$

$$-c_\beta a(\Pi\mathbf{u}(t^k), \mathbf{v}_h) - c_\beta b(\mathbf{v}_h, \Pi p(t^k)) + \alpha_1(\Delta\Pi\mathbf{u}(t^k) - \Delta\Pi\mathbf{u}(t^{k-1}), \mathbf{v}_h) + \alpha_1\sum_{s=1}^{k-1} c_s(\Delta\Pi\mathbf{u}(t^s) - \Delta\Pi\mathbf{u}(t^{s-1}), \mathbf{v}_h)$$

$$= -c_\beta(-(\mathbf{f},\mathbf{v}_h)+a(\Pi\mathbf{u}(t^k),\mathbf{v}_h)-b(\mathbf{v}_h,\Pi p(t^k))-c_\beta\tilde{C}(\mathbf{u}_h^{k-1},\mathbf{u}_h^k,v_h)-(\Pi\mathbf{u}(t^k)-\Pi\mathbf{u}(t^{k-1}),\mathbf{v}_h)$$
$$-\sum_{s=1}^{k-1}c_s(\Pi\mathbf{u}(t^s)-\Pi\mathbf{u}(t^{s-1}),\mathbf{v}_h)+\alpha_1(\Delta\Pi\mathbf{u}(t^k)-\Delta\Pi\mathbf{u}(t^{k-1}),\mathbf{v}_h)+\alpha_1\sum_{s=1}^{k-1}c_s(\Delta\Pi\mathbf{u}(t^s)-\Delta\Pi\mathbf{u}(t^{s-1}),\mathbf{v}_h)$$

Using this equality :

$$a(\Pi\mathbf{u}(t^k),\mathbf{v}_h)+b(\mathbf{v}_h,\Pi p(t^k)-(\mathbf{f},\mathbf{v}_h)=-\tilde{C}(\mathbf{u}(t^k),\mathbf{u}(t^k),\mathbf{v}_h)-(D_t^\beta(\mathbf{u}(t^k)-\alpha_1\Delta\mathbf{u}(t^k)),\mathbf{v}_h)$$

$$= c_\beta(\tilde{C}(\mathbf{u}(t^k),\mathbf{u}(t^k),\mathbf{v}_h)+(D_t^\beta(\mathbf{u}(t^k)-\alpha_1\Delta\mathbf{u}(t^k)),\mathbf{v}_h))-c_\beta\tilde{C}(\mathbf{u}_h^{k-1},\mathbf{u}_h^k,v_h)-(\Pi\mathbf{u}(t^k)-\Pi\mathbf{u}(t^{k-1}),\mathbf{v}_h)$$
$$-\sum_{s=1}^{k-1}c_s(\Pi\mathbf{u}(t^s)-\Pi\mathbf{u}(t^{s-1}),\mathbf{v}_h)+\alpha_1(\Delta\Pi\mathbf{u}(t^k)-\Delta\Pi\mathbf{u}(t^{k-1}),\mathbf{v}_h)+\alpha_1\sum_{s=1}^{k-1}c_s(\Delta\Pi\mathbf{u}(t^s)-\Delta\Pi\mathbf{u}(t^{s-1}),\mathbf{v}_h)$$

We have :

$$-(\Pi\mathbf{u}(t^k)-\Pi\mathbf{u}(t^{k-1}),\mathbf{v}_h)=-(\Pi\mathbf{u}(t^k)-\mathbf{u}(t^k),\mathbf{v}_h)+(\Pi\mathbf{u}(t^{k-1})-\mathbf{u}(t^{k-1}),\mathbf{v}_h)+(\mathbf{u}(t^{k-1})-\mathbf{u}(t^k),\mathbf{v}_h)$$
$$-(\Pi\mathbf{u}(t^k)-\Pi\mathbf{u}(t^{k-1}),\mathbf{v}_h)=-(\eta_1^k-\eta_1^{k-1},\mathbf{v}_h)+(\mathbf{u}(t^{k-1})-\mathbf{u}(t^k),\mathbf{v}_h)$$

By replacing in (5.2) one obtains :

$$(\mathbf{w}_1^k-\mathbf{w}_1^{k-1},\mathbf{v}_h)+\sum_{s=1}^{k-1}c_s(\mathbf{w}_1^s-\mathbf{w}_1^{s-1},\mathbf{v}_h)-\alpha_1(\Delta\mathbf{w}_1^k-\Delta\mathbf{w}_1^{k-1},\mathbf{v}_h)+\alpha_1\sum_{s=1}^{k-1}c_s(\Delta\mathbf{w}_1^s-\Delta\mathbf{w}_1^{s-1},\mathbf{v}_h)$$
$$c_\beta a(\mathbf{w}_1^k,\mathbf{v}_h)+c_\beta b(\mathbf{v}_h,\mathbf{w}_2^k)=c_\beta(\tilde{C}(\mathbf{u}(t^k),\mathbf{u}(t^k),\mathbf{v}_h)+(D_t^\beta(\mathbf{u}(t^k)-\alpha_1\Delta\mathbf{u}(t^k)),\mathbf{v}_h))-c_\beta\tilde{C}(\mathbf{u}_h^{k-1},\mathbf{u}_h^k,v_h)$$
$$-(\eta_1^k-\eta_1^{k-1},\mathbf{v}_h)+(\mathbf{u}(t^{k-1})-\mathbf{u}(t^k),\mathbf{v}_h)-\sum_{s=1}^{k-1}c_s(\Pi\mathbf{u}(t^s)-\Pi\mathbf{u}(t^{s-1}),\mathbf{v}_h)$$
$$+\alpha_1(\Delta\Pi\mathbf{u}(t^k)-\Delta\Pi\mathbf{u}(t^{k-1}),\mathbf{v}_h)+\alpha_1\sum_{s=1}^{k-1}c_s(\Delta\Pi\mathbf{u}(t^s)-\Delta\Pi\mathbf{u}(t^{s-1}),\mathbf{v}_h)$$

We take $\mathbf{v}_h=\mathbf{w}_1^k$ then : $b(\mathbf{w}_1^k,\mathbf{w}_2^k)=0$.
Indeed :

$$b(\mathbf{w}_1^k,\mathbf{w}_2^k)=b(\mathbf{u}_h^k,\mathbf{w}_2^k)-b(\Pi\mathbf{u}(t^k),\mathbf{w}_2^k)$$
$$b(\mathbf{u}_h^k,\mathbf{w}_2^k)=0 \ \ since \ \ \mathbf{w}_2^k\in Y_h$$
$$b(\Pi\mathbf{u}(t^k),\mathbf{w}_2^k)=b(\mathbf{u}(t^k),\mathbf{w}_2^k)=0.$$

Integrating by parts and multiplying by 2 one gets :

$$
\begin{aligned}
&2(\mathbf{w}_1^k - \mathbf{w}_1^{k-1}, \mathbf{w}_1^k) + 2\sum_{s=1}^{k-1} c_s(\mathbf{w}_1^s - \mathbf{w}_1^{s-1}, \mathbf{w}_1^k) + 2\alpha_1(\nabla\mathbf{w}_1^k - \nabla\mathbf{w}_1^{k-1}, \nabla\mathbf{w}_1^k)\\
&+ 2\alpha_1\sum_{s=1}^{k-1} c_s(\nabla\mathbf{w}_1^s - \nabla\mathbf{w}_1^{s-1}, \nabla\mathbf{w}_1^k) + 2c_\beta a(\mathbf{w}_1^k, \mathbf{w}_1^k) = 2c_\beta(\tilde{C}(\mathbf{u}(t^k), \mathbf{u}(t^k), \mathbf{w}_1^k)\\
&+ 2(D_t^\beta(\mathbf{u}(t^k) - \alpha_1\Delta\mathbf{u}(t^k)), \mathbf{w}_1^k)) - 2c_\beta\tilde{C}(\mathbf{u}_h^{k-1}, \mathbf{u}_h^k, \mathbf{w}_1^k) - 2(\eta_1^k - \eta_1^{k-1}, \mathbf{v}_h)\\
&+ 2(\mathbf{u}(t^{k-1}) - \mathbf{u}(t^k), \mathbf{w}_1^k) - 2\sum_{s=1}^{k-1} c_s(\Pi\mathbf{u}(t^s) - \Pi\mathbf{u}(t^{s-1}), \mathbf{w}_1^k) - 2\alpha_1(\nabla\Pi\mathbf{u}(t^k) - \nabla\Pi\mathbf{u}(t^{k-1}), \nabla\mathbf{w}_1^k)\\
&- \alpha_1\sum_{s=1}^{k-1} c_s(\nabla\Pi\mathbf{u}(t^s) - \nabla\Pi\mathbf{u}(t^{s-1}), \nabla\mathbf{w}_1^k)
\end{aligned}
$$

The following identities will be used :

$$
\begin{aligned}
2(\mathbf{u} - \mathbf{v}, \mathbf{u}) &= \|\mathbf{u}\|_0^2 - \|\mathbf{v}\|_0^2 + \|\mathbf{u} - \mathbf{v}\|_0^2\\
2(\mathbf{u} - \mathbf{v}, \mathbf{w}) &= \|\mathbf{u}\|_0^2 - \|\mathbf{v}\|_0^2 + \|\mathbf{w} - \mathbf{v}\|_0^2 - \|\mathbf{w} - \mathbf{u}\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathbf{w}_1^k\|_0^2 - \|\mathbf{w}_1^{k-1}\|_0^2 + \|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0^2 + \alpha_1(\mid \mathbf{w}_1^k \mid_1^2 - \mid \mathbf{w}_1^{k-1} \mid_1^2 + \mid \mathbf{w}_1^k - \mathbf{w}_1^{k-1} \mid_1^2))\\
&+ \sum_{s=1}^{k-1}(\|\mathbf{w}_1^{k-s}\|_0^2 - \|\mathbf{w}_1^{k-s-1}\|_0^2 + \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1}\|_0^2 - \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s}\|_0^2)\\
&+ \alpha_1\sum_{s=1}^{k-1}(\mid \mathbf{w}_1^{k-s} \mid_1^2 - \mid \mathbf{w}_1^{k-s-1} \mid_1^2 + \mid c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1} \mid_1^2 - \mid c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s} \mid_1^2)\\
&+ 2c_\beta a(\mathbf{w}_1^k, \mathbf{w}_1^k) = 2(D_t^\beta(\mathbf{u}(t^k) - \alpha_1\Delta\mathbf{u}(t^k)), \mathbf{w}_1^k)) - 2(\eta_1^k - \eta_1^{k-1}, \mathbf{w}_1^k) + 2(\mathbf{u}(t^{k-1}) - \mathbf{u}(t^k), \mathbf{w}_1^k)\\
&- 2\sum_{s=1}^{k-1} c_s(\Pi\mathbf{u}(t^s) - \Pi\mathbf{u}(t^{s-1}), \mathbf{w}_1^k) - 2\alpha_1(\nabla\Pi\mathbf{u}(t^k) - \nabla\Pi\mathbf{u}(t^{k-1}), \nabla\mathbf{w}_1^k)\\
&- 2\alpha_1\sum_{s=1}^{k-1} c_s(\nabla\Pi\mathbf{u}(t^s) - \nabla\Pi\mathbf{u}(t^{s-1}), \nabla\mathbf{w}_1^k) + 2c_\beta c_k
\end{aligned}
$$

with :

$$
C_k = -\tilde{C}(\mathbf{u}_h^{k-1}, \mathbf{u}_h^k, \mathbf{w}_1^k) + \tilde{C}(\mathbf{u}(t^k), \mathbf{u}(t^k), \mathbf{w}_1^k)
$$

$$
\begin{aligned}
&\|\mathbf{w}_1^k\|_0^2 - \|\mathbf{w}_1^{k-1}\|_0^2 + \|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0^2 + \alpha_1(\mid \mathbf{w}_1^k \mid_1^2 - \mid \mathbf{w}_1^{k-1} \mid_1^2 + \mid \mathbf{w}_1^k - \mathbf{w}_1^{k-1} \mid_1^2))\\
&+ \sum_{s=1}^{k-1}(\|\mathbf{w}_1^{k-s}\|_0^2 - \|\mathbf{w}_1^{k-s-1}\|_0^2 + \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1}\|_0^2 - \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s}\|_0^2)\\
&+ \alpha_1\sum_{s=1}^{k-1}(\mid \mathbf{w}_1^{k-s} \mid_1^2 - \mid \mathbf{w}_1^{k-s-1} \mid_1^2 + \mid c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1} \mid_1^2 - \mid c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s} \mid_1^2) + 2\nu c_\beta \mid \mathbf{w}_1^k \mid_1^2 \leqslant\\
&2\|\eta_1^k - \eta_1^{k-1}\|_0\|\mathbf{w}_1^k\|_0 + 2\|D_t^\beta(\mathbf{u}(t^k) - \alpha_1\Delta\mathbf{u}(t^k))\|_0\|\mathbf{w}_1^k\|_0 + 2\alpha_1 \mid \Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}) \mid_1 \mid \mathbf{w}_1^k \mid_1\\
&+ 2\sum_{s=1}^{k-1} c_s\|\Pi\mathbf{u}(t^s) - \Pi\mathbf{u}(t^{s-1})\|_0\|\mathbf{w}_1^k\|_0 + 2\alpha_1\sum_{s=1}^{k-1} c_s \mid \Pi\mathbf{u}(t^s) - \Pi\mathbf{u}(t^{s-1}) \mid_1 \mid \mathbf{w}_1^k \mid_1 + 2c_\beta C_k
\end{aligned}
$$

