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SOLVABILITY FOR SOLUTION FOR DIFFUSION FRACTIONAL NONLINEAR PARABOLIC DIRICHLET PROBLEMS

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Abstract Through our investigation into the initial boundary value diffusion fractionnaire spatio-temporal (STFDE) parabolic problem class, we establish that a unique weak solution exists within a functional weighted Sobolev space, By utilizing a priori estimate and an iterative process, we demonstrate the validity of our results, which are based on previous findings regarding the linear problem.

1 Introduction

In recent times, researchers have shown significant interest in fractional differential equations (FDEs) and their diverse applications. These equations involve extending the conventional integer order derivative to an arbitrary order, specifying relationships in time and space through a power law memory kernel of nonlocal nature. This generalization proves to be a potent tool for characterizing the memory aspects of various substances and the inheritance properties. The exploration of FDEs spans multiple domains, giving rise to a burgeoning field of scientific research. This encompasses new theoretical analyses and applications in a wide array of areas such as viscoelasticity, electrochemistry, signal processing, electromagnetics, porous media, electrical networks, electromagnetic theory, probability, signal and image processing, as well as numerical methods for fractional order dynamical systems. The applications of FDEs extend to diverse physical processes, reflecting their versatility and significance in contemporary scientific investigations.

The past few years have seen a lot of progress in the study of fractional differential equations.This development can be attributed to the numerous recent papers and monographs that have explored this field. More information can be found in the Kilbas monographs [8], and other works in the theory of fractional differential equations(see e.g. [9], [10],[11] ,[12],[13], [18],[25, 26],[27],[28],[32],[33],). The general references in Baleanu et al.[34],Additionally, the included references in those works have also been influential in this area of research.Nonetheless, there are various phenomena that can be more accurately described using the conditions of Dirichlet .

The use of conditions of Dirichlet type has proven effective in addressing numerous complex issues, including those encountered in porous media, electromagnetic, and environmental science. The application of classical theories and methods to the examine the fractional paroles and hyperbolic problems is a difficult domain, resulting in a few publications in this field. Therefore, the current work aims to the demonstration of the existence and unique solvability of the solutions to Dirichlet fractional problems, this area that has not received much.

2 Preliminaries

Let the interval $\Delta = [0;T]$ be a bounded and closed . For any $0 < \alpha < 1$ the Caputo and Riemann Liouville derivative are, defined in the following manner :

To define the left and right Caputo fractional derivatives of order α we have the following notation :

$$^{C}D_t^{\alpha}\mu(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial\mu(x,s)}{\partial s}\frac{1}{(t-s)^{\alpha}}ds \tag{2.1}$$

and

$$^{C}_{t}D^{\alpha}\mu(x,t) = \frac{-1}{\Gamma(1-\alpha)}\int_t^T \frac{\partial\mu(x,s)}{\partial s}\frac{1}{(s-t)^{\alpha}}ds \tag{2.2}$$

The left and right fractional derivatives of Riemann-Liouville of order α are given in the following form :

$$^{R}D_t^{\alpha}\mu(x,t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{\mu(x,s)}{(t-s)^{\alpha}}ds \tag{2.3}$$

and

$$^{R}_{t}D^{\alpha}\mu(x,t) = \frac{-1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{\mu(x,s)}{(s-t)^{\alpha}}ds \tag{2.4}$$

According to several authors, Caputo version of the equation is considered more natural since it enables simpler handling inhomogeneous initial conditions.Then the two definitions (2.1) and (2.3) are linked by the following relationship is possible to check this by performing a simple calculation :

$$^{R}_{a}D_t^{\alpha}\mu(x,t) =^{C}_{a} D_t^{\alpha}\mu(x,t) + \frac{\mu(x,0)}{\Gamma(1-\alpha)t^{\alpha}} \tag{2.5}$$

2.1 Definition 1 :[41]

For any real $\theta > 0$ and finite interval $\Omega = [a,b]$ of the real axis R,We establish the definition of the semi-norm as follows :

$$|\mu|^2_{^{l}H^{\theta}(\Omega)} = \left\|^{R}D_t^{\theta}\mu\right\|^2_{L^2(\Omega)}$$

and norm :

$$\|\mu\|^2_{^{l}H^{\theta}(\Omega)} = \|\mu\|^2_{L^2(\Omega)} + |\mu|^2_{^{l}H^{\theta}(\Omega)} \tag{2.6}$$

After that, we define the space: $^{l}H^{\theta}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ regarding to the next norm, we define $\|.\|_{^{l}H^{\theta}(\Omega)}$.

2.2 Definition 2 [41]:

For any real $\theta > 0$ and finite interval $\Omega = [a,b]$ of the real axis R ,we define the semi-norm :

$$|\mu|^2_{^{r}H^{\theta}(\Omega)} = \left\|^{R}_{t}D^{\theta}\mu\right\|^2_{L^2(\Omega)}$$

and norm :

$$\|\mu\|^2_{^{r}H^{\theta}(\Omega)} = \|\mu\|^2_{L^2(\Omega)} + |\mu|^2_{^{r}H^{\theta}(\Omega)} \tag{2.7}$$

we then define $^{l}H^{\theta}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|.\|_{^{l}H^{\theta}(\Omega)}$.

