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CONFORMAL RICCI SOLITON IN SASAKIAN MANIFOLD

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Abstract *The object of the present paper is to study the conformal curvature tensor, the conharmonic curvature tensor, the Ricci curvature tensor and the projective curvature tensor in the Sasakian manifold, admitting conformal Ricci Soliton. We have studied the conformally semi-symmetric Sasakian manifold, admitting conformal Ricci soliton. We have found that Ricci conharmonically symmetric Sasakian manifold admitting conformal Ricci soliton is a quadratic equation. We have proved that a conformally symmetric Sasakian manifold M with respect to projective curvature tensor admitting conformal Ricci soliton is an η Einstein manifold. We have also studied Ricci projectively symmetric sasakian manifold.*

1 Introduction

The Ricci flow concept and its proof were introduced by Hamilton [19] in the year 1982. It was degined to answer the Thurston's geometric conjecture, according to it each closed three manifold admits a geometric decomposition. Categorization of all compact manifold with positive curvature operator in the fourth dimension was done by Hamilton [21]. After which, the Ricci flow became one of the most powerful tools to study Riemannian manifolds, especially those having positive curvatures.

The Ricci flow is presented as

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

for a compact Riemannian manifold M with Riemannian metric g. Ricci soliton has come as the limit of the solutions of Ricci flow [8, 9, 10]. The solution for the Ricci flow is known as a Ricci soliton in case it moves only by a one parameter group of diffeomorphism and scaling. Ramesh Sharma [22, 3] begin study of Ricci soliton for compact manifold and later it was studied by M. M. Tripathi [16], Bejan, Crasmareanu [4] analysed Ricci soliton in contact metric manifolds. The Ricci soliton equation is presented as

$$\pounds_X g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar.

A.E. Fischer [1] in the year 2005 established a new concept known as conformal Ricci flow, a variation of the classical Ricci flow equation which has revised the unit volume constraint of that equation to a scalar curvature constraint. After that a conformal geometry has played a prevalent role to constraint the scalar curvature and equation are vector field sum of a conformal flow equation and a Ricci flow equation, the equations resulting to this is said as conformal Ricci flow equations. The new equations are presented as

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \tag{1.3}$$

and $R(g) = -1$, *where* p *is a scalar non-dynamical field (time dependent scalar field),* $R(g)$ *is the scalar curvature of the manifold and* n *is the dimension of manifold.*

N. Basu and A. Bhattacharyya [28, 31] in 2015 brought the notion of conformal Ricci soliton and the equation is as follows

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{1.4}$$

The above equation is the generalization of the Ricci soliton equation and it also assures the conformal Ricci flow equation [27, 30].

A Riemannian manifold is said to be locally symmetric if its curvature tensor R *satisfies* $\nabla R = 0$, *where* ∇ *is Levi-Civita connection on the Riemannian manifold. As a generalization of locally symmetric spaces, many geometers have considered semi symmetric spaces and their generalization. A Riemannian manifold is said to be semi symmetric if its curvature tensor* R *satisfies* $R(X,Y).R = 0$ *for all* $X, Y \in TM$, *where* $R(X,Y)$ *acts on* R *as a derivation.*

In this paper, we have studied conformal curvature tensor, conharmonic curvature tensor, Ricci curvature tensor, projective curvature tensor in Sasakian manifold [11, 12] admitting conformal Ricci soliton. We have studied Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).\widetilde{C} = 0$, *is* η *Einstein manifold.We have proved Sasakian manifold admitting conformal Ricci soliton and* $P(\xi, X).\widetilde{C} = 0$, *is* η *Einstein manifold. We have found that Sasakian manifold admitting conformal Ricci soliton and* $P(\xi, X).S = 0$, *is an Einstein manifold. We have studied Sasakian manifold admitting conformal Ricci soliton [23, 24, 25] and* $R(\xi, X).P = 0$, *is an Einstein manifold. We have found that a Ricci conharmonically symmetric Sasakian manifold admitting conformal Ricci soliton is a quadratic equation.*

2 PRELIMINARIES

Let M *be a* $(2n+1)$ *dimensional connected almost metric manifold with an almost contact metric structure* (ϕ, ξ, η, g) *where* ϕ *is a* $(1,1)$ *tensor field,* ξ *is a covariant vector field,* η *is a* $1-$ *form and* g *is compatible Riemannian metric [6, 7] such that*

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \tag{2.2}$$

$$\eta(\xi) = 1 \backslash \phi\xi = 0 \backslash \eta o \phi = 0 \backslash \eta(\phi X) = 0, \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.5}$$

for all $X, Y \in \chi(M)$.