**Other formulations of $C_k$ :**

$$C_k = \tilde{C}(\mathbf{u}(t^k) - \mathbf{u}^k, \mathbf{u}(t^k), \mathbf{w}_1^k) + \tilde{C}(\mathbf{u}^k - \mathbf{u}^{k-1}, \mathbf{u}(t^k), \mathbf{w}_1^k) + \tilde{C}(\mathbf{u}^{k-1}, \mathbf{u}(t^k) - \mathbf{u}^k, \mathbf{w}_1^k) \quad (5.4)$$

We have

$$\mathbf{w}_1^k + \eta_1^k = \mathbf{u}^k - \mathbf{u}(t^k)$$
$$\mathbf{w}_1^k - \mathbf{w}_1^{k-1} = (\mathbf{u}^k - \mathbf{u}^{k-1}) - (\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}))$$

$$C_K = -\tilde{C}(\mathbf{w}_1^k + \eta_1^k, \mathbf{u}(t^k), \mathbf{w}_1^k) + \tilde{C}(\mathbf{w}_1^k - \mathbf{w}_1^{k-1}, \mathbf{u}(t^k), \mathbf{w}_1^k) + \tilde{C}(\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}), \mathbf{u}(t^k), \mathbf{w}_1^k) + \tilde{C}(\mathbf{u}^{k-1}, \mathbf{u}(t^k) - \mathbf{u}^k, \mathbf{w}_1^k)$$

since $\tilde{C}(\mathbf{u}^{k-1}, \mathbf{w}_1^k, \mathbf{w}_1^k) = 0$ then

$$\tilde{C}(\mathbf{u}^{k-1}, \eta_1^k, \mathbf{w}_1^k) = \tilde{C}(\mathbf{u}^{k-1} - \mathbf{u}(t^{k-1}), \eta_1^k, \mathbf{w}_1^k) + \tilde{C}(\mathbf{u}(t^{k-1}), \eta_1^k, \mathbf{w}_1^k) \quad (5.5)$$

By replacing in (5.4) one has

$$C_K = -\tilde{C}(\mathbf{w}_1^k + \eta_1^k, \mathbf{u}(t^k), \mathbf{w}_1^k) + \tilde{C}(\mathbf{w}_1^k - \mathbf{w}_1^{k-1}, \mathbf{u}(t^k), \mathbf{w}_1^k) + \tilde{C}(\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}), \mathbf{u}(t^k), \mathbf{w}_1^k) - \tilde{C}(\mathbf{w}_1^{k-1} + \eta_1^{k-1}, \eta_1^k, \mathbf{w}_1^k) - \tilde{C}(\mathbf{u}(t^{k-1}), \eta_1^k, \mathbf{w}_1^k)$$

**Estimation of $C_k$**

(i)

$$\tilde{C}(\mathbf{w}_1^k + \eta_1^k, \mathbf{u}(t^k), \mathbf{w}_1^k) \leqslant c(\|\mathbf{w}_1^k + \eta_1^k\|_0)(|\ \mathbf{w}_1^k\ |_1\ (1+\alpha_2) + \alpha_2)$$
$$\tilde{C}(\mathbf{w}_1^k + \eta_1^k, \mathbf{u}(t^k), \mathbf{w}_1^k) \leqslant c(\|\mathbf{w}_1^k\|_0 + \|\eta_1^k\|_0)(|\ \mathbf{w}_1^k\ |_1\ (1+\alpha_2) + \alpha_2)$$
$$\tilde{C}(\mathbf{w}_1^k + \eta_1^k, \mathbf{u}(t^k), \mathbf{w}_1^k) \leqslant c(\|\mathbf{w}_1^k\|_0 + h^3)(|\ \mathbf{w}_1^k\ |_1\ (1+\alpha_2) + \alpha_2)$$

(ii)

$$\tilde{C}(\mathbf{w}_1^k - \mathbf{w}_1^{k-1}, \mathbf{u}(t^k), \mathbf{w}_1^k) \leqslant c(\|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0)(|\ \mathbf{w}_1^k\ |_1\ (1+\alpha_2) + \alpha_2)$$

(iii)

$$\tilde{C}(\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}), \mathbf{u}(t^k), \mathbf{w}_1^k) \leqslant c(\|\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}\|_0)(|\ \mathbf{w}_1^k\ |_1\ (1+\alpha_2) + \alpha_2)$$
$$\tilde{C}(\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}), \mathbf{u}(t^k), \mathbf{w}_1^k) \leqslant c(C_{3,k}\sqrt{\Delta t})(|\ \mathbf{w}_1^k\ |_1\ (1+\alpha_2) + \alpha_2)$$

(iv)

$$\tilde{C}(\mathbf{w}_1^{k-1} + \eta_1^{k-1}, \eta_1^k, \mathbf{w}_1^k) \leqslant c(|\ \mathbf{w}_1^{k-1} + \eta_1^{k-1}\ |_1)(|\ \eta_1^k\ |_1|\ \mathbf{w}_1^k\ |_1 + \alpha_2\ |\ \Delta\eta_1^k\ |_1\ (|\ \mathbf{w}_1^k\ |_1 + |\ \eta_1^k\ |_1))$$
$$\tilde{C}(\mathbf{w}_1^{k-1} + \eta_1^{k-1}, \eta_1^k, \mathbf{w}_1^k) \leqslant c(|\ \mathbf{w}_1^{k-1}\ |_1 + |\ \eta_1^{k-1}\ |_1)(|\ \eta_1^k\ |_1|\ \mathbf{w}_1^k\ |_1 + \alpha_2\ |\ \Delta\eta_1^k\ |_1\ (|\ \mathbf{w}_1^k\ |_1 + |\ \eta_1^k\ |_1))$$
$$\tilde{C}(\mathbf{w}_1^{k-1} + \eta_1^{k-1}, \eta_1^k, \mathbf{w}_1^k) \leqslant c(|\ \mathbf{w}_1^{k-1}\ |_1 + h^2)((h^2 + \alpha_2)\ |\ \mathbf{w}_1^k\ |_1 + \alpha_2 h^2)$$

$$\tilde{C}(\mathbf{u}(t^{k-1}), \eta_1^k, \mathbf{w}_1^k) = \frac{1}{2}(c(\mathbf{u}(t^{k-1}), \eta_1^k, \mathbf{w}_1^k) - c(\mathbf{u}(t^{k-1}), \mathbf{w}_1^k, \eta_1^k)) - \frac{\alpha_2}{2}(c(\mathbf{u}(t^{k-1}), \Delta\eta_1^k, \mathbf{w}_1^k) - c(\mathbf{u}(t^{k-1}), \mathbf{w}_1^k, \Delta\eta_1^k) - c(\mathbf{u}(t^{k-1}), \Delta\eta_1^k, \eta_1^k) + c(\mathbf{u}(t^{k-1}), \eta_1^k, \Delta\eta_1^k))$$

$$\tilde{C}(\mathbf{u}(t^{k-1}), \eta_1^k, \mathbf{w}_1^k) = -c(\mathbf{u}(t^{k-1}), \mathbf{w}_1^k, \eta_1^k) - \frac{\alpha_2}{2}(c(\mathbf{u}(t^{k-1}), \Delta\eta_1^k, \mathbf{w}_1^k) - c(\mathbf{u}(t^{k-1}), \mathbf{w}_1^k, \Delta\eta_1^k) - c(\mathbf{u}(t^{k-1}), \Delta\eta_1^k, \eta_1^k) + c(\mathbf{u}(t^{k-1}), \eta_1^k, \Delta\eta_1^k))$$

$$\tilde{C}(\mathbf{u}(t^{k-1}),\eta_1^k,\mathbf{w}_1^k) \leqslant c \mid \mathbf{w}_1^k \mid_1 \|\eta_1^k\|_0 + c\frac{\alpha_2}{2}(\mid \mathbf{w}_1^k \mid_1 \|\Delta\eta_1^k\|_0 + c \mid \mathbf{w}_1^k \mid_1 \|\Delta\eta_1^k\|_0 + c \mid \Delta\eta_1^k \mid_1 \|\eta_1^k\|_0 + c \mid \eta_1^k \mid_1 \|\Delta\eta_1^k\|_0)$$

The summation leads to :

$$\tilde{C}(\mathbf{u}(t^{k-1}),\eta_1^k,\mathbf{w}_1^k) \leqslant c((h^3+\alpha_2 h) \mid \mathbf{w}_1^k \mid_1 + \alpha_2(h+h^2+h^3))$$

Finally, one obtains the following :

$$C_k \leqslant c((1+\alpha_2)\|\mathbf{w}_1^k\|_0 + h + h^2 + h^3 + h\alpha + (1+\alpha_2)\|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0 + (\alpha_2+h^2) \mid \mathbf{w}_1^{k-1} \mid_1 + (1+\alpha_2)C_{3,k}\sqrt{\Delta t}) \mid \mathbf{w}_1^k \mid_1 + c\alpha_2\|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0 + c\alpha_2\|\mathbf{w}_1^k\|_0 + c\alpha_2 h^2 \mid \mathbf{w}_1^{k-1} \mid_1 + c\alpha_2(h^3+h^2+h) + C_{3,k}\sqrt{\Delta t}(1+\alpha_2)$$

**Estimation of :**$\|\eta_1^k - \eta_1^{k-1}\|_0$
We have the following :

$$\|\frac{\partial}{\partial t}(\mathbf{u} - \Pi\mathbf{u}(t))\|_0 \leqslant ch^3(\|\frac{\partial \mathbf{u}}{\partial t}\|_{H^3(\Omega)} + \|\frac{\partial p}{\partial t}\|_{H^1(\Omega)})$$

and

$$\eta_1^k - \eta_1^{k-1} = \int_{t^{k-1}}^{t^k} (\frac{\partial \eta_1}{\partial t})dt$$
$$\|\eta_1^k - \eta_1^{k-1}\|_0^2 = \int_\Omega \mid \int_{t^{k-1}}^{t^k} (\frac{\partial}{\partial t}(\Pi\mathbf{u}(t) - \mathbf{u}(t)))dt \mid^2 dx$$
$$\|\eta_1^k - \eta_1^{k-1}\|_0^2 \leqslant \Delta t \int_{t^{k-1}}^{t^k} \|\frac{\partial}{\partial t}(\Pi\mathbf{u}(t) - \mathbf{u}(t))\|_0^2 dt$$
$$\|\eta_1^k - \eta_1^{k-1}\|_0 \leqslant C_{1,k}\sqrt{\Delta t}h^3,$$

where $C_{1,k}$ are chosen such that :

$$\sum_{k=1}^m (C_{1,k})^2 \leqslant c(\|\frac{\partial \mathbf{u}}{\partial t}\|^2_{L^2(0,T,H^3(\Omega))} + \|\frac{\partial p}{\partial t}\|^2_{L^2(0,T,H^1(\Omega))}) \tag{5.6}$$