2.3 Definition 3 :

For any real $\theta > 0$ and finite interval $\Omega = [a, b]$ of the real axis R,we define the semi-norm :

$$|\mu|^2_{^cH^\theta(\Omega)} = \left| \frac{({}^RD_t^\theta \mu, {}_t^R D^\theta \mu)_{L^2(\Omega)}}{\cos(\theta\pi)} \right|$$

and norm:

$$\|\mu\|^2_{^cH^\theta(\Omega)} = \|\mu\|^2_{L^2(\Omega)} + |\mu|^2_{^cH^\theta(\Omega)} \tag{2.8}$$

2.4 Lemma 1 :[36, 41]

For any real $\theta \in R_+$, if $u \in {}^lH^\theta(\Omega)$ and $v \in C_0^\infty(\Omega)$,then :

$$({}^RD_t^\theta \mu(t), \xi(t))_{L^2(\Omega)} = (\mu(t), {}_t^R D^\theta \xi(t))_{L^2(\Omega)}$$

2.5 Lemma 2 :[36, 41]

for $0 < \theta < 2$, $\theta \neq 1$ and $u \in H_0^{\frac{\theta}{2}}(\Omega)$,then :

$${}^RD_t^\theta \xi(t) = {}^R D_t^{\frac{\theta}{2}} \xi {}^RD_t^{\frac{\theta}{2}} \xi(t)$$

2.6 Lemma 3 :[36, 41]

For any real $\theta \in R_+$and $\theta \neq n + \frac{1}{2}$, the semi norms :$|.|_{^lH^\theta(\Omega)}$,$|.|_{^rH^\theta(\Omega)}$ and $|.|_{^cH^\theta(\Omega)}$ are equivalent :

$$|.|_{^lH^\theta(\Omega)} \cong |.|_{^rH^\theta(\Omega)} \cong |.|_{^cH^\theta(\Omega)}$$

2.7 Lemma 4 :[36, 41]

For any real $\theta > 0$ the space ${}^RH_0^\theta(\Omega)$ With regards to the norm (1,7) is complete.

2.8 Definition 4 :

Let us indicate $L^2(0, T, L^2(0, 1)) = L_2(Q)$ The space consisting of functions that are square-integrable in the Bochner sense, with the scalar product :

$$(\mu, \xi)_{L^2(0,T,L^2(0,1))} = \int_0^T ((\mu, .), (\xi, .))_{L^2(0,1)} \, dt \tag{2.9}$$

3 Determining the solvability of fractional diffusion Dirichlet problems

3.1 Presenting of the problem

Let $T \geq 0$, $\Sigma \subset \mathbb{R}$ and

$$Q = \Sigma \times I \text{ with } \Sigma = (-1; 1) \text{ and } I = (0; T)$$

We consider the nonlinear fractional problem

$$\begin{cases} {}^RD_t^\alpha u(x,t) - p_1.{}^RD_x^\beta u(x,t) - p_2.{}_x^RD^\beta u(x,t) + a(x,t)u(x,t) = f(x,t,u,{}^R D_x^\beta u), \forall (x,t) \in Q \\ u(x,0) = 0 \qquad \forall x \in (-1,1), \\ u(-1,t) = u(1,t) = 0 \qquad \forall t \in (0,T). \end{cases} \tag{P$_1$}$$

where p_1, p_2 let there be two constants that satisfy:

$$p_1 + p_2 = 1 \text{ and } 0 < p_1, p_2 < 1$$

where the function $a(x,t)$

$$a_0 < a(x,t) < a_1; a_0, a_1 \in \mathbb{R}_*^+ \text{ and } \forall (x,t) \in Q$$

4 Our study of the associated linear problem

In this part,We establish the existence and uniqueness of a strong solution to the linear problem by means of an a priori estimate and the density of the set of values produced by the operator generated by the problem.

$$\begin{cases} {}^R D_t^{\alpha} u(x,t) - p_1.{}^R D_x^{\beta} u(x,t) - p_2.{}_x^R D^{\beta} u(x,t) + a(x,t)u(x,t) = f(x,t), \forall (x,t) \in Q \\ u(x,0) = 0 \qquad \forall x \in (-1,1), \\ u(-1,t) = u(1,t) = 0 \qquad \forall t \in (0,T). \end{cases} \tag{P$_1$}$$

Whose diffusion problem is represented as follows

$$\pounds u =^R D_t^{\alpha} u(x,t) - p_1^R D_x^{\beta} u(x,t) - p_2.{}_x^R D^{\beta} u(x,t) + a(x,t)u(x,t) = f(x,t) \tag{4.1}$$

with the initial condition

$$lu = u(x,0) = 0 \ \ \forall x \in [-1,1] \tag{4.2}$$

the Dirichlet boundary conditions

$$u(-1,t) = u(1,t) = 0 \quad \forall t \in [0,T] \tag{4.3}$$

where $f(x,t)$The given functions, with α and β satisfy the following assumptions:
$0 \le \alpha \le 1$, $0 \le \beta < 1$, $(x,t) \in \bar{Q}$.

5 A priori estimate

The operator L acts from E to F defined as follows. The Banach space E consists of all functions $u(x,t)$ with the finite norm

$$\|u\|_E^2 = \left\| {}^R D_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q)}^2 + \left\| {}^R D_x^{\frac{\beta}{2}} u \right\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 . \tag{5.1}$$

The Hilbert space F consists of the vector valued functions $F = f$ with the norm

$$\|\xi\|_F^2 = \|f\|_{L^2(Q)}^2 \tag{5.2}$$

5.1 A priori bound of linear problem

Theorem 5.1. *If the assumptions $A1$ are satisfied then for any function $u \in D(L)$, there exists a positive constant c independent of u such that*

$$\left\| {}^R D_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q)}^2 + \left\| {}^R D_x^{\frac{\beta}{2}} u \right\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 \le k \left(\|f\|_{L^2(Q)}^2 \right), \tag{5.3}$$

and $D(L)$ is the domain of definition of the operator L defined by

$$D(L) = \{u \in L^2(Q) \ / \ {}^R D_t^{\frac{\alpha}{2}} u, {}^R D_x^{\frac{\beta}{2}} u \in L^2(Q)\},$$

satisfying conditions (4.3).