If Sasakian manifold M *satisfies*

$$\bigtriangledown_X \xi = -\phi X, (\bigtriangledown_X \eta)Y = g(X, \phi Y), \tag{2.6}$$

where $\bigtriangledown$ *denotes the Riemannian connection in* M.

In a Sasakian manifold [18, 26] the following relations hold

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{2.7}$$

$$R(\xi, X)Y = (\bigtriangledown_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.9}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.10}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{2.11}$$

$$S(X,\xi) = 2n\eta(X), Q\xi = 2n\xi \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n\eta(X)\eta(Y) \tag{2.13}$$

where R is a Riemannian curvature, S is the Ricci tensor and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$ for all $X, Y \in \chi(M)$.

Now from definition of Lie derivative, we have

$$\pounds_\xi g(X,Y) = \bigtriangledown_\xi g(X,Y) + g(\bigtriangledown_\xi X, Y) + g(X, \bigtriangledown_\xi Y)$$

Using (2.3) and (2.6), we have

$$\pounds_\xi g(X,Y) = 0 \tag{2.14}$$

Applying conformal Ricci soliton equation (1.4) in (2.14), we obtain

$$S(X,Y) = Ag(X,Y) \tag{2.15}$$

where $A = \frac{1}{2}[2\lambda - (p + \frac{2}{n})]$, which shows that manifold is Einstein manifold. Also

$$QX = AX, \tag{2.16}$$

$$S(X,\xi) = A\eta(X), \tag{2.17}$$

$$S(\xi,\xi) = A, \tag{2.18}$$

Using these results we shall prove some important results of Sasakian manifold in the following sections.

Example *We consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(r_i, s_i, t)(i = 1, 2, 3, ..., n)$ and its usual contact form*

$$\eta = \frac{1}{2}(dt - \sum i = 1^n s_i dr_i).$$

The characteristic vector field ξ is given by $2\frac{\partial}{\partial t}$ and its Riemannian metric g and tensor field ϕ are given by

$$g = \frac{1}{4}(\eta \otimes \eta + \sum_{i=1}^{n}((dr_i)^2 + (ds_i)^2), \phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & s_i & 0 \end{bmatrix}, i = 1, 2, 3, ..., n$$

This gives a contact metric structure on $\mathbb{R}^{2n+1}$.The vector fields $E_i = 2\frac{\partial}{\partial s_i}, E_{n+i} = 2(\frac{\partial}{\partial r_i} + s_i\frac{\partial}{\partial t}), \xi$ form a $\phi-$ basis for the contact metric structure. On the other hand, it can be shown that $\mathbb{R}^{2n+1}(\phi, \xi, \eta, g)$ is a Sasakian manifold.

3 Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).\widetilde{C} = 0$

Let M be a $(2n+1)$ dimensional Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). The conformal curvature tensor $\widetilde{C}$ on M is defined by [5, 20]

$$\begin{aligned} \widetilde{C}(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1}$$

where r is scalar curvature.

Now we prove the following theorem:

Theorem 3.1. *If a Sasakian manifold admits conformal Ricci soliton and is Weyl conformally semi symmetric i.e. $R(\xi, X).\widetilde{C} = 0$, then the manifold is an η -Einstein manifold where $\widetilde{C}$ is conformal curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

Proof. Let M be a $(2n+1)$ dimensional Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). So we have $r = -1$ [1].

After putting $r = -1$ and $Z = \xi$ in (3.1), we have

$$\begin{aligned} \widetilde{C}(X,Y)\xi &= R(X,Y)\xi - \frac{1}{2n-1}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX \\ &\quad -g(X,\xi)QY] - \frac{1}{2n(2n-1)}[g(Y,\xi)X - g(X,\xi)Y], \end{aligned} \tag{3.2}$$

Using (2.2), (2.9), (2.16) and (2.17) in (3.2), we get

$$\widetilde{C}(X,Y)\xi = \{1 - \frac{2A}{2n-1} - \frac{1}{2n(2n-1)}\}[\eta(Y)X - \eta(X)Y], \tag{3.3}$$