**Estimation of :**$\|\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}\|_0$
Let us note that

$$\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1} = \int_{t^{k-1}}^{t^k} \frac{\partial}{\partial t}\Pi_h^1\mathbf{u}(t)dt = \int_{t^{k-1}}^{t^k} \Pi_h^1\frac{\partial \mathbf{u}(t)}{\partial t}dt$$

$$\|\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}\|_0 \leqslant \int_{t^{k-1}}^{t^k} \|\Pi_h^1\frac{\partial \mathbf{u}(t)}{\partial t}\|_0 dt$$
$$\|\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}\|_0 \leqslant \sqrt{\Delta t}\|\Pi_h^1\frac{\partial \mathbf{u}(t)}{\partial t}\|_{L^2(I_k,L^2(\Omega))}$$

and also that :

$$\|\Pi_h^1\frac{\partial \mathbf{u}(t)}{\partial t}\|_{L^2(\Omega)} \leqslant \|\Pi_h^1\frac{\partial \mathbf{u}(t)}{\partial t} - \frac{\partial \mathbf{u}(t)}{\partial t}\|_{L^2(\Omega)} + \|\frac{\partial \mathbf{u}(t)}{\partial t}\|_{L^2(\Omega)}$$
$$\|\Pi_h^1\frac{\partial \mathbf{u}(t)}{\partial t}\|_{L^2(\Omega)} \leqslant ch^2(\|\frac{\partial \mathbf{u}(t)}{\partial t}\|_{H^3(\Omega)} + \|\frac{\partial p(t)}{\partial t}\|_{H^1(\Omega)}) + \|\frac{\partial \mathbf{u}(t)}{\partial t}\|_{L^2(\Omega)}$$
$$\|\Pi_h^1\frac{\partial \mathbf{u}(t)}{\partial t}\|_{L^2(\Omega)} \leqslant c(\|\frac{\partial \mathbf{u}(t)}{\partial t}\|_{H^3(\Omega)} + \|\frac{\partial p(t)}{\partial t}\|_{H^1(\Omega)})$$
$$\|\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}\|_0 \leqslant c\sqrt{\Delta t}(\|\frac{\partial \mathbf{u}(t)}{\partial t}\|_{L^2(I_k,H^3(\Omega))} + \|\frac{\partial p(t)}{\partial t}\|_{L^2(I_k,H^1(\Omega))}))$$
$$\|\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}\|_0 \leqslant C_{3,k}\sqrt{\Delta t},$$

where $C_{3,k}$ are chosen such that :

$$\sum_{k=1}^{k=m}(C_{3,k})^2 \leqslant c(\|\frac{\partial \mathbf{u}}{\partial t}\|^2_{L^2(0,T,H^3(\Omega))} + \|\frac{\partial p}{\partial t}\|^2_{L^2(0,T,H^1(\Omega))}) \tag{5.7}$$

**Estimation of :**$\mid \Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}) \mid_1$
we know that

$$\Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}) = \int_{t^{k-1}}^{t^k} \frac{\partial}{\partial t}(\Pi\mathbf{u}(t))dt = \int_{t^{k-1}}^{t^k} \Pi(\frac{\partial}{\partial t}\mathbf{u}(t))dt$$

$$\mid \Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}) \mid_1 \leqslant \int_{t^{k-1}}^{t^k} \mid \Pi\frac{\partial}{\partial t}(\mathbf{u}(t)) \mid_1 dt \leqslant \sqrt{\Delta t}(\int_{t^{k-1}}^{t^k} \mid \Pi\frac{\partial}{\partial t}\mathbf{u}(t) \mid_1^2)^{1/2}dt.$$

Nevertheless :

$$\mid \Pi\frac{\partial \mathbf{u}}{\partial t} \mid_1 \leqslant \mid \Pi\frac{\partial \mathbf{u}}{\partial t} - \frac{\partial \mathbf{u}}{\partial t} \mid_1 + \mid \frac{\partial \mathbf{u}}{\partial t} \mid_1$$
$$\mid \Pi\frac{\partial \mathbf{u}}{\partial t} \mid_1 \leqslant ch(\|\frac{\partial \mathbf{u}}{\partial t}\|_3 + \|\frac{\partial p}{\partial t}\|_1) + \mid \frac{\partial \mathbf{u}}{\partial t} \mid_1$$
$$\mid \Pi\frac{\partial \mathbf{u}}{\partial t} \mid_2 \leqslant c(\|\frac{\partial \mathbf{u}}{\partial t}\|_3 + \|\frac{\partial p}{\partial t}\|_1)$$

$$\mid \Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}) \mid_1 \leqslant c\sqrt{\Delta t}(\|\frac{\partial \mathbf{u}}{\partial t}\|_{L^2(I_k,H^3(\Omega))} + \|\frac{\partial p}{\partial t}\|_{L^2(I_k,H^1(\Omega))})$$
$$\mid \Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}) \mid_1 \leqslant c\sqrt{\Delta t}C_{4,k},$$

where $C_{4,k}$ are chosen such that

$$\sum_{k=1}^{m}(C_{4,k})^2 \leqslant c(\|\frac{\partial \mathbf{u}}{\partial t}\|^2_{L^2(0,T,H^3(\Omega))} + \|\frac{\partial p}{\partial t}\|^2_{L^2(0,T,H^1(\Omega))}) \tag{5.8}$$

We are now able to estimate $\mathbf{w}_1^k$. From (5.2) and all the previous estimates, we obtain :

$$\begin{aligned}
&\|\mathbf{w}_1^k\|_0^2 - \|\mathbf{w}_1^{k-1}\|_0^2 + \|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0^2 + \alpha_1(\mid \mathbf{w}_1^k \mid_1^2 - \mid \mathbf{w}_1^{k-1} \mid_1^2 + \mid \mathbf{w}_1^k - \mathbf{w}_1^{k-1} \mid_1^2)) \\
&+ \sum_{s=1}^{k-1}(\|\mathbf{w}_1^{k-s}\|_0^2 - \|\mathbf{w}_1^{k-s-1}\|_0^2 + \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1}\|_0^2 - \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s}\|_0^2) \\
&+ \alpha_1\sum_{s=1}^{k-1}(\mid \mathbf{w}_1^{k-s} \mid_1^2 - \mid \mathbf{w}_1^{k-s-1} \mid_1^2 + \mid c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1} \mid_1^2 - \mid c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s} \mid_1^2) \\
&+ 2\nu c_\beta \mid \mathbf{w}_1^k \mid_1^2 \leqslant 2\|\eta_1^k - \eta_1^{k-1}\|_0\|\mathbf{w}_1^k\|_0 + 2\|D_t^\beta(\mathbf{u}(t^k) - \alpha_1\Delta\mathbf{u}(t^k))\|_0\|\mathbf{w}_1^k\|_0 \\
&+ 2\alpha_1 \mid \Pi\mathbf{u}(t^k) - \Pi\mathbf{u}(t^{k-1}) \mid_1 \mid \mathbf{w}_1^k \mid_1 + 2\sum_{s=1}^{k-1} c_s\|\Pi\mathbf{u}(t^s) - \Pi\mathbf{u}(t^{s-1})\|_0\|\mathbf{w}_1^k\|_0 \\
&+ 2\alpha_1\sum_{s=1}^{k-1} c_s \mid \Pi\mathbf{u}(t^s) - \Pi\mathbf{u}(t^{s-1}) \mid_1 \mid \mathbf{w}_1^k \mid_1 + 2c_\beta C_k
\end{aligned}$$

$$\|\mathbf{w}_1^k\|_0^2 - \|\mathbf{w}_1^{k-1}\|_0^2 + \|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0^2 + \alpha_1(|\ \mathbf{w}_1^k\ |_1^2 - |\ \mathbf{w}_1^{k-1}\ |_1^2 + |\ \mathbf{w}_1^k - \mathbf{w}_1^{k-1}\ |_1^2))$$
$$+ \sum_{s=1}^{k-1}(\|\mathbf{w}_1^{k-s}\|_0^2 - \|\mathbf{w}_1^{k-s-1}\|_0^2 + \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1}\|_0^2 - \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s}\|_0^2)$$
$$+ \alpha_1\sum_{s=1}^{k-1}(|\ \mathbf{w}_1^{k-s}\ |_1^2 - |\ \mathbf{w}_1^{k-s-1}\ |_1^2 + |\ c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1}\ |_1^2 - |\ c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s}\ |_1^2)$$
$$+ 2\nu c_\beta\ |\ \mathbf{w}_1^k\ |_1^2 \leqslant\ 2C_{1,k}\sqrt{\Delta t}h^3\|\mathbf{w}_1^k\|_0 + 2\Delta t^{1-\beta}(1+\alpha_1)C_{2,k}\|\mathbf{w}_1^k\|_0 + 2c\alpha_1\sqrt{\Delta t}C_{4,k}\ |\ \mathbf{w}_1^k\ |_1$$
$$+ 2c\sum_{s=1}^{k-1}c_s\sqrt{\Delta t}C_{3,s}\|\mathbf{w}_1^k\|_0 + 2c\alpha_1\sum_{s=1}^{k-1}c_s\sqrt{\Delta t}C_{4,s}\ |\ \mathbf{w}_1^k\ |_1 + 2c_\beta C_k$$

We have the following inequations :

$$2c(\alpha_2+1)c_\beta\|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0\ |\ \mathbf{w}_1^k\ |_1 \leqslant \frac{c_\beta}{\varepsilon_{11}}\|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0^2 + \varepsilon_{11}c_\beta\ |\ \mathbf{w}_1^k\ |_1^2$$
$$+ \frac{\alpha_2 c_\beta c_p^2}{\varepsilon_{12}}\ |\ \mathbf{w}_1^k - \mathbf{w}_1^{k-1}\ |_1^2 + \alpha_2\varepsilon_{12}c_\beta\ |\ \mathbf{w}_1^k\ |_1^2$$
$$c\alpha_2 c_\beta\|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0 \leqslant \frac{\alpha_2}{2}\ |\ \mathbf{w}_1^k - \mathbf{w}_1^{k-1}\ |_1^2 + \alpha_2 c_p^2 c_\beta^2$$
$$2c(\alpha_2+1)c_\beta\|\mathbf{w}_1^k\|_0\ |\ \mathbf{w}_1^k\ |_1 \leqslant \frac{cc_\beta}{\varepsilon_1}\|\mathbf{w}_1^k\|_0^2 + \varepsilon_1 c_\beta\ |\ \mathbf{w}_1^k\ |_1^2 + \frac{c\alpha_2 c_\beta}{\varepsilon_2}\|\mathbf{w}_1^k\|_0^2 + \varepsilon_2\alpha_2 c_\beta\ |\ \mathbf{w}_1^k\ |_1^2$$
$$2cc_\beta(\alpha_2+h^2)\ |\ \mathbf{w}_1^{k-1}\ |_1|\ \mathbf{w}_1^k\ |_1 \leqslant c\alpha_2\frac{c_\beta}{\varepsilon_{10}}\ |\ \mathbf{w}_1^{k-1}\ |_1^2 + \varepsilon_{10}c_\beta\alpha_2\ |\ \mathbf{w}_1^k\ |_1^2$$
$$+ c\frac{h^4 c_\beta}{\varepsilon_9}\ |\ \mathbf{w}_1^{k-1}\ |_1^2 + \varepsilon_9 c_\beta\ |\ \mathbf{w}_1^k\ |_1^2$$
$$2c_\beta(h^3+h^2+h+h\alpha_2)\ |\ \mathbf{w}_1^k\ |_1 \leqslant \frac{c_\beta(h^3+h^2+h+h\alpha_2)^2}{\varepsilon_8} + \varepsilon_8 c_\beta\ |\ \mathbf{w}_1^k\ |_1^2$$
$$2c_\beta h^2\alpha_2\ |\ \mathbf{w}_1^{k-1}\ |_1 \leqslant \frac{c_\beta\alpha_2 h^2}{\varepsilon_7} + \varepsilon_7\alpha_2 c_\beta h^2\ |\ \mathbf{w}_1^{k-1}\ |_1^2$$
$$2c_\beta C_{3,k}\sqrt{\Delta t}(1+\alpha_2)\ |\ \mathbf{w}_1^k\ |_1 \leqslant \frac{c_\beta\sqrt{\Delta t}C_{3,k}^2}{\varepsilon_5} + \varepsilon_5\sqrt{\Delta t}c_\beta\ |\ \mathbf{w}_1^k\ |_1^2$$
$$+ \frac{\alpha_2 c_\beta\sqrt{\Delta t}C_{3,k}^2}{\varepsilon_6} + \alpha_2\varepsilon_6\sqrt{\Delta t}c_\beta\ |\ \mathbf{w}_1^k\ |_1^2$$
$$2(C_{1,k}h^3 + c_1 C_{3,k})\sqrt{\Delta t}\|\mathbf{w}_1^k\|_0 \leqslant \frac{1}{\varepsilon_{13}}\sqrt{\Delta t}(C_{1,k}h^3 + c_1C_{3,k})^2 + \varepsilon_{13}\sqrt{\Delta t}\ |\ \mathbf{w}_1^k\ |_1^2$$
$$2\alpha_1(C_{4,k}\Delta t^{\frac14} + c_1 C_{4,k}\Delta t^{\frac14})\Delta t^{\frac14}\ |\ \mathbf{w}_1^k\ |_1 \leqslant \frac{\alpha_1\Delta t^{1-\beta}}{\varepsilon_3}(C_{4,k} + c_1 C_{4,k})^2 + \varepsilon_3\alpha_1 c_\beta\ |\ \mathbf{w}_1^k\ |_1^2$$
$$2\Delta t^{1-\beta}C_{2,k}(1+\alpha_1)\|\mathbf{w}_1^k\|_0 \leqslant \varepsilon_4\Delta t^{1-\beta}C_{2,k}^2(1+\alpha_1) + \frac{1}{\varepsilon_4}\Delta t^{1-\beta}(1+\alpha_1)\|\mathbf{w}_1^k\|_0^2$$