Proof. Taking the scalar product in $L^2(Q)$ of Eq. (4.1) and the operator

$$Mu = u,$$

where $Q^\tau = \Sigma \times (0,T)$, we have

$$(\pounds u, Mu)_{L^2(Q^\tau)} = \left({}^R D_t^\alpha u, u\right)_{L^2 Q^\tau} - p_1 \left({}^R D_x^\alpha u, u\right)_{L^2 Q^\tau} - p_2 \left({}_x^R D^\alpha u, u\right)_{L^2 Q^\tau} + (a, u)_{L^2(Q^\tau)} = \left(\tilde{f}, u\right)_{L^2(Q^\tau)}, \tag{5.4}$$

Successive integration by parts can be applied to the integrals on the right-hand side of the equation (5.4), yields

$$\begin{aligned} \left({}^R D_x^\alpha u, u\right)_{L^2(Q^\tau)} &= \left({}^R D_x^{\frac{\alpha}{2}} u, {}_x^R D^{\frac{\alpha}{2}} u\right)_{L^2(Q)} \\ &= \cos\left(\frac{\alpha\pi}{2}\right) |u|^2_{{}^C H^{\frac{\beta}{2}}(Q)} \\ &= \cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^R D_x^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)}, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} -p_1 \left({}^R D_x^\beta u, u\right)_{L^2(Q^\tau)} &= \left({}^R D_x^{\frac{\beta}{2}} u, {}_x^R D^{\frac{\beta}{2}} u\right)_{L^2(Q)} \\ &= -p_1 \cos\left(\frac{\beta\pi}{2}\right) |u|^2_{{}^C H^{\frac{\beta}{2}}(Q)} \\ &= -p_1 \cos\left(\frac{\beta\pi}{2}\right) \left\| {}^R D_x^{\frac{\beta}{2}} u \right\|^2_{L^2(Q)}, \end{aligned} \tag{5.6}$$

$$\begin{aligned} -p_2 \left({}_x^R D^\beta u, u\right)_{L^2(Q^\tau)} &= \left({}_x^R D^{\frac{\beta}{2}} u, {}^R D_x^{\frac{\beta}{2}} u\right)_{L^2(Q)} \\ &= -p_2 \cos\left(\frac{\beta\pi}{2}\right) |u|^2_{{}^C H^{\frac{\beta}{2}}(Q)} \\ &= -p_2 \cos\left(\frac{\beta\pi}{2}\right) \left\| {}_x^R D^{\frac{\beta}{2}} u \right\|^2_{L^2(Q)} \\ &= -p_2 \cos\left(\frac{\beta\pi}{2}\right) \left\| {}^R D_x^{\frac{\beta}{2}} u \right\|^2_{L^2(Q)} \end{aligned} \tag{5.7}$$

Substituting (5.5), (5.6)and (5.7) into (5.4),

$$\cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^R D_t^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)} - (p_1 + p_2) \cos\left(\frac{\beta\pi}{2}\right) \left\| {}^R D_x^{\frac{\beta}{2}} u \right\|^2_{L^2(Q)} + (a, u)_{L^2(Q^\tau)} \le \left(\tilde{f}, u\right), \tag{5.8}$$

estimate the last term on the right-hand side of (5.8) by applying Cauchy inequality with ε, ($|ab| \le \dfrac{a^2}{2\varepsilon} + \dfrac{\varepsilon b^2}{2}$), then we get

$$\begin{aligned} &\cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^R D_t^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)} - (p_1 + p_2) \cos\left(\frac{\beta\pi}{2}\right) \left\| {}^R D_x^{\frac{\beta}{2}} u \right\|^2_{L^2(Q)} + a_0 \|u\|^2_{L^2(Q)} \\ \le\ & \frac{1}{2\varepsilon} \|f\|^2_{L^2(Q)} + \frac{\varepsilon}{2} \|u\|^2_{L^2(Q)}, \end{aligned} \tag{5.9}$$

Then the estimate (5.9) becomes

$$\left\| {}^{R}D_{t}^{\frac{\alpha}{2}} u \right\|_{L^{2}(Q)}^{2} + \left\| {}^{R}D_{x}^{\frac{\beta}{2}} u \right\|_{L^{2}(Q)}^{2} + \|u\|_{L^{2}(Q)}^{2}$$

$$\leq \frac{1}{2\varepsilon \min\{\cos\left(\frac{\alpha\pi}{2}\right), -(p_1+p_2)\cos\left(\frac{\beta\pi}{2}\right), \left(a_0 - \frac{\varepsilon}{2}\right)\}} \|f\|_{L^{2}(Q)}^{2},$$

So, finally we get

$$\left\| {}^{R}D_{t}^{\frac{\alpha}{2}} u \right\|_{L^{2}(Q)}^{2} + \left\| {}^{R}D_{x}^{\frac{\beta}{2}} u \right\|_{L^{2}(Q)}^{2} + \|u\|_{L^{2}(Q)}^{2}$$

$$\leq k \|f\|_{L^{2}(Q)}^{2},$$

Where

$$k = \frac{1}{2\varepsilon \min\{\cos\left(\frac{\alpha\pi}{2}\right), -(p_1+p_2)\cos\left(\frac{\beta\pi}{2}\right), \left(a_0 - \frac{\varepsilon}{2}\right)\}}$$

So, we have

$$\|u\|_{E} \leq k \|Lu\|_{F} \tag{5.10}$$

Let $R(L)$ be the range of the operator L. However, given the lack of information pertaining to $R(L)$, except that $R(L) \subset F$, It is essential to expand L, so that estimate (5.10) The statement is valid for the extension, and its range encompasses the entire space F. We first state the following proposition. □

Proposition 5.2. *The operator $L : E \longrightarrow F$ has a closure*

Proof. let $(u_n)_{n\in\mathbb{N}} \subset D(L)$ a sequence where :

$$u_n \longrightarrow 0 \qquad \text{in } E$$

and

$$Lu_n \longrightarrow \left(\tilde{f}; 0\right) \qquad \text{in } F \tag{5.11}$$

we must proof that

$$f \equiv 0\,.$$

The convergence of u_n to 0 in E drives :

$$u_n \longrightarrow 0 \qquad \text{in } D'(Q)\ . \tag{5.12}$$

Based on the continuity of the derivative of $D'(Q)$ in $D'(Q)$. The relation (5.12) involved

$$u_n \longrightarrow 0 \qquad \text{in } D'(Q)\ , \tag{5.13}$$

Moreover, The convergence of u_n to f in $L^2(Q)$ give

$$u_n \longrightarrow f \qquad \text{in } D'(Q) \tag{5.14}$$

As we have the the uniqueness of the limit in $D'(Q)$, we conclude from (5.13) and (5.14) that

$$f = 0.$$

Then, we get L the operator is closable, with the domain of definition $D(L)$.

