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GENERALIZATIONS OF 2-ABSORBING δ-PRIMARY IDEALS IN COMMUTATIVE RINGS

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Abstract In this paper, I introduced the concept of an almost 2-absorbing δ-primary ideal which unifies the concept of an almost 2-absorbing ideal and an almost 2-absorbing primary ideal in a commutative ring. I also defined and study the concept of a ϕ-2-absorbing δ-primary ideal in a commutative ring and proved some characterizations.

1 Introduction

The concept of prime ideals in rings are generalized in different directions D. D. Anderson and M. Bataineh [1]. Darani and Yousefian [4], studied generalization of primary ideals in a commutative ring. The study of an expansion of ideal and δ-primary ideals in a commutative ring is carried out by Zhao [8], where δ is a mapping with some additional properties. Fahid and Zhao [5] defined 2-absorbing δ-primary ideals in a commutative ring. Badawi and Fahid [3], defined weakly δ-primary ideals and weakly 2-absorbing δ-primary ideals of commutative rings. They defined expansion of ideals in product of rings. Also they defined δ-twin zero and δ-triple zero. S. K. Nimbhorkar and J. Y. Nehete [7], studied generalizations of δ-primary elements in Multiplicative Lattices. They defined Almost δ-primary elementss, n almost δ-primary elements and studied some characterizations.

In this paper I used expansion of ideals to define an almost 2-absorbing δ-primary ideal which unifies the concept of an almost 2-absorbing ideal and an almost 2-absorbing primary ideal in one frame. I defined an n almost n-absorbing δ-primary ideal. Also I introduced ϕ-2-absorbing δ-primary ideal in a commutative ring and proved some results. I introduced ϕ-δ-triple-zero and some results based on it.

In this paper, all the rings are commutative rings. I shall use the notation $Id(R)$ to denote the set of all ideals of the commutative ring R.

2 preliminaries

We recall some definitions of a ring R.
An ideal P is called a proper ideal of R if $P \neq R$.

Definition 2.1. A proper ideal P of R is called a prime ideal if for $a, b \in R$, $ab \in P$ implies that either $a \in P$ or $b \in P$.

Definition 2.2. The radical of an ideal I is defined as,
$\sqrt{I} = \{x \in R | x^n \in I, n \in \mathbb{N}\}$.

Definition 2.3. A proper ideal P of R is called a primary ideal if for $a, b \in R$, $ab \in P$ implies that either $a \in P$ or $b \in \sqrt{P}$.

The following definitions are from Zhao Dongsheng and Fahid [5].

Definition 2.4. An expansion of ideals, or an ideal expansion, is a function $\delta : Id(R) \to Id(R)$, satisfying the conditions (i) $I \subseteq \delta(I)$ and (ii) $J \subseteq K$ implies $\delta(J) \subseteq \delta(K)$, for all $I, J, K \in Id(R)$.

Example 2.5. (1) The identity function $\delta_0 : Id(R) \to Id(R)$, where $\delta_0(I) = I$ for every $I \in Id(R)$, is an expansion of ideals.
(2) The function $\mathbf{B}$ that assigns the biggest ideal R to each ideal is an expansion of ideals.
(3) For each proper ideal P, the mapping $\mathbf{M} : Id(R) \to Id(R)$, defined by $\mathbf{M}(P) = \cap\{I \in Id(R) | P \subseteq I, I$ is a maximal ideal other than $R\}$, and $\mathbf{M}(R) = R$. Then $\mathbf{M}$ is an expansion of ideals.
(4) For each ideal I define $\delta_1(I) = \sqrt{I}$, the radical of I. Then $\delta_1(I)$ is an expansion of ideals.

Definition 2.6. Let δ be an expansion of ideals of R. A proper ideal I of R is called a 2-absorbing δ-primary ideal if $abc \in I$, $ab \notin I$ and $bc \notin \delta(I)$, then $ac \in \delta(I)$, for all $a, b, c \in R$.

Definition 2.7. Let δ be an expansion of ideals of R. A proper ideal I of R is called a weakly 2-absorbing δ-primary ideal if $0 \neq abc \in I$, then $ab \in I$ or $bc \in \delta(I)$ or $ac \in \delta(I)$ for all $a, b, c \in R$.

Definition 2.8. Let δ be an expansion of ideals on R and n be a positive integer. A proper ideal I of R is called a n-absorbing δ-primary ideal of R if for $a_1, \ldots, a_{n+1} \in R$, $a_1 a_2 \ldots a_{n+1} \in I$, then there are n number of the a_i's whose product is in $\delta(I)$.

3 Almost 2-absorbing δ-primary ideals

We define an almost 2-absorbing δ-primary ideal in a commutative ring.

Definition 3.1. Let δ be an expansion function. Let D be a proper ideal of R. Then D is said to be an n-almost 2-absorbing δ-primary ideal if $xyz \in D - D^n$ implies that either $xy \in D$ or $yz \in \delta(D)$ or $xz \in \delta(D)$, for $x, y, z \in R$. If $n = 2$, then D is called an almost 2-absorbing δ-primary ideal of R.

In the following theorem we prove a relationships between an almost 2-absorbing δ-primary ideal and δ-primary ideals.