Considering

$$e = 1 - \frac{2A}{2n-1} - \frac{1}{2n(2n-1)},$$

(3.3) becomes

$$\widetilde{C}(X,Y)\xi = e[\eta(Y)X - \eta(X)Y] \tag{3.4}$$

and

$$g(\widetilde{C}(X,Y)\xi, Z) = e[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \tag{3.5}$$

which implies

$$-\eta(\widetilde{C}(X,Y)Z) = e[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]. \tag{3.6}$$

Now we consider that the Sasakian manifold [2] admits conformal Ricci soliton and is conformally semi symmetric i.e. $R(\xi,X).\widetilde{C} = 0$ holds in M, which implies

$$\begin{aligned} &R(\xi,X)(\widetilde{C}(Y,Z)W) - \widetilde{C}(R(\xi,X)Y,Z)W \\ &-\widetilde{C}(Y,R(\xi,X)Z)W - \widetilde{C}(Y,Z)R(\xi,X)W = 0 \end{aligned} \tag{3.7}$$

for all vector field X, Y, Z, W on M.

Using (2.8) in (3.7) and putting $W = \xi$, we get

$$\begin{aligned} &g(X,\widetilde{C}(Y,Z)\xi)\xi - \eta(\widetilde{C}(Y,Z)\xi)X - g(X,Y)\widetilde{C}(\xi,Z)\xi \\ &+\eta(Y)\widetilde{C}(X,Z)\xi - g(X,Z)\widetilde{C}(Y,\xi)\xi + \eta(Z)\widetilde{C}(Y,X)\xi \\ &-g(X,\xi)\widetilde{C}(Y,Z)\xi + \eta(\xi)\widetilde{C}(Y,Z)X = 0 \end{aligned} \tag{3.8}$$

Taking inner product with ξ in (3.8) and using (2.3), we get

$$\begin{aligned} &g(X,\widetilde{C}(Y,Z)\xi) - g(X,Y)\eta(\widetilde{C}(\xi,Z)\xi) + \eta(Y)\eta(\widetilde{C}(X,Z)\xi) \\ &-g(X,Z)\eta(\widetilde{C}(Y,\xi)\xi) + \eta(Z)\eta(\widetilde{C}(Y,X)\xi) \\ &-g(X,\xi)\eta(\widetilde{C}(Y,Z)\xi) + \eta(\xi)\eta(\widetilde{C}(Y,Z)X) = 0 \end{aligned} \tag{3.9}$$

Using (3.4) in (3.9), we have

$$e[\eta(Z)g(X,Y) - \eta(Y)g(X,Z)] + \eta(\widetilde{C}(Y,Z)X) = 0 \tag{3.10}$$

Putting $Z = \xi$ in (3.10) and using (2.3), we get

$$e\{g(X,Y) - \eta(Y)\eta(X)\} + \eta(\widetilde{C}(Y,\xi)X) = 0 \tag{3.11}$$

Now from (3.1), we get

$$\begin{aligned} \eta(\widetilde{C}(Y,\xi)X &= \eta(X)\eta(Y) - g(X,Y) \\ &\quad -\frac{A}{2n-1}[2\eta(X)\eta(Y) - S(X,Y) - g(X,Y)] \\ &\quad -\frac{1}{2n(2n-1)}[\eta(X)\eta(Y) - g(X,Y)] \end{aligned} \tag{3.12}$$

After putting (3.12) in (3.11), we get

$$\begin{aligned}&e\{g(X,Y)-\eta(Y)\eta(X)\}+\eta(X)\eta(Y)\\&-g(X,Y)-\frac{A}{2n-1}[2\eta(X)\eta(Y)-S(X,Y)\\&-g(X,Y)]-\frac{1}{2n(2n-1)}[\eta(X)\eta(Y)-g(X,Y)]=0\end{aligned}\tag{3.13}$$

Simplifying (3.13), we get

$$\begin{aligned}S(X,Y)&=\frac{2n-1}{A}[1-B-\frac{A}{2n-1}-\frac{1}{2n(2n-1)}]g(X,Y)\\&+\frac{2n-1}{A}[B-1+\frac{2A}{2n-1}+\frac{1}{2n(2n-1)}]\eta(X)\eta(Y)\end{aligned}$$

the above equation can be written in the form

$$S(X,Y)=f_1g(X,Y)+f_2\eta(X)\eta(Y)\tag{3.14}$$

where

$$f_1=\frac{2n-1}{A}[1-B-\frac{A}{2n-1}-\frac{1}{2n(2n-1)}]$$

and

$$f_2=\frac{2n-1}{A}[B-1+\frac{2A}{2n-1}+\frac{1}{2n(2n-1)}]$$

So, from (3.14), we conclude that the manifold becomes an η Einstein manifold. □

4 Sasakian manifold admitting conformal Ricci soliton and $P(\xi,X).\widetilde{C}=0$

Definition 4.1. Let M be a $(2n+1)$ dimensional Sasakian manifold admitting a conformal Ricci soliton (g,V,λ). The Weyl projective curvature tensor P on M is given by [5, 29]