Definition 5.3. A solution of the operator equation

$$\bar{L}u = \mathcal{F}$$

A solution is deemed strong for problems (4.1) – (4.2).The a priori estimate (4.3) the solutions are capable of being extended to strong solutions, i.e., we have the estimate

$$\left\| {}^{R}D_{t}^{\frac{\alpha}{2}} u \right\|_{L^{2}(Q)}^{2} + \left\| {}^{R}D_{x}^{\frac{\beta}{2}} u \right\|_{L^{2}(Q)}^{2} + \|u\|_{L^{2}(Q)}^{2}$$

$$\leq k \|f\|_{L^{2}(Q)}^{2}, \tag{5.15}$$

□

We deduce from the estimate (5.15) :

Corollary 5.4. *The range $R(\bar{L})$ of the operator $\bar{L}$ is closed in F and the operator is equal to its closure $\overline{R(L)}$ of $R(L)$, that is $R(\bar{L}) = \overline{R(L)}$.*

Proof. Let $\vartheta \in \overline{R(L)}$, so there is a cauchy sequence $(\vartheta_n)_{n\in\mathbb{N}}$in F Consisting of the elements within the set $R(L)$ such as

$$\lim_{n\longrightarrow+\infty} \vartheta_n = \vartheta.$$

This leads to the creation of a corresponding sequence $u_n \in D(L)$ such as

$$\vartheta_n = Lu_n.$$

The estimate 5.15 we get

$$\|u_p - u_q\|_E \leq C \|Lu_p - Lu_q\|_F \to 0, \tag{5.16}$$

Where p, q As the value approaches infinity, we can conclude that $(u_n)_{n\in\mathbb{N}}$is a cauchy sequence in E, so like E is a Banach space, it exists $u \in E$ such as

$$\lim_{n\longrightarrow+\infty} u_n = u \text{ in } E.$$

According to the definition of $\bar{L}$ ($\lim_{n\longrightarrow+\infty} u_n = u$ in E ; If $\lim_{n\longrightarrow+\infty} Lu_n = \lim_{n\longrightarrow+\infty} \vartheta_n = \vartheta$, then $\lim_{n\longrightarrow+\infty} \bar{L}u_n = \vartheta$ as like $\bar{L}$ and is closed, so $\bar{L}u = \vartheta$), the function u check :

$$v \in D\left(\bar{L}\right),\ \bar{L}v = \vartheta.$$

Then $\vartheta \in R(\bar{L})$, so

$$\overline{R(L)} \subset R(\bar{L})$$

Therefore, we can infer that $R(\bar{L})$The operator is a closed set due to being Banach (any complete subspace of a metric space not necessarily complete is closed). It remains to show the reverse inclusion. Either $\vartheta \in R(\bar{L})$ then it exists a cauchy sequence $(\vartheta_n)_{n\in\mathbb{N}}$in F constituted of the elements of the set $R(\bar{L})$ such that

$$\lim_{n\longrightarrow+\infty} \vartheta_n = \vartheta$$

or $\vartheta \in R(\bar{L})$, because $R(\bar{L})$ is a closed subset a completed F, So $R(\bar{L})$ is complet. There is then a corresponding sequence $u_n \in D(\bar{L})$ such that

$$\bar{L}u_n = \vartheta_n.$$

We get from (5.13):

$$\|u_p - u_q\|_E \leq C \left\|\bar{L}u_p - \bar{L}u_q\right\|_F \to 0, \tag{5.17}$$

Where p, q as the value approaches infinity, we can conclude that $(u_n)_{n\in\mathbb{N}}$is a cauchy sequence in E, so like E is a Banach space, it exists $u \in E$ such as

$$\lim_{n\longrightarrow+\infty} u_n = u \ \text{ in } \mathbb{E}.$$

Once more, a related sequence arises $(Lu_n)_{n\in\mathbb{N}} \subset R(L)$ such as

$$\bar{L}u_n = Lu_n \text{ on } R\left(L\right), \forall n \in \mathbb{N}.$$

So

$$\lim_{n\longrightarrow+\infty} Lu_n = \vartheta,$$

Consequently $\vartheta \in \overline{R\left(L\right)}$, From this, we can deduce that :

$$R\left(\bar{L}\right) \subset \overline{R\left(L\right)}.$$

□

5.2 Existence of solution

Theorem 5.5. *Let the assumptions A_1 be satisfied. Then for all $F = (f, 0) \in F$, there exists a unique strong solution $u = \bar{L}^{-1} = \overline{L^{-1}}$ of the problem (4.1)-(4.2).*

Proof. We have

$$(Lu, \lambda)_F = \int_Q lu.\gamma dxdt \tag{5.18}$$

Where

$$\lambda = (\gamma, 0) .$$

Si for $\gamma \in L^2(Q)$ and and for all $u \in D_0(L) = \{u,\ u \in D(L) : \ell u = 0\}$, we have

$$\int_Q lu.\gamma dxdt = 0$$

By putting $w = u$, Applying the same estimation of the section 1, we obtain

$$\cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^R D_t^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)} - (p_1 + p_2) \cos\left(\frac{\beta\pi}{2}\right) \left\| {}^R D_x^{\frac{\beta}{2}} u \right\|^2_{L^2(Q)} + a_0 \|u\|^2_{L^2(Q)} = 0,$$

we get

$$\|u\| \leq 0 \Rightarrow u = 0 .$$

So, it's give $u = \gamma = 0$. □

Corollary 5.6. *If for any function $u \in D(L)$, we have the following estimate:*

$$\|u\|_E \leq C \, \| | \|_F ,$$

Then the solution of the problem (P_1) if it exists, it is unique.