By replacing

$$\|\mathbf{w}_1^k\|_0^2 - \|\mathbf{w}_1^{k-1}\|_0^2 + (1 - \frac{c_\beta}{\varepsilon_{11}})\|\mathbf{w}_1^k - \mathbf{w}_1^{k-1}\|_0^2$$
$$+ \sum_{s=1}^{k-1}(\|\mathbf{w}_1^{k-s}\|_0^2 - \|\mathbf{w}_1^{k-s-1}\|_0^2 + \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1}\|_0^2 - \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s}\|_0^2)$$
$$+ \alpha_1(|\ \mathbf{w}_1^k\ |_1^2 - |\ \mathbf{w}_1^{k-1}\ |_1^2) + (\alpha_1 - \frac{\alpha_2}{2} - \frac{\alpha_2 c_\beta c_p^2}{\varepsilon_{12}})\ |\ \mathbf{w}_1^k - \mathbf{w}_1^{k-1}\ |_1^2))$$
$$+ \alpha_1 \sum_{s=1}^{k-1}(|\ \mathbf{w}_1^{k-s}\ |_1^2 - |\ \mathbf{w}_1^{k-s-1}\ |_1^2 + |\ c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1}\ |_1^2 - |\ c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s}\ |_1^2) + 2\nu c_\beta\ |\ \mathbf{w}_1^k\ |_1^2 \leqslant$$
$$+ c_\beta((\varepsilon_1 + \varepsilon_8 + \varepsilon_9 + \varepsilon_{11}) + \sqrt{\Delta t}(\varepsilon_5 + \varepsilon_{13}) + \alpha_2(\varepsilon_2 + \varepsilon_{10} + \varepsilon_{12}) + \varepsilon_3\alpha_1 + \varepsilon_6\alpha_2\sqrt{\Delta t})\ |\ \mathbf{w}_1^k\ |_1^2$$
$$+ c_\beta(\frac{\alpha_2}{\varepsilon_{10}} + \frac{h^4}{\varepsilon_9} + \varepsilon_7\alpha_2 h^2)\ |\ \mathbf{w}_1^{k-1}\ |_1^2 + (\frac{1}{\varepsilon_4}\Delta t^{1-\beta}(1 + \alpha_1) + c_\beta(\frac{1}{\varepsilon_1} + \frac{\alpha_2}{\varepsilon_2}))\|\mathbf{w}_1^k\|_0^2$$
$$+ \frac{c\sqrt{\Delta t}}{\varepsilon_{13}}(C_{1,k}h^3 + c_1 C_{3,k})^2 + \frac{\alpha_1}{\varepsilon_3}\Delta t^{1-\beta}C_{4,k}^2 + \frac{c_\beta\alpha_2 h^2}{\varepsilon_7} + \frac{c_\beta(h^3 + h^2 + h + h\alpha_2)^2}{\varepsilon_8} + \alpha_2 c_\beta^2 c_p^2$$
$$+ \alpha_2 c_\beta(h^3 + h^2 + h) + c_\beta(1 + \alpha_2)\sqrt{\Delta t}C_{3,k} + \frac{c_\beta\sqrt{\Delta t}C_{3,k}^2}{\varepsilon_5} + \frac{\alpha_2 c_\beta\sqrt{\Delta t}C_{3,k}^2}{\varepsilon_6} + \varepsilon_4\Delta t^{1-\beta}C_{2,k}^2(1 + \alpha_1)$$

Taking values for epsilons :

$$\varepsilon_1 = \varepsilon_8 = \varepsilon_9 = \varepsilon_{11} = \frac{\nu}{8}$$
$$\varepsilon_2 = \varepsilon_{10} = \varepsilon_{12} = \frac{\nu}{6}$$
$$\varepsilon_5 = \varepsilon_{13} = \frac{\nu}{4}$$
$$\varepsilon_3 = \varepsilon_6 = \frac{\nu}{3}$$
$$\varepsilon_7 = 1$$
$$\varepsilon_4 = 2\nu$$

We take $(1 - \frac{8c_\beta}{\nu}) > 0$, $(\alpha_1 - \frac{\alpha_2}{2} - \frac{6\alpha_2 c_\beta c_p^2}{\nu}) > 0$, $\Delta t \leq 1$, $\beta_0 = Min(\beta, 1-\beta)$
By replacing each value of epsilon and minoration of positive terms leads to :

$$(1 - c\Delta t^{\beta_0}\frac{(2 + \alpha_1 + \alpha_2)}{2\nu})\|\mathbf{w}_1^k\|_0^2 - \|\mathbf{w}_1^{k-1}\|_0^2 + +\alpha_1(|\ \mathbf{w}_1^k\ |_1^2 - |\ \mathbf{w}_1^{k-1}\ |_1^2)$$
$$+ \sum_{s=1}^{k-1}(\|\mathbf{w}_1^{k-s}\|_0^2 - \|\mathbf{w}_1^{k-s-1}\|_0^2 + \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1}\|_0^2 - \|c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s}\|_0^2)$$
$$+ \alpha_1 \sum_{s=1}^{k-1}(|\ \mathbf{w}_1^{k-s}\ |_1^2 - |\ \mathbf{w}_1^{k-s-1}\ |_1^2 + |\ c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s-1}\ |_1^2 - |\ c_s\mathbf{w}_1^k - \mathbf{w}_1^{k-s}\ |_1^2)$$
$$+ \nu c_\beta(1 - \frac{2\alpha_2 + \alpha_1}{3})\ |\ \mathbf{w}_1^k\ |_1^2 \leqslant\ +c_\beta(\frac{\alpha_2}{\nu} + \frac{h^4}{\nu} + \alpha_2 h^2)\ |\ \mathbf{w}_1^{k-1}\ |_1^2 + \frac{\sqrt{\Delta t}}{\nu}(C_{1,k}h^3 + c_1 C_{3,k})^2$$
$$+ \frac{\alpha_1}{\nu}\Delta t^{1-\beta}C_{4,k}^2 + c_\beta\alpha_2 h^2 + \frac{c_\beta(h^3 + h^2 + h + h\alpha_2)^2}{\nu} + \alpha_2 c_\beta^2 c_p^2 + \alpha_2 c_\beta(h^3 + h^2 + h)$$
$$+ c_\beta(1 + \alpha_2)\sqrt{\Delta t}C_{3,k} + \frac{c_\beta\sqrt{\Delta t}C_{3,k}^2}{\nu} + \frac{\alpha_2 c_\beta\sqrt{\Delta t}C_{3,k}^2}{\nu} + \Delta t^{1-\beta}C_{2,k}^2(1 + \alpha_1)$$

The summation on k=2 to m and minorations of positive terms leads to

$$\left(1 - c\Delta t^{\beta_0}\frac{(2+\alpha_1+\alpha_2)}{2\nu}\right)\|\mathbf{w}_1^m\|_0^2 - \|\mathbf{w}_1^1\|_0^2 + 2\alpha_1(|\ \mathbf{w}_1^m\ |_1^2 - |\ \mathbf{w}_1^1\ |_1^2)$$
$$+ c_\beta\left(\nu(1 - \frac{2\alpha_2+\alpha_1}{3}) - \frac{\alpha_2}{\nu} - \frac{h^4}{\nu} - \alpha_2 h^2\right)|\ \mathbf{w}_1^k\ |_1^2 \leqslant\ c_\beta\left(\frac{\alpha_2}{\nu} + \frac{h^4}{\nu} + \alpha_2 h^2\right)|\ \mathbf{w}_1^1\ |_1^2$$
$$+ \left(c\Delta t^{\beta_0}\frac{(2+\alpha_1+\alpha_2)}{2\nu}\right)\sum_{k=2}^{m}\|\mathbf{w}_1^k\|_0^2 + c(\alpha_1\Delta t^{1-\beta}\sum_{k=1}^{m}C_{4,k}^2 + \sqrt{\Delta t}h^6\sum_{k=1}^{m}C_{1,k}^2 + \sqrt{\Delta t}c_1^2\sum_{k=1}^{m}C_{3,k}^2$$
$$+ ((h^3+h^2+h)^2 + h^2\alpha_2^2)\sum_{k=1}^{m}\Delta t^\beta + (1+\alpha_2)\Delta t^\beta\sqrt{\Delta t}\sum_{k=1}^{m}C_{3,k}^2 + (\alpha_2 h^2 + \alpha_2\Delta t^\beta + \alpha_2(h^3+h^2+h)$$
$$+ \sqrt{\Delta t}(1+\alpha_2))\sum_{k=1}^{m}\Delta t^\beta + (1+\alpha_1)\Delta t^{1-\beta}\sum_{k=1}^{m}C_{2,k}^2$$

Using the majoration on $\sum_k C_{i,k}^2$ leads to

$$\left(1 - c\Delta t^{\beta_0}\frac{(2+\alpha_1+\alpha_2)}{2\nu}\right)\|\mathbf{w}_1^m\|_0^2 - \|\mathbf{w}_1^1\|_0^2 + 2\alpha_1(|\ \mathbf{w}_1^m\ |_1^2 - |\ \mathbf{w}_1^1\ |_1^2)$$
$$+ c_\beta\left(\nu(1 - \frac{2\alpha_2+\alpha_1}{3}) - \frac{\alpha_2}{\nu} - \frac{h^4}{\nu} - \alpha_2 h^2\right)\sum_{k=2}^{m}|\ \mathbf{w}_1^k\ |_1^2 \leqslant c_\beta\left(\frac{\alpha_2}{\nu} + \frac{h^4}{\nu} + \alpha_2 h^2\right)|\ \mathbf{w}_1^1\ |_1^2$$
$$+ \left(c\Delta t^{\beta_0}\frac{(2+\alpha_1+\alpha_2)}{2\nu}\right)\sum_{k=2}^{m-1}\|\mathbf{w}_1^k\|_0^2 + c(\alpha_1\Delta t^{1-\beta} + \sqrt{\Delta t}h^6 + \sqrt{\Delta t}c_1^2 + ((h^3+h^2+h)^2 + h^2\alpha_2^2)$$
$$+ (1+\alpha_2)\Delta t^\beta\sqrt{\Delta t} + (\alpha_2 h^2 + \alpha_2\Delta t^\beta + \alpha_2(h^3+h^2+h) + \sqrt{\Delta t}(1+\alpha_2)) + (1+\alpha_1)\Delta t^{1-\beta})$$