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X-S(X,Z)Y]\tag{4.1}$$

Now we prove the following theorem:

Theorem 4.2. *If a Sasakian manifold M admits conformal Ricci soliton and $P(\xi,X).\widetilde{C}=0$ holds, then the manifold becomes an η Einstein manifold, where P is projective curvature tensor and $\widetilde{C}$ is conformal curvature tensor.*

Proof. We know that from (4.1),

$$\begin{aligned}\widetilde{C}(\xi,X)Y&=R(\xi,X)Y\\&-\frac{1}{2n-1}[S(X,Y)\xi-S(\xi,Y)X+g(X,Y)Q\xi-g(\xi,Y)QX]\\&-\frac{1}{2n(2n-1)}[g(X,Y)\xi-g(\xi,Y)X],\end{aligned}\tag{4.2}$$

Since, for conformal Ricci soliton the scalar curvature $r=-1$ [1].

Taking inner product with respect to ξ on (4.2) and using (2.2), (2.8), (2.16) and (2.17), we have

$$\begin{aligned}\eta(\widetilde{C}(\xi,X)Y)&=[1-A-\frac{1}{2n(2n-1)}]g(X,Y)\\&+[-1+\frac{2A}{2n-1}+\frac{1}{2n(2n-1)}]\eta(X)\eta(Y)\\&-\frac{1}{2n-1}S(X,Y)\end{aligned}\tag{4.3}$$

The above equation can be written as

$$\eta(\widetilde{C}(\xi, X)Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y) - \gamma S(X, Y) \tag{4.4}$$

where

$$\alpha = 1 - A - \frac{1}{2n(2n-1)}$$

$$\beta = -1 + \frac{2A}{2n-1} + \frac{1}{2n(2n-1)}$$

and

$$\gamma = \frac{1}{2n-1}$$

Now

$$P(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n}[S(X, Y)\xi - S(\xi, Y)X] \tag{4.5}$$

Using (2.8), (2.17) in (4.5), we get

$$P(\xi, X)Y = g(X, Y)\xi - \eta(Y)X - \frac{1}{2n}[S(X, Y)\xi - A\eta(Y)X] \tag{4.6}$$

Here, we consider that the tensor derivative of $\widetilde{C}$ by $P(\xi, X)$ is zero i.e. conformally symmetric with respect to projective curvature tensor i.e. $P(\xi, X).\widetilde{C} = 0$ holds. So

$$\begin{aligned} &P(\xi, X)\widetilde{C}(Y, Z)W - \widetilde{C}(P(\xi, X)Y, Z)W \\ &-\widetilde{C}(Y, P(\xi, X)Z)W - \widetilde{C}(Y, Z)P(\xi, X)W = 0 \end{aligned} \tag{4.7}$$

for all vector fields X, Y, Z, W on M.

Using (4.6) in (4.7) and putting $W = \xi$, we have

$$\begin{aligned} &g(X, e(\eta(Z)Y - \eta(Y)Z))\xi - \frac{1}{2n}S(X, e(\eta(Z)Y - \eta(Y)Z))\xi \\ &-g(X, Y)\widetilde{C}(\xi, Z)\xi + \eta(Y)\widetilde{C}(X, Z)\xi + \frac{1}{2n}S(X, Y)\widetilde{C}(\xi, Z)\xi \\ &-\frac{A}{2n}\eta(Y)\widetilde{C}(X, Z)\xi - g(X, Z)\widetilde{C}(Y, \xi)\xi + \eta(Z)\widetilde{C}(Y, X)\xi \\ &+\frac{1}{2n}S(X, Z)\widetilde{C}(Y, \xi)\xi - \frac{A}{2n}\eta(Z)\widetilde{C}(Y, X)\xi - \eta(X)\widetilde{C}(Y, Z)\xi \\ &+\widetilde{C}(Y, Z)X + \frac{A}{2n}\eta(X)\widetilde{C}(Y, Z)\xi - \frac{A}{2n}\widetilde{C}(Y, Z)X = 0 \end{aligned} \tag{4.8}$$