Proof. Let u_1 and u_2 two possible remedies to the problem (P_1)

$$\begin{cases} Lu_1 = \\ Lu_2 = \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0,$$

and as L is linear we then obtain

$$L(u_1 - u_2) = 0,$$

according to (5.17) :

$$\|u_1 - u_2\|^2_E \leq c \, \|0\|^2_F = 0,$$

Which give

$$u_1 = u_2.$$

□

6 The solvability of the nonlinear problem through a weak solution

The focus of this section is to demonstrate the existence and uniqueness of the solution to the nonlinear problem (Pr) :

$$\begin{cases} {}^R D_t^{\alpha} u(x,t) - p_1.{}^R D_x^{\beta} u(x,t) - p_2.{}^R_x D^{\beta} u(x,t) + a(x,t)u(x,t) = f(x,t,u,{}^R D_x^{\beta} u), \forall (x,t) \in Q \\ u(x,0) = 0 \qquad \forall x \in (-1,1), \\ u(-1,t) = u(1,t) = 0 \qquad \forall t \in (0,T). \end{cases} \tag{P_2}$$

Putting

$$u = \chi$$

The following nonlocal linear problem can be solved by using χ as a solution. :

In addition, the following nonlocal and nonlinear problem is satisfied by the solution.

$$\pounds\chi = {}^{R}D_{t}^{\alpha}\chi(x,t) - p_{1}.{}^{R}D_{x}^{\beta}\chi(x,t) - p_{2}.{}_{x}^{R}D^{\beta}\chi(x,t) + a(x,t)\chi(x,t) = f(x,t,\chi,{}^{R}D_{x}^{\beta}\chi) \quad (6.1)$$

$$\chi(x,0) = 0, \quad \forall x \in (-1,1), \quad (6.2)$$
$$\chi(0,t) = \chi(1,t) = 0 \quad \forall t \in (0,T). \quad (6.3)$$

Since f is Lipchitzian, there exists a positive constant k , that satisfies

$$\| f\left(x,t,u_{1},{}^{R}D_{x}^{\beta}u_{1}\right) - f\left(x,t,u_{2},{}^{R}D_{x}^{\beta}u_{2}\right) \|_{L^{2}(Q)} \leq k\left(\|u_{1}-u_{2}\|_{L^{2}(Q)} + \left\|{}^{R}D_{x}^{\beta}u_{1} - {}^{R}D_{x}^{\beta}u_{2}\right\|_{L^{2}(Q)}\right). \quad (6.4)$$

Forming a repeating sequence that begins with $\chi^{(0)} = 0$.

The sequence $\left(\chi^{(n)}\right)_{n\in\mathbb{N}}$ the following definition characterizes: given the element $\chi^{(n-1)}$, then for $n = 1,2,3,...$, our objective is to find a solution to the following problem.:

$$\begin{cases} {}^{R}D_{t}^{\alpha}\chi^{n} - p_{1}.{}^{R}D_{x}^{\beta}\chi^{n} - p_{2}.{}_{x}^{R}D^{\beta}\chi^{n} + a(x,t)\chi^{n} = f(x,t,\chi^{(n-1)},{}^{R}D_{x}^{\alpha}\chi^{(n-1)}) \\ \chi^{(n)}(x,0) = 0 \\ \chi^{(n)}(0,t) = \chi^{(n)}(1,t) = 0. \end{cases}, \quad (P_{3})$$

As established in the study of the linear problem prior to this one, fixing n to a constant value leads to the problem (P_{5}) admits a unique solution $\chi^{(n)}(x,t)$.

By postulating

$$z^{(n)}(x,t) = \chi^{(n+1)}(x,t) - \chi^{(n)}(x,t),$$

Therefore, a new problem must be tackled:

$$\begin{cases} {}^{R}D_{t}^{\alpha}z^{n} - p_{1}.{}^{R}D_{x}^{\beta}z^{n} - p_{2}.{}_{x}^{R}D^{\beta}z^{n} + a(x,t)z^{n} = \zeta^{(n-1)}(x,t) \\ z^{(n)}(x.0) = 0 \\ z^{(n)}(0,t) = z^{(n)}(1,t) = 0. \end{cases}, \quad (P_{4})$$

Or

$$\zeta^{(n-1)}(x,t) = f\left(x,t,\chi^{(n)},{}^{R}D_{x}^{\alpha}\chi^{(n-1)}\right) - f\left(x,t,\chi^{(n-1)},{}^{R}D_{x}^{\alpha}\chi^{(n-1)}\right).$$

Multiply

$${}^{R}D_{t}^{\alpha}z^{n} - p_{1}.{}^{R}D_{x}^{\beta}z^{n} - p_{2}.{}_{x}^{R}D^{\beta}z^{n} + a(x,t)z^{n} = \zeta^{(n-1)}(x,t)$$

by $z^{(n)}$,and integrate it on Q we get :

$$\int_{Q}({}^{R}D_{t}^{\alpha}z^{(n)}z^{(n)})dxdt - p_{1}\int_{Q}({}^{R}D_{x}^{\beta}z^{(n)}z^{(n)})dxdt - p_{2}\int_{Q}({}_{x}^{R}D^{\beta}z^{(n)}z^{(n)})dxdt + a_{0}\int_{Q}z^{(n)}z^{(n)}dxdt$$
$$\preceq \int_{Q}\zeta^{(n-1)}z^{(n)}dxdt.$$