Let us introduce these inequations :

$$1 - c\Delta t^{\beta_0}\frac{(2+\alpha_1+\alpha_2)}{2\nu} > \frac{1}{2},$$
$$\kappa = \nu(1 - \frac{2\alpha_2+\alpha_1}{3}) - \frac{\alpha_2}{\nu} - \frac{h^4}{\nu} - \alpha_2 h^2 >\ 0,$$

and also

$$(1 - \frac{8c_\beta}{\nu})\ > 0,$$
$$(\alpha_1 - \frac{\alpha_2}{2} - \frac{6\alpha_2 c_\beta c_p^2}{\nu})\ >\ 0.$$

These conditions holds if :

$$\Delta t^\beta\ \leq Min\left(\frac{\nu}{8\Gamma(2-\beta)}, \frac{\nu}{2+\alpha_1+\alpha_2}, \frac{\nu(2\alpha_1-\alpha_2)}{12\alpha_2 c_p^2\Gamma(2-\beta)}\right),$$
$$h\ \leq \sqrt{\frac{(1-\frac{2\alpha_2+\alpha_1}{3})\nu^2 - \alpha_2}{\alpha_2\nu}},$$
$$\nu\ \geq \sqrt{\frac{4\alpha_2}{\alpha_2^2 + 4(1-\frac{2\alpha_2+\alpha_1}{3})}},$$
$$\alpha_1\ > \frac{\alpha_2}{2}.$$

$$\frac{1}{2}\|\mathbf{w}_1^m\|_0^2 + \kappa\sum_{k=2}^{m}|\ \mathbf{w}_1^k\ |_1^2 \leqslant c\ |\ \mathbf{w}_1^1\ |_1^2 + \left(c\Delta t^{\beta_0}\frac{(2+\alpha_1+\alpha_2)}{2\nu}\right)\sum_{k=2}^{m-1}\|\mathbf{w}_1^k\|_0^2$$
$$+ c\left((h^3+\Delta t^\beta)^2 + \Delta t^{\beta'} + \alpha_2((h^3+\Delta t^\beta + \sqrt{\Delta t} + \Delta t^\beta\sqrt{\Delta t}) + c_1^2\sqrt{\Delta t} + (\alpha_2)^2h^2 + \alpha_1\Delta t^{\beta'}\right)$$

Based with lemma of Gronwall discreet, the following inequation is resulted :

$$\frac{1}{2}\|\mathbf{w}_1^m\|_0^2 + \kappa \sum_{k=2}^{m} |\ \mathbf{w}_1^k\ |_1^2 \leqslant$$

$$c\left( |\ \mathbf{w}_1^1\ |_1^2 + (h^3 + \Delta t^\beta)^2 + \Delta t^{\beta'} + \alpha_2(h^3 + \Delta t^\beta + \sqrt{\Delta t} + \Delta t^\beta \sqrt{\Delta t}) + c_1^2\sqrt{\Delta t} + \alpha_2^2 h^2 + \alpha_1 \Delta t^{\beta'} \right)$$

## 6 Numerical results

The study of the flow of this new fluid model involves the integration of various physical and fluid variables that influence the fluid velocity and pressure. The following numerical solutions are obtained by implementing the procedure described in this article, introduced in the open source finite element software FreeFEM++ with the use of the P2-P3 libraries. To show the effectiveness of the proposed numerical scheme, the comparison of the numerically obtained results with the available exact ones is first done. Then, the famous lid-driven cavity of polymer aqueous solutions is studied. The obtained results are analyzed graphically.

### 6.1 Exact solution

Fore the sake of comparison with an analytically exact solution of the main problem (1.1) we introduce the following components of the velocity

$$u_1(x,y,t) = -x^2(x-1)^2 y(y-1)(2y-1)t,$$
$$u_2(x,y,t) = y^2(y-1)^2 x(x-1)(2x-1)t,$$
$$p(x,y,t) = (x-0.5)(y-0.5).$$

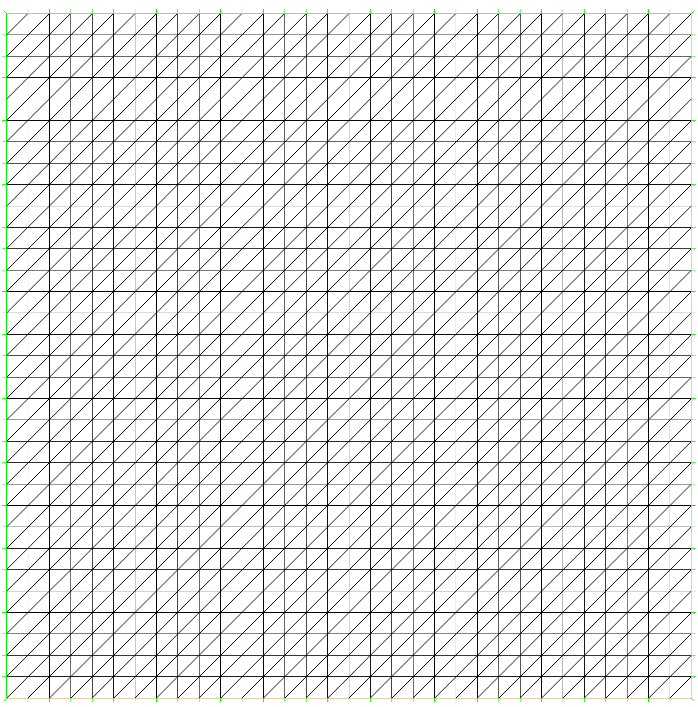

**Figure 1.** Uniform regular triangulation of the square domaine with h=1/32

These velocities satisfy the two conditions of (1.1). By substituting into equation (1.1), the following right-hand sides **f**=($f_1$, $f_2$) are obtained and will be used in the numerical scheme.

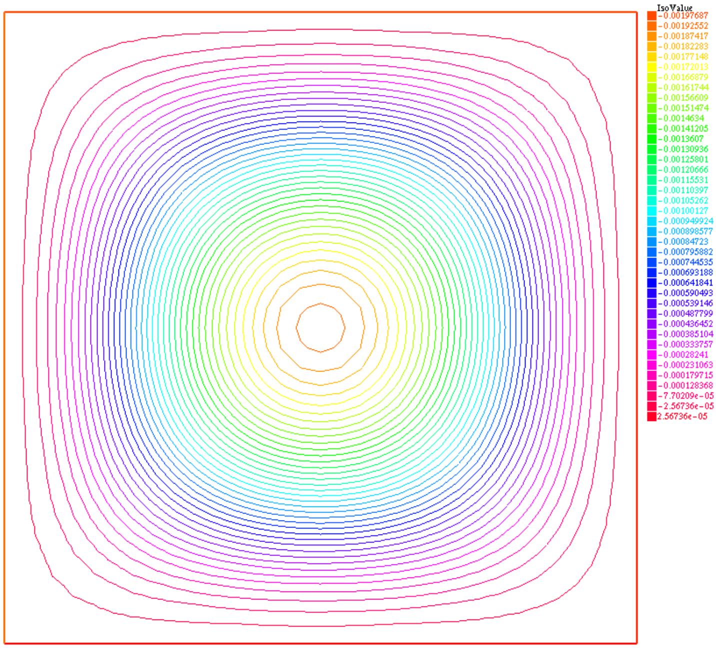

**Figure 2.** Steamlines of the exact solution for $\nu = 100$, $\Delta t = \frac{1}{2^5}$, $h = \frac{1}{2^5}$, $\beta = 0.5$,$\alpha_1 = \alpha_2 = 0.01$

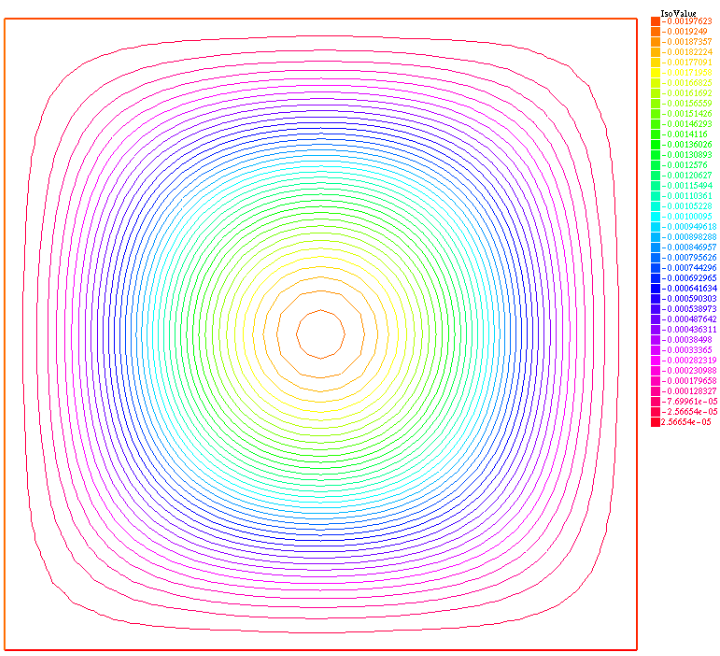

**Figure 3.** Steamlines of the numerical solution for $\nu = 100$, $\Delta t = \frac{1}{2^5}$, $h = \frac{1}{2^5}$, $\beta = 0.5$,$\alpha_1 = \alpha_2 = 0.01$

Fig.1 represents the regular and uniform mesh used to solve the problem in a square domain, respecting the conditions taken into account to preserve stability and convergence results. In Figure 9, we present the resulting errors of velocity by comparing the exact solution to the approximated ones using the $L^2$ norm for different values of spatial and temporal discretizations h and $\Delta t$ with $\nu = 100$, $\alpha_1 = 0.02$, $\alpha_2 = 0.01$, $\beta = 0.5$. It can be clearly observed from Fig.9 that

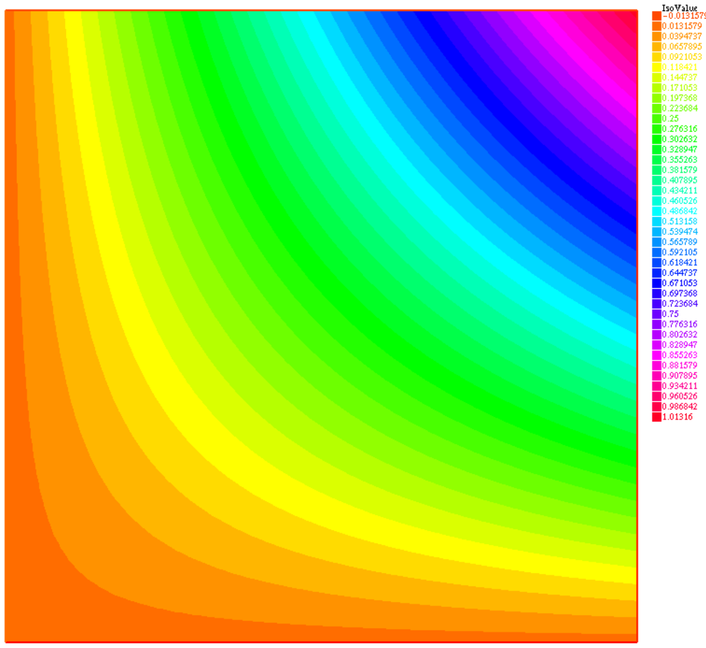


**Figure 4.** Pressure contours of the exact solution for $\nu = 100$, $\Delta t = \frac{1}{2^5}$, $h = \frac{1}{2^5}$, $\beta = 0.5$,$\alpha_1 = \alpha_2 = 0.01$