Taking inner product with respect to ξ in above equation, we get

$$\begin{aligned} &e[\eta(Z)g(X, Y) - \eta(Y)g(X, Z)] - \frac{e}{2n}S(X, Y)\eta(Z) \\ &+\frac{e}{2n}S(X, Z)\eta(Y) + e(1 - \frac{A}{2n})[g(Y, Z)\eta(X) \\ &-g(X, Z)\eta(Y)] = 0 \end{aligned} \tag{4.9}$$

Putting $Z = \xi$ in above equation and using (2.2), (2.3) and (2.17), we get

$$eg(X, Y) - e\eta(X)\eta(Y) - \frac{e}{2n}S(X, Y) + \frac{Ae}{2n}\eta(X)\eta(Y) = 0 \tag{4.10}$$

which implies

$$S(X, Y) = 2ng(X, Y) + (A - 2n)\eta(X)\eta(Y)$$

Therefore, we conclude that the manifold becomes an η-Einstein manifold. □

5 Sasakian manifold admitting conformal Ricci soliton and $P(\xi, X).S = 0$

Theorem 5.1. *If a Sasakian manifold M admits conformal Ricci soliton and $P(\xi, X).S = 0$ holds i.e. the manifold is Ricci projectively symmetric, then the manifold is an Einstein manifold, where P is projective curvature tensor and S is Ricci tensor.*

Proof. Let M be a $(2n+1)$ dimensional Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). Now the equation (4.1) can be written as

$$P(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n}[S(X, Y)\xi - S(\xi, Y)X] \tag{5.1}$$

and

$$P(\xi, X)Z = R(\xi, X)Z - \frac{1}{2n}[S(X, Z)\xi - S(\xi, Z)X] \tag{5.2}$$

Now we assume that the manifold is Ricci projectively symmetric i.e. $P(\xi, X).S = 0$ holds in M, which gives

$$S(P(\xi, X)Y, Z) + S(Y, P(\xi, X)Z) = 0 \tag{5.3}$$

Using (2.2), (2.8), (5.1), (5.2) in (5.3), we have

$$\begin{aligned}
&A[g(X, Y)\eta(Z) - g(X, Z)\eta(Y)] - \frac{A}{2n}[S(X, Y)\eta(Z) \\
&-A\eta(Y)g(X, Z)] + A[g(X, Z)\eta(Y) - g(X, Y)\eta(Z)] \\
&-\frac{A}{2n}[S(X, Z)\eta(Y) - Ag(X, Y)\eta(Z)] = 0
\end{aligned} \tag{5.4}$$

Putting $Z = \xi$, we get

$$S(X, Y) = Ag(X, Y)$$

which proves that the manifold is an Einstein manifold. □

6 Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).P = 0$

Theorem 6.1. *If a Sasakian manifold M admits conformal Ricci soliton and $R(\xi, X).P = 0$ holds i.e. the manifold is projectively semi-symmetric, then the manifold is an Einstein manifold, where P is projective curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

Proof. Let M be a $(2n+1)$ dimensional Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). We assume that the manifold is projectively semi-symmetric i.e. $R(\xi, X).P = 0$ holds in M, which implies

$$\begin{aligned}
&R(\xi, X)(P(Y, Z)W) - P(R(\xi, X)Y, Z)W \\
&-P(Y, R(\xi, X)Z)W - P(Y, Z)R(\xi, X)W = 0
\end{aligned} \tag{6.1}$$

for all vector field X, Y, Z, W on M.

Using (2.8) in (6.1) and putting $W = \xi$, we get

$$\begin{aligned}
&g(X, \eta(Z)Y - \eta(Y)Z)\xi) - \frac{A}{2n}[\eta(Z)g(X, Y)\xi \\
&-\eta(Y)g(X, Z)\xi] - \eta(P(Y, Z)\xi)X - g(X, Y)P(\xi, Z)\xi \\
&+\eta(Y)P(X, Z)\xi - g(X, Z)P(Y, \xi)\xi + \eta(Z)P(Y, X)\xi \\
&-\eta(X)P(Y, Z)\xi + P(Y, Z)X = 0
\end{aligned} \tag{6.2}$$

Taking inner product with respect to ξ and using (2.2) and (2.3), we get

$$(1 - \frac{A}{2n})[g(X, Y)\eta(Z) - g(X, Z)\eta(Y)] + \eta(P(Y, Z)X) = 0 \tag{6.3}$$

Putting $Z = \xi$ in (6.3), we get

$$S(X, Y) = Ag(X, Y)$$

which proves that the manifold is an Einstein manifold. □

7 Sasakian manifold admitting conformal Ricci soliton and $T(\xi, X).S = 0$

Definition 7.1. Let M be a $(2n+1)$ dimensional Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). The conharmonic curvature tensor T on M is defined by [14, 15]

$$\begin{aligned} T(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \end{aligned} \tag{7.1}$$

for all $X, Y, Z \in \chi(M)$, R is the curvature tensor and Q is the Ricci operator.