By employing integration by parts and considering the initial and boundary conditions, a solution can be obtained, which is given by :

$$\cos\left(\tfrac{\alpha\pi}{2}\right)\left\|{}^{C}D_{t}^{\frac{\alpha}{2}}z^{(n)}\right\|_{L^{2}(Q)}^{2} - p_{1}\cos\left(\tfrac{\beta\pi}{2}\right)\left\|{}^{R}D_{x}^{\frac{\beta}{2}}z^{(n)}\right\|_{L^{2}(Q)}^{2} - p_{2}\cos\left(\tfrac{\beta\pi}{2}\right)\left\|{}_{x}^{R}D^{\frac{\beta}{2}}z^{(n)}\right\|_{L^{2}(Q)}^{2}$$
$$+a_{0}\left\|z^{(n)}\right\|_{L^{2}(Q)}^{2} \preceq \int_{Q_{\tau}}\zeta^{(n-1)}(x,t)\cdot z^{(n)}(x,t)\,dxdt.$$

After applying the Cauchy Schwarz inequality to the second part of the equation, we obtain the following result:

$$\begin{aligned}
&\int_Q \zeta^{(n-1)}(x,t)\cdot z^{(n)}(x,t)\,dxdt\\
\leqslant\ & \frac{1}{2\varepsilon}\int_{Q^\tau} \mid \zeta^{(n-1)}(x,t)\mid^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}(x,t)\right)^2 dxdt,\\
\leqslant\ & \frac{1}{2\varepsilon}\int_Q \mid f\left(x,t,\chi^{(n)},{}^R D_x^\beta\chi^{(n)}\right) - f\left(x,t,\chi^{(n-1)},{}^R D_x^\beta\chi^{(n-1)}\right)\mid^2 dxdt + \frac{\varepsilon}{2}\int_Q\left(z^{(n)}(x,t)\right)^2 dxdt
\end{aligned}$$

Like f Lipschtizienne, we find :

$$\begin{aligned}
\leqslant\ & \frac{k^2}{2\varepsilon}\Big(\int_Q(\mid\chi^{(n)}-\chi^{(n-1)}\mid)^2dxdt + \int_Q(\mid^R D_x^\beta\chi^{(n)} -{}^R D_x^\beta\chi^{(n-1)}\mid)^2dxdt\Big) + \frac{\varepsilon}{2}\int_Q\left(z^{(n)}(x,t)\right)^2 dxdt,\\
\leqslant\ & \frac{k^2}{2\varepsilon}\Big(\int_Q(\mid z^{(n-1)}\mid)^2dxdt + \int_Q(\mid^R D_x^\beta z^{(n-1)}\mid)^2dxdt\Big) + \frac{\varepsilon}{2}\int_Q\left(z^{(n)}(x,t)\right)^2 dxdt,\\
\leqslant\ & \frac{k^2}{\varepsilon}\Big(\int_Q(\mid z^{(n-1)}\mid)^2dxdt + \int_Q(\mid^R D_x^\beta z^{(n-1)}\mid)^2dxdt)\Big) + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}(x,t)\right)^2 dxdt,\\
\leqslant\ & \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|^2_{L^2(Q)} + \frac{k^2}{\varepsilon}\cos(\frac{\alpha\pi}{2})\left\|{}^C D_x^{\frac{\alpha}{2}} z^{(n-1)}\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|z^{(n)}\right\|^2_{L^2(Q)}
\end{aligned}$$

we get :

$$\begin{aligned}
&\cos(\frac{\alpha\pi}{2})\left\|{}^C D_t^{\frac{\alpha}{2}} z^{(n)}\right\|^2_{L^2(Q)} - (p_1+p_2)\cos\left(\frac{\beta\pi}{2}\right)\left\|{}^R D_x^{\frac{\beta}{2}} z^{(n)}\right\|^2_{L^2(Q)} + a_0\left\|z^{(n)}\right\|^2_{L^2(Q)}\\
\leqslant\ & \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|^2_{L^2(Q)} + \frac{k^2}{\varepsilon}\cos(\frac{\alpha\pi}{2})\left\|{}^R D_x^{\frac{\beta}{2}} z^{(n-1)}\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|z^{(n)}\right\|^2_{L^2(Q)},
\end{aligned}$$

We integrate on t, we obtain :

$$\begin{aligned}
&\cos(\frac{\alpha\pi}{2})\left\|{}^C D_t^{\frac{\alpha}{2}} z^{(n)}\right\|^2_{L^2(Q)} - (p_1+p_2)\cos\left(\frac{\beta\pi}{2}\right)\left\|{}^R D_x^{\frac{\beta}{2}} z^{(n)}\right\|^2_{L^2(Q)} + (a_0-\frac{\varepsilon}{2})\left\|z^{(n)}\right\|^2_{L^2(Q)}\\
\leqslant\ & \frac{k^2}{\varepsilon}\Big(\left\|z^{(n-1)}\right\|_{L^2_{(Q)}} + \cos(\frac{\beta\pi}{2})\left\|{}^R D_x^{\frac{\beta}{2}} z^{(n-1)}\right\|^2_{L^2(Q)}\Big)
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\left\|{}^C D_t^{\frac{\alpha}{2}} z^{(n)}\right\|^2_{L^2(Q)} + \left\|{}^R D_x^{\frac{\beta}{2}} z^{(n)}\right\|^2_{L^2(Q)} + \left\|z^{(n)}\right\|^2_{L^2(Q)}\\
\leqslant\ & \frac{k^2\cos(\frac{\beta\pi}{2})}{\varepsilon\min\{\cos(\frac{\alpha\pi}{2}), -(p_1+p_2)\cos\left(\frac{\beta\pi}{2}\right), (a_0-\frac{\varepsilon}{2})\}}\Big(\left\|z^{(n-1)}\right\|_{L^2_{(Q)}} + \left\|{}^R D_x^{\frac{\beta}{2}} z^{(n-1)}\right\|^2_{L^2(Q)}\Big)
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\left\|{}^C D_t^{\frac{\alpha}{2}} z^{(n)}\right\|^2_{L^2(Q)} + \left\|{}^R D_x^{\frac{\beta}{2}} z^{(n)}\right\|^2_{L^2(Q)} + \left\|z^{(n)}\right\|^2_{L^2(Q)}\\
\leqslant\ & \frac{k^2\cos(\frac{\beta\pi}{2})}{\varepsilon\min\{\cos(\frac{\alpha\pi}{2}), -(p_1+p_2)\cos(\frac{\beta\pi}{2}), (a_0-\frac{\varepsilon}{2})}\Big(\left\|{}^C D_t^{\frac{\alpha}{2}} z^{(n-1)}\right\|^2_{L^2(Q)} + \left\|{}^R D_x^{\frac{\beta}{2}} z^{(n-1)}\right\|^2_{L^2(Q)} + \left\|z^{(n-1)}\right\|^2_{L^2
\end{aligned}$$