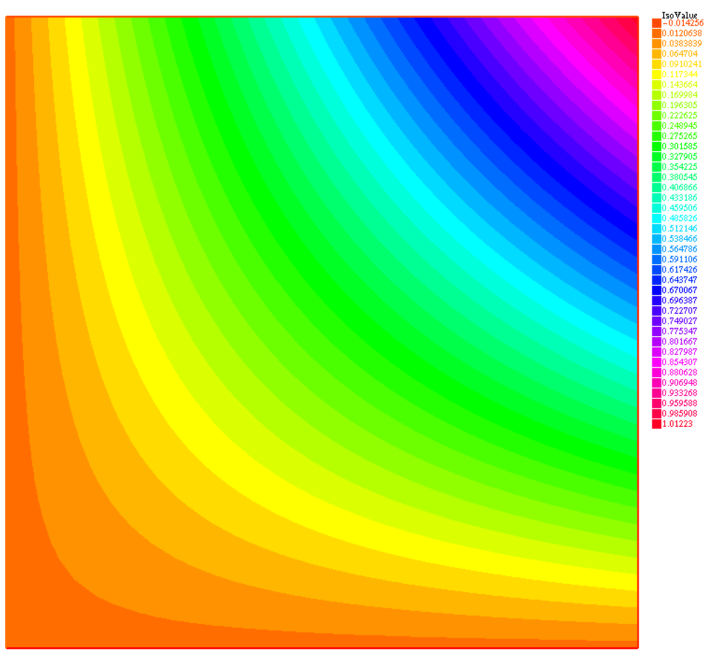


**Figure 5.** Pressure contours of the numerical solution for $\nu = 100$, $\Delta t = \frac{1}{2^5}$, $h = \frac{1}{2^5}$, $\beta = 0.5$,$\alpha_1 = \alpha_2 = 0.01$

the scheme converges perfectly to the exact solution when the time step $\Delta t$ is decreased. Figures 4, 5, 2, 3, show, respectively, the streamlines and pressure contours obtained from the exact and numerical solutions of the problem for fixed values of $\alpha$ and h. It is clearly demonstrated that the streamlines are almost the same and the numerical scheme converges to the exact solution presented below.

To validate the stability criterion, Figure 16 represents the error variation of the velocity over

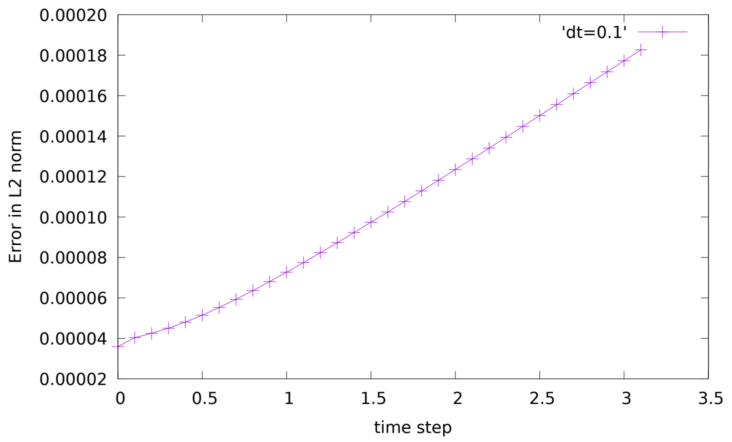


**Figure 6.** *

**(a)**

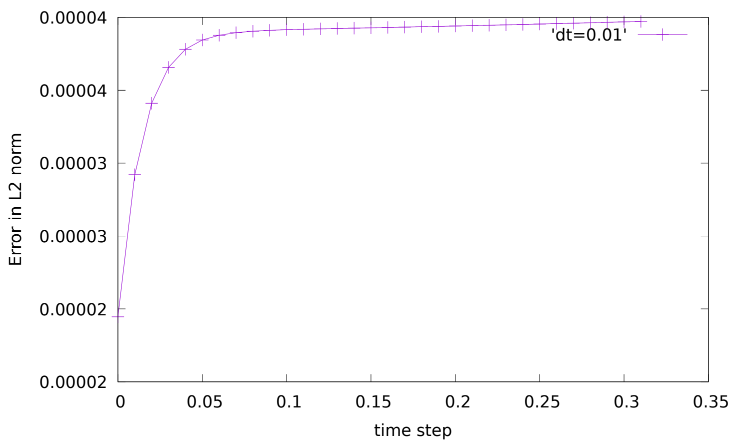


**Figure 7.** *

**(b)**

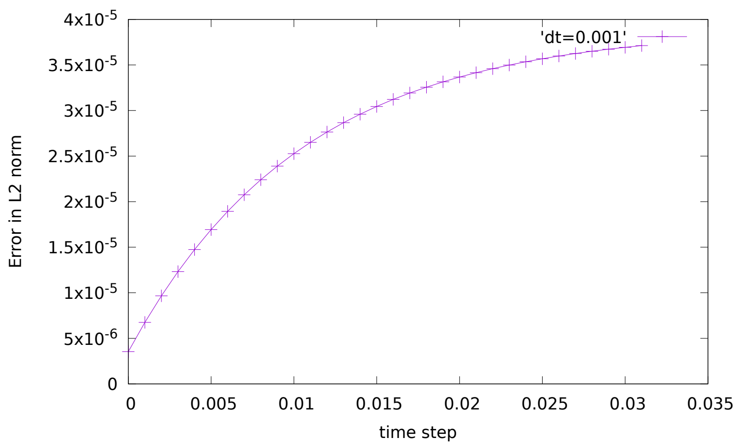


**Figure 8.** *

**(c)**

**Figure 9.** Error $L^2$ for (a) $\Delta t = 0.1$, (b) $\Delta t = 0.01$, (c)$\Delta t = 0.001$, with $\nu = 100$, $\alpha_1 = 0.02$, $\alpha_2 = 0.01$,$\beta = 0.5$ , $h = \frac{1}{2^5}$

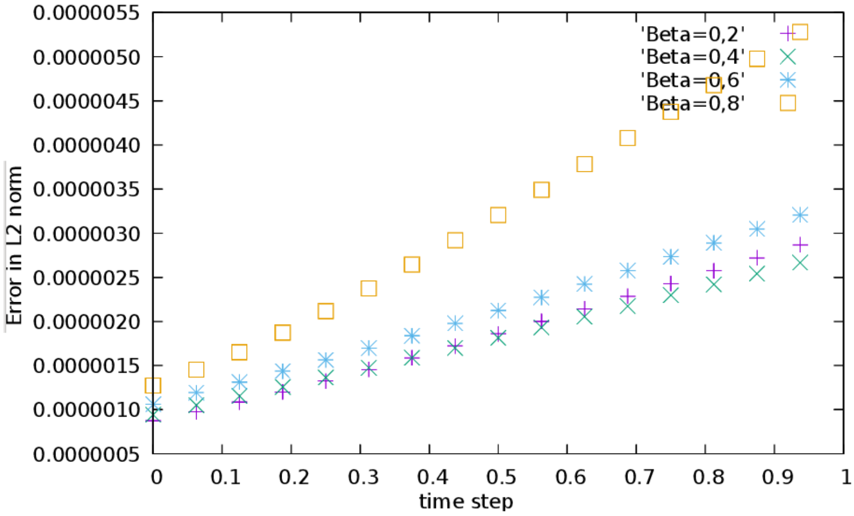


**Figure 10.** Errors at variatious $\beta$ with respect to time iterations for $\nu = 100$, $\Delta t = \frac{1}{2^4}$, $h = \frac{1}{2^5}$, $\alpha_1 = 0.02,\ \alpha_2 = 0.01$

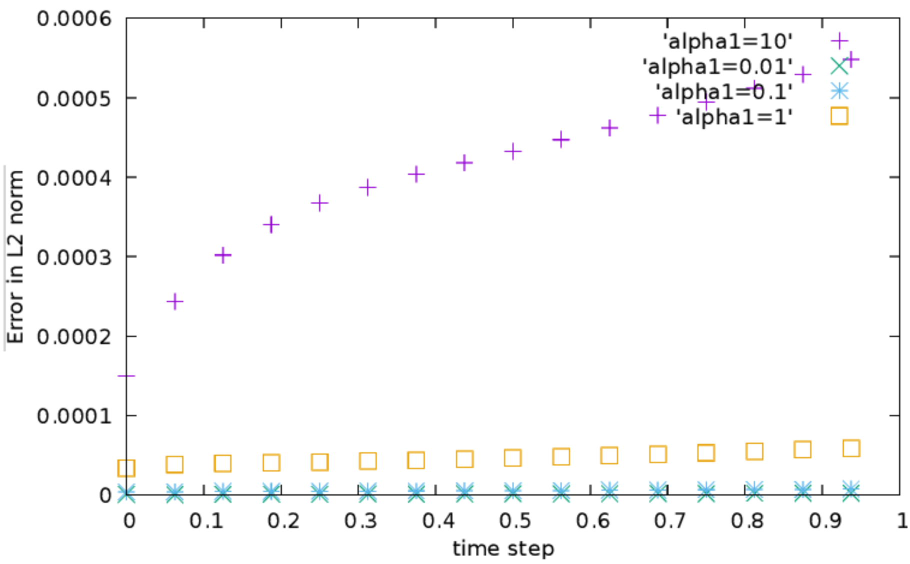


**Figure 11.** Errors at variatious $\alpha_1$ with respect to time iterations for $\nu = 100$, $\Delta t = \frac{1}{2^4}$, $h = \frac{1}{2^5}$, $\beta = 0.5$,$\alpha_2 = 0.2$

time for different values of the fractional order $\beta$. To show non-stability, Figure 20represents the error variation with respect to time for fixed values of viscosity and time step and for various values of the fractional order $\beta$ that do not follow the stability condition. When $\nu \to \infty$ and $\alpha_1 >> \alpha_2$, we obtain that the scheme is unconditionally stable for $0 \leq \Delta t \leqslant \infty$ and for all h, see Figure 12 as well as for $\alpha_1 \approx 0$. The error is minimal for $\beta = 0.4 \pm \varepsilon$ see Fig. 10, and in Figure 11, allow us tu deduce that when $\alpha_1 \to 0$ the error tends to zero.

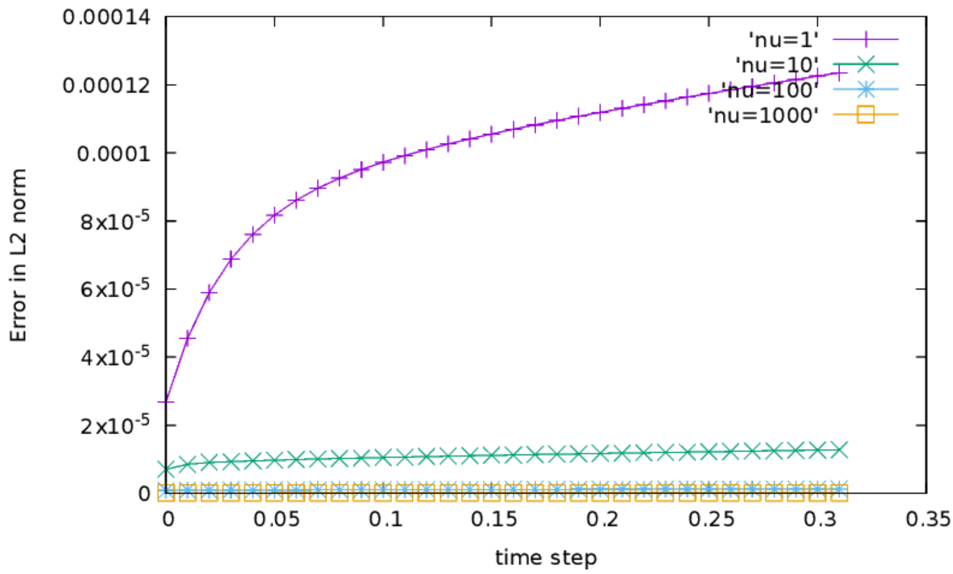


**Figure 12.** Errors at variatious viscosity with respect to time iterations for $\Delta t = 0.01$, $h = \frac{1}{2^5}$, $\beta = 0.5$,$\alpha_1 = 0.02$, $\alpha_2 = 0.01$