Theorem 7.2. *If a Sasakian manifold M admits conformal Ricci soliton and the manifold is conharmonically Ricci symmetric i.e. $T(\xi, X).S = 0$ then the Ricci operator Q satisfies the quadratic equation $EQ^2 - Q + D = 0$ for all $X \in \chi(M)$ where E, D are constants, T is conharmonic curvature tensor and S is a Ricci tensor.*

Proof. Let M be a $(2n+1)$ dimensional Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). From (7.1) we can write

$$\begin{aligned} T(\xi,X)Y &= R(\xi,X)Y - \frac{1}{2n-1}[S(X,Y)\xi - S(\xi,Y)X \\ &\quad + g(X,Y)Q\xi - g(\xi,Y)QX] \end{aligned} \tag{7.2}$$

Using (2.8), (2.17) in (7.2), we have

$$\begin{aligned} T(\xi,X)Y &= [g(X,Y)\xi - \eta(Y)X] - \frac{1}{2n-1}[S(X,Y)\xi \\ &\quad - A\eta(Y)X + Ag(X,Y)\xi - \eta(Y)QX] \end{aligned} \tag{7.3}$$

Similarly, we have

$$\begin{aligned} T(\xi,X)Z &= [g(X,Z)\xi - \eta(Z)X] - \frac{1}{2n-1}[S(X,Z)\xi \\ &\quad - A\eta(Z)X + Ag(X,Z)\xi - \eta(Z)QX] \end{aligned} \tag{7.4}$$

Now we consider that the tensor derivative of S by $T(\xi, X)$ is zero i.e. $T(\xi, X).S = 0$. Then the Sasakian manifold M admitting conformal Ricci soliton is conharmonically Ricci symmetric. It gives

$$S(T(\xi,X)Y, Z) + S(Y, T(\xi,X)Z) = 0 \tag{7.5}$$

Using (7.3) and (7.4) in (7.5), we get

$$\begin{aligned} & S([g(X,Y)\xi - \eta(Y)X] - \frac{1}{2n-1}[S(X,Y)\xi \\ & - A\eta(Y)X + Ag(X,Y)\xi - \eta(Y)QX], Z) \\ & + S(Y, [g(X,Z)\xi - \eta(Z)X] - \frac{1}{2n-1}[S(X,Z)\xi \\ & - A\eta(Z)X + Ag(X,Z)\xi - \eta(Z)QX]) = 0 \end{aligned} \tag{7.6}$$

Putting $Z = \xi$ and using (2.2) and (2.3) in (7.5), we get

$$(A - \frac{A^2}{2n-1})g(X,Y) - S(X,Y) + \frac{1}{2n-1}S(Y,QX) = 0 \tag{7.7}$$

Which implies

$$Dg(X,Y) - S(X,Y) + ES(QX,Y) = 0$$

Where $D = (A - \frac{A^2}{2n-1})$ and $E = \frac{1}{2n-1}$, which implies

$$QX = DX + EQ^2X \quad \forall\, Y \in \chi(M), \tag{7.8}$$

i.e.

$$EQ^2 - Q + D = 0 \quad \forall\, X$$

□

8 Conclusion

This paper aims is to study the conformal curvature tensor, the conharmonic curvature tensor, the Ricci curvature tensor and the projective curvature tensor in the Sasakian manifold, admitting conformal Ricci Soliton. Also, studied the conformally semi-symmetric Sasakian manifold, admitting conformal Ricci soliton. Further, it identified that the Ricci conharmonically symmetric Sasakian manifold admitting conformal Ricci soliton is a quadratic equation. Again it shows that a conformally symmetric Sasakian manifold M with respect to projective curvature tensor admitting conformal Ricci soliton is an η Einstein manifold. We have also studied Ricci projectively symmetric sasakian manifold.Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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