(6.6)

Putting :

$$c=\max\left\{1,\frac{k^2\cos(\frac{\beta\pi}{2})}{\varepsilon\min\{\cos(\frac{\alpha\pi}{2}),-(p_1+p_2)\cos(\frac{\beta\pi}{2}),(a_0-\frac{\varepsilon}{2})}\right\}.$$

So, we get :

$$\left\|z^{(n)}\right\|^2_{L^2(Q)}\leqslant c\left\|z^{(n-1)}\right\|^2_{L^2(Q)}$$

As we have :

$$\sum_{i=1}^{n-1}z^{(i)}=\chi^{(n)}$$

Based on the convergence criterion of the series, we can conclude that the series S $=\sum\limits_{n=1}^{\infty}z^{(n)}$ converges if $|c|<1$, which implies :

$$\left|\frac{k^2\cos(\frac{\beta\pi}{2})}{\varepsilon\min\{\cos(\frac{\alpha\pi}{2}),-(p_1+p_2)\cos(\frac{\beta\pi}{2}),(a_0-\frac{\varepsilon}{2})}\right|<1$$

$$k<\sqrt{\varepsilon\min\{\cos(\frac{\alpha\pi}{2}),-(p_1+p_2)\cos(\frac{\beta\pi}{2}),(a_0-\frac{\varepsilon}{2})}<1$$

then $\chi^{(n)}$ converges on an element of V , we call χ. We will show that in $L^2(Q)$:

$$\lim_{n\longrightarrow\infty}\chi^{(n)}(x,t)=\chi(x,t)$$

is a solution to the problem (P_4) showing that χ chacket :

$$A\left(\chi,v\right)=\int\limits_Q f(x,t,\chi,^R D_t^{\alpha}\chi)\cdot v(x,t)dxdt\quad\forall\, v\in O.$$

Where

$$O=\left\{v\in C^1(Q),v(0.t)=v(1.t)=0,\forall t\in(0.T)\right\},$$

and

$$A(\chi^n,v)=(^RD_t^{\frac{\alpha}{2}}\chi^n(x,t),^R_t D^{\frac{\alpha}{2}}v(x,t))_{L^2(Q)}-p_1(^RD_x^{\frac{\beta}{\acute{e}}}\chi^n(x,t),^R_x D^{\frac{\beta}{2}}v(x,t))_{L^2(Q)}$$

(6.7)

$$-p_2(^R_xD^{\frac{\beta}{2}}\chi^n(x,t),^R D_x^{\frac{\beta}{2}}v(x,t))_{L^2(Q)}+\int_Q a(x,t)\chi^n(x,t)\cdot v_x(x,t)dxdt$$

we have :

$$A\left(\chi^{(n)}-\chi,v\right)=$$

$$(^RD_t^{\frac{\alpha}{\acute{e}}}(\chi^n-\chi)(x,t),^R_t D^{\frac{\alpha}{2}}v(x,t))_{L^2(Q)}-p_1(^RD_x^{\frac{\beta}{\acute{e}}}(\chi^n-\chi)(x,t),^R_x D^{\frac{\beta}{2}}v(x,t))_{L^2(Q)}$$
$$-p_2(^R_xD^{\frac{\beta}{\acute{e}}}(\chi^n-\chi)(x,t),^R D_x^{\frac{\beta}{2}}v(x,t))_{L^2(Q)}+\int_Q a(x,t)(\chi^n-\chi)(x,t)\cdot v_x(x,t)$$

We apply the Cauchy Schwartz inequality, we find :

$$A(\chi^{(n)}-\chi,v)\leq\|v\|_V\left\|\left(\chi^{(n)}-\chi\right)_t\right\|_V+\|v\|_V\left\|\left(\chi^{(n)}-\chi\right)_x\right\|_V$$

On the other hand, as

$$\chi^{(n)} \longrightarrow \chi \quad \text{in } V,$$

So

$$\begin{aligned} \chi^{(n)} &\longrightarrow \chi \quad \text{in } L^2(Q), \\ \chi_t^{(n)} &\longrightarrow \chi_t \quad \text{in } L^2(Q), \\ \chi_x^{(n)} &\longrightarrow \chi_x \quad \text{in } L^2(Q), \end{aligned}$$

Let's go to the limit when $n \longrightarrow +\chi$, we get :$\lim_{n \longrightarrow +\chi} A(\chi^{(n)} - \chi, v) = 0$

References

[1] Oldham, KB, Spanier, J: The Fractional Calculus. Academic Press, New York (1974)

[2] Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)

[3] He, JH: Some applications of nonlinear fractional di¤erential equations and their approximations. Bull Sci Technol15, 86-90 (1999)

[4] He, JH: Nonlinear oscillation with fractional derivative and its applications. In: International Conference on Vibrating Engineering'98, Dalian, China, pp. 288-291 (1998)

[5] He, JH: Approximate analytical solution for seepage zow with fractional derivatives in porous media. Comput Methods Appl Mech Eng167, 57-68 (1998)

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[7] Metzler, R, Klafter, J: The random walk's guide to anomalous dixousion: a fractional dynamics approach. Phys Rep339, 1-77 (2000)

[8] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations,J.Wiley, New York, 1993.