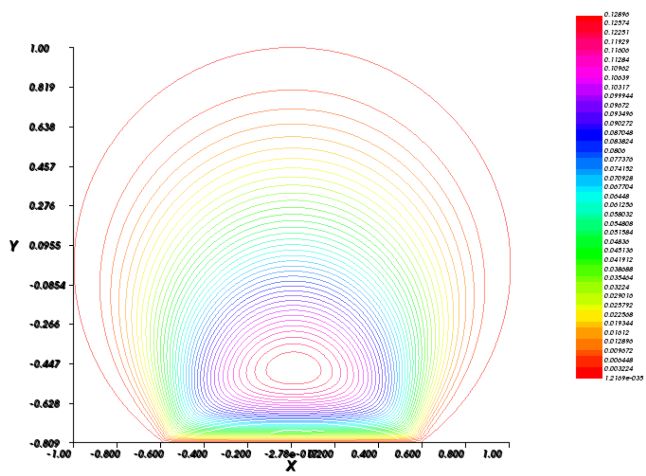


**Figure 22.** *

**(a)**

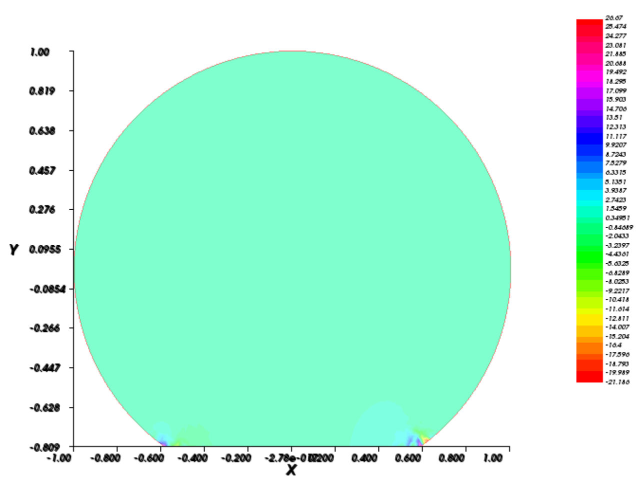


**Figure 23.** *

**(b)**

**Figure 24.** Streamlines (a) and pressure contour (b) of the numerical solution for the lid-driven cavity

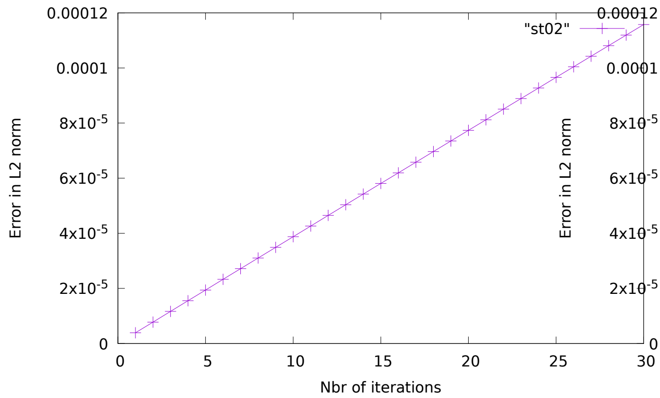


**Figure 13.** *
(a)

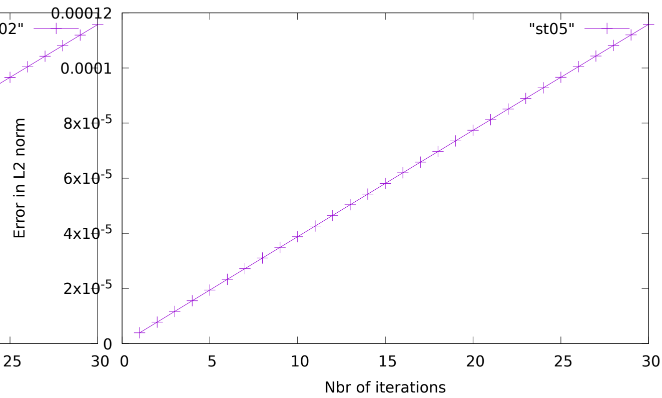


**Figure 14.** *
(b)

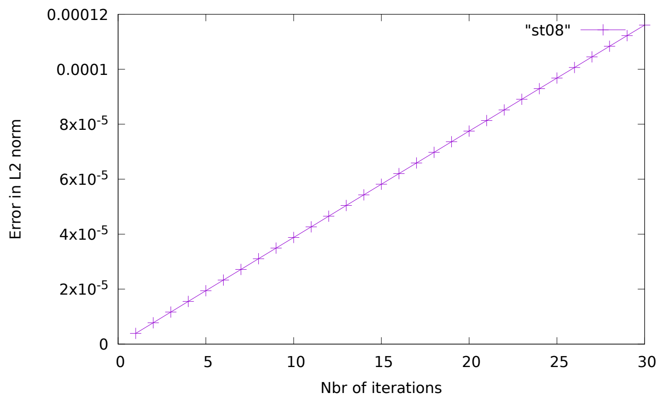


**Figure 15.** *
(c)

**Figure 16.** Error $L^2$ for **a**.$\beta = 0.2$, **b**.$\beta = 0.5$, **c**.$\beta = 0.8$, with $\nu = 0.01$, $\Delta t = 0.001$, $h = \frac{1}{2^5}$, $\alpha_1 = 0.02$, $\alpha_2 = 0.01$

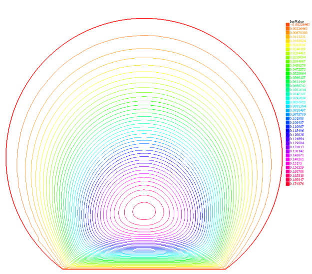

**Figure 25.** *
**(a)**

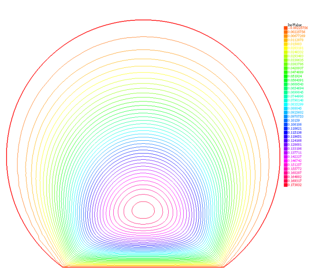

**Figure 26.** *
**(b)**

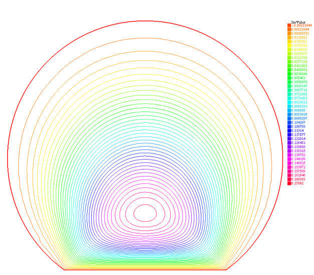

**Figure 27.** *
**(c)**

**Figure 28.** (**P**)Streamlines for, (a) $\beta = 0.1$, (b) $\beta = 0.5$, (c)$\beta = 0.9$, with $\nu = 0.01$, $\alpha_1 = 1$, $\alpha_2 = 0.01$, $\Delta t = 0.001$, $h = \frac{1}{2^5}$

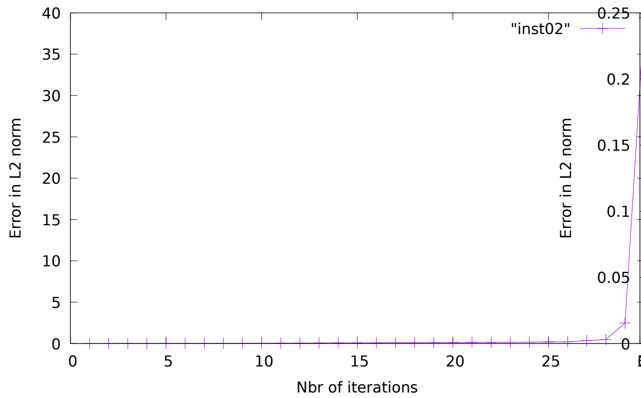


**Figure 17.** *
(a)

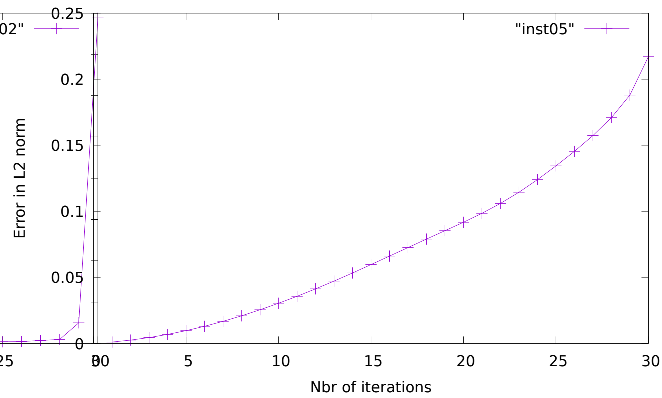


**Figure 18.** *
(b)

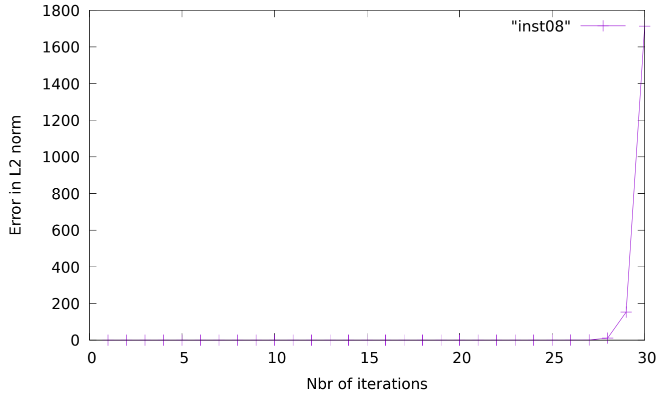


**Figure 19.** *
(c)

**Figure 20.** Error $L^2$ for **a**.$\beta = 0.2$, **b**.$\beta = 0.5$, **c**.$\beta = 0.8$, with $\nu = 0.001$, $\Delta t = 0.2$, $h = \frac{1}{2^5}$, $\alpha_1 = 0.02$, $\alpha_2 = 0.01$

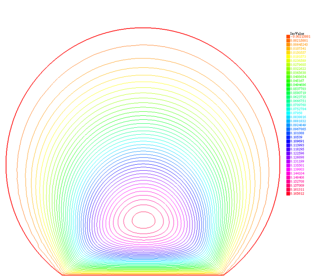

**Figure 29.** *
**(a)**

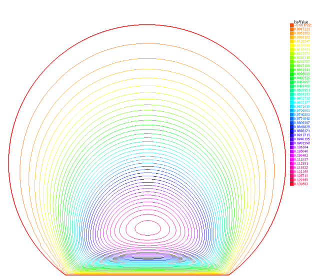

**Figure 30.** *
**(b)**

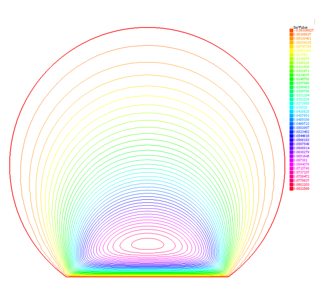

**Figure 31.** *
**(c)**

**Figure 32.** **(FKV)** Streamlines for, (a) $\beta = 0.1$, (b) $\beta = 0.5$, (c)$\beta = 0.9$, with $\nu = 0.01$, $\alpha_1 = 0.01$, $\alpha_2 = 0$, $\Delta t = 0.001$, $h = \frac{1}{2^5}$

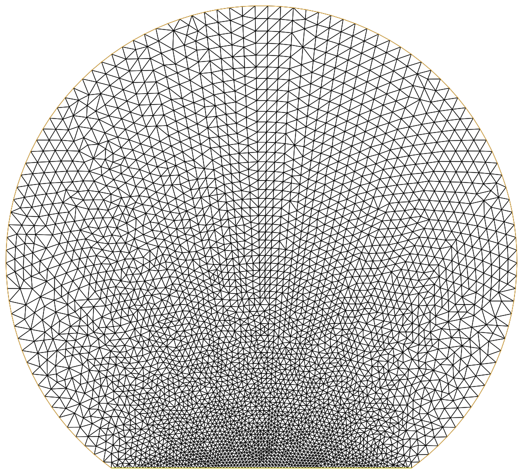

**Figure 21.** Non-regular triangulation of the complex two-dimensional domain

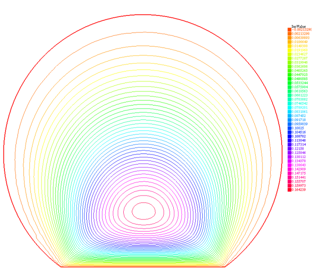

**Figure 33.** *
**(a)**

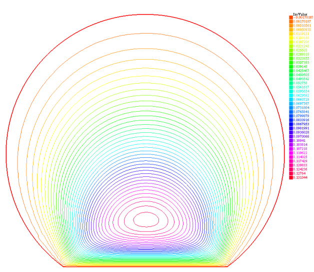

**Figure 34.** *
**(b)**

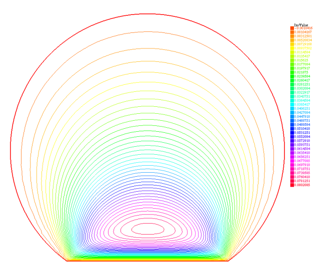

**Figure 35.** *
**(c)**

**Figure 36.** (**FAQS**)Streamlines for, (a) $\beta = 0.1$, (b) $\beta = 0.5$, (c)$\beta = 0.9$, with $\nu = 0.01$, $\alpha_1 = \alpha_2 = 0.01$, $\Delta t = 0.001$, $h = \frac{1}{2^5}$

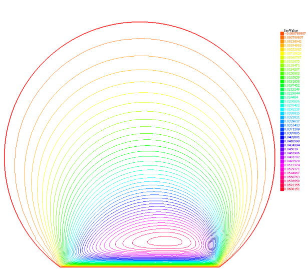

**Figure 37.** *
**(a)**

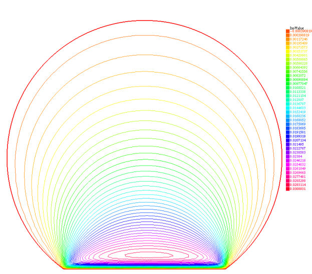

**Figure 38.** *
**(b)**

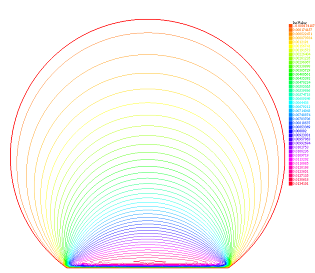

**Figure 39.** *
**(c)**

**Figure 40.** (**FNSE**) Streamlines for, (a) $\beta = 0.1$, (b) $\beta = 0.5$, (c)$\beta = 0.9$, with $\nu = 0.01$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\Delta t = 0.001$, $h = \frac{1}{2^5}$

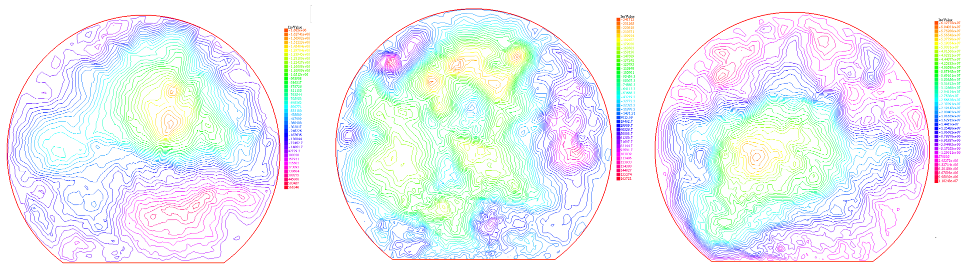

**Figure 41.** * **Figure 42.** * **Figure 43.** *

**(a)** **(b)** **(c)**

**Figure 44.** ($\alpha_1 < \alpha_2$) Streamlines for, (a) $\beta = 0.1$, (b) $\beta = 0.5$, (c)$\beta = 0.9$, with $\nu = 0.01$, $\alpha_1 = 0.01$, $\alpha_2 = 1$, $\Delta t = 0.001$, $h = \frac{1}{2^5}$

### 6.2 Application : Lid-driven cavity

As an application to a real problem, we consider the well-known lid-driven cavity problem. This example allows to test the effectiveness of the numerical scheme elaborated in this paper. The scheme has been applied with the following conditions: $\Delta t = 0.001$ as time step and h = 0.0707 as the maximum triangle size of the domain mesh. Figure 24 present streamlines and pressure contours numerically obtained. It is clearly shown that the flow behavior is different from one fluid to another and it is observed that the flow creates parallel rolls. The flow is similar for, Fractional Aqueuse Solution of Polymers **(FASP)**, Fractional Kelvin Voigt **(FKV)** and Fractional Navier-Stokes Equations **(FNSE)**, but they differ in the position of the center of the rolls. Additionally, higher pressure values occur near the end of the cavity and low values appear at the entrance of the cavity. It is clearly observed from these figures that the fractional parameter $\beta$ has a strong effect on the dynamic fluid flow. An increase of the value of $\beta$ slows down the flow in which the center of the fluid circulation tends to the center of the domain slowly, see Figs. 32,40,28,36. Instability is observed when the condition $(\alpha_1 > \frac{\alpha_2}{2})$ is not satisfied, see Fig.44.

## 7 Conclusion

A mathematical model has been elaborated to study the dynamic behavior of new model of fluid flow. A semi-implicit numerical scheme with fractional derivative order is introduced and analyzed. The effects of various geometric and fluid parameters on the problem are studied. The stability and convergence results of the presented scheme were demonstrated. A finite element numerical code has been elaborated based on the open source software **FreeFEM++**. An application to the case of a lid-driven cavity is made to study the influence of the fractional parameter $\beta$ on the flow. The obtained main results are summarized as follows : – A new form $c$ of the nonlinear term is given. This new form enabled us to obtain the coercivity and to be able to prove the existence and uniqueness in the discrete case, – to satisfy the convergence, we need to reserve the following conditions :

$$\Delta t^{\beta} \leq min(\frac{\nu}{8\Gamma(2-\beta)}, \frac{\nu}{2+\alpha_1+\alpha_2}, \frac{\nu(2\alpha_1-\alpha_2)}{12\alpha_2 c_p^2\Gamma(2-\beta)}), \tag{7.1}$$

and

$$h \leq \sqrt{\frac{(1-\frac{2\alpha_2+\alpha_1}{3})\nu^2-\alpha_2}{\alpha_2\nu}}, \tag{7.2}$$

$$\nu \geq \sqrt{\frac{4\alpha_2}{\alpha_2^2+4(1-\frac{2\alpha_2+\alpha_1}{3})}}, \tag{7.3}$$

$$\alpha_1 > \frac{\alpha_2}{2}. \tag{7.4}$$

We can clearly see that in condition (7.4) of the figure (44), there is no convergence.

## References

[1] C.F.M. Coimbra. *Mechanics with variable-order differential operators* Annalen der Physik, **12(1112)**:692–703, (2003).

[2] L. I. Sedov and P. G. Hodge. *Introduction to the Mechanics of a Continuous Medium*. Journal of Applied Mechanics, **33(1)**:238, (1966).

[3] A. P. Oskolkov. *The uniqueness and global solvability of boundary-value problems for the equations of motion for aqueous solutions of polymers*. J. Sovi. Math., **8**:427–455, (1977).

[4] AP Oskolkov. *A certain nonstationary quasilinear system with a small parameter that regularizes the system of navier-stokes equations*. Probl. Mat. Anal, **4**:78–87, (1973).

[5] Anatolii Petrovich Oskolkov. *Certain model nonstationary systems in the theory of non-newtonian fluids*. Trudy Matematicheskogo Instituta imeni VA Steklova, **127**:32–57 (1975).

[6] Anatolii Petrovich Oskolkov. *Some nonstationary linear and quasilinear systems occurring in the investigation of the motion of viscous fluids*. Boundary value problems of mathematical physics and related questions in the theory of functions, **9(59)**:133–177, 1976.

[7] Anatolii Petrovich Oskolkov. *On the theory of nonstationary flows of nonlinear visco-elastlc liquids*. Zapiski Nauchnykh Seminarov POMI, **120**:142–158, (1982).

[8] Slimane Azoug, Hamzah Bakhti, Lahcen Azrar, and Tarik Ali-Ziane. *Numerical analysis of a semi-implicit scheme for non-Newtonian fluid flows of polymer aqueous solutions*. Submitted to journal of Applied Mathematics, (2018).

[9] C. Amrouche and E. H. Ouazar. Energy solutions for polymer aqueous solutions in two dimension. Nonlinear Analysis, **68**:3233–3245, (2008).

[10] Evgenii S Baranovskii and Mikhail A Artemov. Model for aqueous polymer solutions with damping term: Solvability and vanishing relaxation limit. Polymers, **14(18)**:3789, (2022).

[11] Jacques-Louis Lions. *Quelques résultats d'existence dans des équations aux dérivées partielles non linéaires*. Bulletin de la Société Mathématique de France, **87**:245–273, (1959).

[12] Jun Zhang and JinRong Wang. *Numerical analysis for navier–stokes equations with time fractional derivatives*. Applied Mathematics and Computation, **336**:481–489, (2018).

[13] Sunil Kumar, Deepak Kumar, Saeid Abbasbandy, and MM Rashidi. *Analytical solution of fractional navier–stokes equation by using modified laplace decomposition method*. Ain Shams Engineering Journal, **5(2)**:569–574, (2014).

[14] Faraj Y Ishak. *Existence and uniqueness solution for fractional volterra equation with fractional antiperiodic boundary conditions*. Palestine Journal of Mathematics, **12(1)**, (2023).

[15] JC Prajapati, AD Patel, KN Pathak, and AK Shukla. *Fractional calculus approach in the study of instability phenomenon in fluid dynamics*. Palestine Journal of Mathematics, **1(2)**:95–103, (2012).

[16] Ali Khalouta. *Closed-form solutions to some nonlinear fractional partial differential equations arising in mathematical sciences.* Palestine Journal of Mathematics, **11**:113–126, (2022).

[17] Reshu Agarwal. *An analytical study of non-newtonian visco-inelastic fluid flowbetween two stretchable rotating disks*. Palestine Journal of Mathematics, **11**, (2022).

[18] Slimane Azoug, Hamzah Bakhti, and Lahcen Azrar. *Blood flow simulation through two-dimensional complex stenosed arteries using viscoelastic oldroyd-b fluid*. In 2018 International Conference on Electronics, Control, Optimization and Computer Science (ICECOCS), pages 1–4. IEEE, (2018).

[19] Woldemar Voigt. *Ueber innere reibung fester körper, insbesondere der metalle*. Annalen der Physik, **283(12)**:671–693, (1892).

[20] Pedro D Damazio, Patricia Manholi, and Ana L Silvestre. *Lq-theory of the kelvin–voigt equations in bounded domains*. Journal of Differential Equations, **260(11)**:8242–8260, (2016).

[21] Evgenii S Baranovskii. *Strong solutions of the incompressible navier–stokes–voigt model*. Mathematics, 8(2):181, (2020).

[22] Slimane Azoug, Hamzah Bakhti, Lahcen Azrar, and Tarik Ali-Ziane. *Stability and convergence analysis of a semi-implicit fractional fem-scheme for non-newtonian fluid flows of polymer aqueous solutions with fractional time-derivative*. Computational and Applied Mathematics, **39(2)**:1–32, (2020).

[23] P. Zhuang, F. Liu, V. Anh, and I. Turner. *Numerical Methods for the Variable-Order Fractional Advection-Diffusion Equation with a Nonlinear Source Term.* SIAM Journal on Numerical Analysis, **47(3)**, :1760–1781, (2009).

## Author information

S. Azoug, ESGEN, Higher Shool of Management and Digital Economy, Kolea, Algeria
AMNEDP, Faculty of Mathematics, USTHB, Algiers, Algeria.
E-mail: az_slimane@hotmail.com

A. Ouardani, AMNEDP, Faculty of Mathematics, USTHB, Algiers, Algeria.
E-mail: asmalouardani@yahoo.fr

T. Aliziane, AMNEDP, Faculty of Mathematics, USTHB, Algiers, Algeria.
E-mail: mohamed_taliziane@gmail.com

Received: 2023-05-24
Accepted: 2023-12-16