[9] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon, 1993.

[10] R.P. Agarwal, M. Benchohra, S. Hamani, Boundary value problems for fractional differential equations, Adv. Stud. Contemp. Math. 16 (2008) 181–196.

[11] A. Anguraj, P. Karthikeyan, Existence of solutions for fractional semilinear evolution boundary value problem, Commun. Appl. Anal. 14 (2010) 505–514.

[12] M. Belmekki, M. Benchohra, L. Gorniewicz, Semilinear functional di¤erential equations with fractional order and in. . . nite delay, Fixed Point Theory 9 (2008) 423–439.

[13] M. Belmekki, M. Benchohra, Existence results for fractional order semilinear functional differential equations, Proc. A. Razmadze Math. Inst. 146 (2008) 9–20.

[14] M. Benchohra, J.R. Graef, S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, Appl. Anal. 87 (2008) 851–863.

[15] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for dixoerential equations with fractional order, Surv. Math. Appl. 3 (2008) 1–12.

[16] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, J. Math. Anal. Appl. 338 (2008) 1340–1350.

[17] V. Daftardar-Gejji, H. Jafari, Boundary value problems for fractional diffusion-wave equation,Aust. J. Math. Anal. Appl. 3 (2006) 1–8.

[18] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, J. Math. Anal. Appl. 204 (1996) 609–625.

[19] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl.265 (2002) 229–248.

[20] K. Diethelm, G. Walz, Numerical solution of fractional order differential equations by extrapolation, Numer. Algorithms 16 (1997) 231–253.

[21] A.M.A. El-Sayed, Fractional order evolution equations, J. Fract. Calc. 7 (1995) 89–100.

[22] A.M.A. El-Sayed, Fractional order diffusion-wave equations, Internat. J. Theoret. Phys. 35(2) (1996) 311–322.

[23] A.M.A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, Nonlinear Anal. Theory Methods Appl. 33 (1998) 181–186.

[24] K.M. Furati, N. Tatar, Behavior of solutions for a weighted Cauchy-type fractional differential problem, J. Fract. Calc. 28 (2005) 23–42.

[25] K.M. Furati, N. Tatar, An existence result for a nonlocal fractional differential problem,J.Fract. Calc. 26 (2004) 43–51.

[26] E.R. Kaufmann, E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, Electron. J. Qual. Theory Di¤er. Equ. 3 (2007) 1–11.

[27] A.A. Kilbas, S.A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, Di¤er. Equ. 41 (1) (2005) 84–89.

[28] S.M. Momani, S.B. Hadid, Some comparison results for integro-fractional differential inequalities, J. Fract. Calc. 24 (2003) 37–44.

[29] S.M. Momani, S.B. Hadid, Z.M. Alawenh, Some analytical properes of solutions of differential equations of noninteger order, Int. J. Math. Math. Sci. 13 (2004) 697–701.

[30] I. Podlubny, I. Petraš, B.M. Vinagre, P. O'Leary, L. Dorcak, Analogue realizations offractional-order controllers. Fractional order calculus and its applications, Nonlinear Dynam.29 (2002) 281–296.

[31] C. Yu, G. Gao, Existence of fractional differential equations, J. Math. Anal. Appl. 310(1)(2005) 26–29.

[32] T. Oussaeif, A. Bouziani, Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions, Vol. 2014 (2014), No. 179, pp. 1-10.

[33] D. Baleanu, J.A. Tenreiro Machado, Z.B. Guvenc, New Trends in Nanotechnology and Fractional Calculus Applications, Springer-Verlag, London, 2010.

[34] J. Sabatier, Om P. Agrawal, J.A. Tenreiro Machado, Z.B. Guvenc, Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, SpringerVerlag, London, 2007.

[35] J. Tenreiro Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 1140–1153.

[36] X. J. Li and C. J. Xu, "Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation,"Communications in Computational Physics, vol.8, no.5, pp.1016–1051, 2010.

[37] X. J. Li and C. J. Xu, "A space-time spectral method for the time fractional diffusion equation,"SIAM Journal on Numerical Analysis, vol.47, no.3, pp. 2108–2131, 2009.

[38] O. Taki-Eddine, B Abdelfatah; A priori estimates for weak solution for a time-fractional nonlinear reaction-diffusion equations with an integral condition,Chaos, Solitons & Fractals,V 103 P 79–89; 2017

[39] Oussaeif T-E, Bouziani A. Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions. Electron J Differ Equ 2014;2014(179):1–10

[40] R Imad, O Taki-Eddine, Solvability of a solution and controllability of partial fractional differential systems, Journal of Interdisciplinary Mathematics, 2021,doi.org/10.1080/09720502.2020.1838754

[41] R Imad, O Taki-Eddine and B abdelouahab, solvability of a solution and controllability for nonlinear fractional differential equations, bulletin of the institute of mathematics vol. 15(2020), no. 3, pp. 237-249

[42] Benaoua A, O taki-eddine and Rezzoug I; Unique solvability of a Dirichlet problem for a fractional parabolic equation using energy-inequality method; Methods Funct. Anal. Topology 26 (2020), no. 3, 216-226.

[43] Salih Ibrahim and Shayma Adil Murad SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH SOME EXISTENCE AND STABILITY RESULTS Vol 12(1), 2023) , 482–492.

[44] Faraj Y. Ishak ;EXISTENCE AND UNIQUENESS SOLUTION FOR FRACTIONAL VOLTERRA EQUATION WITH FRACTIONAL ANTI-PERIODIC BOUNDARY CONDITIONS ;Vol. 12(1)(2023) , 791–799.

[45] Adel Lachouri and Nesrine Gouri ; Existence and uniqueness of positive solutions for a class of fractional integro-differential equations ;Vol. 11(3)(2022) , 167–174 .

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