Theorem 3.2. *Let D be a proper ideal of R. Then the following statements hold:*
(1) If D is an almost δ-primary ideal of R, then D is an almost 2-absorbing δ-primary ideal.
(2) If D is a 2-absorbing δ-primary ideal of R, then D is an almost 2-absorbing δ-primary ideal.
(3) If D is a weakly δ-primary ideal of R, then D is an almost 2-absorbing δ-primary ideal.
(2) If D is a weakly 2-absorbing δ-primary ideal of R, then D is an almost 2-absorbing δ-primary ideal.

Proof. (1) Let $pqr \in D - D^2$ for $p, q, r \in R$. As D is almost δ-primary, we get either $pq \in D$ or $r \in \delta(D)$. If $pq \in D$, then the proof is clear. If $r \in \delta(D)$ then $pr \in \delta(D)$ and $qr \in \delta(D)$. Thus D is an almost 2-absorbing δ-primary ideal.
(2) Let $xyz \in D - D^2$ for $x, y, z \in R$. As D is a 2-absorbing δ-primary, we get $xy \in D$ or $yz \in \delta(D)$ or $xz \in \delta(D)$. Hence D is an almost 2-absorbing δ-primary ideal.
(3) Let $abc \in D - D^2$ for $a, b, c \in R$. Then $abc \in D - \{0\}$. As D is weakly δ-primary, we get either $ab \in D$ or $c \in \delta(D)$. If $ab \in D$, then the proof is clear. If $c \in \delta(D)$ then $ac \in \delta(D)$ and $bc \in \delta(D)$. Thus D is an almost 2-absorbing δ-primary ideal.
(4) Let $abc \in D - D^2$ for $a, b, c \in R$. Then $abc \in D - \{0\}$. As D is a weakly 2-absorbing δ-primary, we get $ab \in D$ or $bc \in \delta(D)$ or $ac \in \delta(D)$. Hence D is an almost 2-absorbing δ-primary ideal. □

In the following result we show that the radical of an almost 2-absorbing δ-primary ideal is also an almost 2-absorbing δ-primary.

Proposition 3.3. *Let D be a proper ideal of R such that $\sqrt{\delta(D)} = \delta(\sqrt{D})$. If D is an almost 2-absorbing δ-primary ideal of R then $\sqrt{D}$ is an almost 2-absorbing δ-primary ideal of R.*

Proof. Let $p, q, r \in R$ be such that $pqr \in \sqrt{D} - (\sqrt{D})^2$. As D is almost 2-absorbing δ-primary ideal of R and $(pqr)^n \in D - D^2$ implies that either $(pq)^n \in D$ or $(qr)^n \in \delta(D)$ or $(pr)^n \in \delta(D)$. It follows that either $pq \in \sqrt{D}$ or
$qr \in \sqrt{\delta(D)} = \delta(\sqrt{D})$ or $pr \in \sqrt{\delta(D)} = \delta(\sqrt{D})$. Thus $\sqrt{D}$ is an almost 2-absorbing δ-primary ideal of R. □

Definition 3.4. (Zhao Dongsheng [8]) Let J and K be ideals of a ring R, the residual division of J and K is defined as the set $(J : K) = \{x \in R | xy \in J \text{ for all } y \in K\}$.
Similarly, we can define $(J : a) = \{x \in R | ax \in J\}$.

In the following theorem, we give a characterization for an almost 2-absorbing δ-primary ideal.

Theorem 3.5. *For a proper ideal D of R, the following statements are equivalent:*
(1) D is an almost 2-absorbing δ-primary ideal of R;
(2) for every $p, q \in R$ such that $pq \notin \delta(D)$, $(D : pq) \subseteq (D : p) \cup (\delta(D) : q) \cup (D^2 : pq)$.

Proof. (1) $\Rightarrow$ (2) Let $x \in R$ be such that $x \in (D : pq)$ for some $pq \notin \delta(D)$. Then $xpq \in D$. If $xpq \in D^2$, then $x \in (D^2 : pq)$. Suppose that $xpq \notin D^2$. Since $pq \notin \delta(D)$ and D is almost 2-absorbing δ-primary. We get either $xp \in D$ or $xq \in \delta(D)$ it follows that either $x \in (D : p)$ or $x \in (\delta(D) : q)$. Hence $(D : pq) \subseteq (D : p) \cup (\delta(D) : q) \cup (D^2 : pq)$.
(2) $\Rightarrow$ (1) Let $pqr \in D - D^2$. Then $p \in (D : qr)$. Suppose that $qr \notin \delta(D)$.
Since $(D : qr) \subseteq (D : q) \cup (\delta(D) : r) \cup (D^2 : qr)$, we get $p \in (\delta(D) : r) \cup (D : q) \cup (D^2 : qr)$. If $p \in (\delta(D) : r)$, then $pr \in \delta(D)$. If $p \in (D : q)$, then $pq \in D$. If $p \in (D^2 : qr)$, then $pqr \in D^2$, a contradiction. Hence we get either $pr \in \delta(D)$ or $pq \in D$. Therefore, D is an almost 2-absorbing δ-primary ideal. □

Proposition 3.6. *Let D be a proper ideal of R. If I, J, K are ideals of R with $IJK \subseteq D - D^2$, then $IJ \subseteq D$ or $JK \subseteq \delta(D)$ or $IK \subseteq \delta(D)$ implies that D is an almost 2-absorbing δ-primary ideal of R .*

Proof. Let $abc \in D - D^2$, where $a, b, c \in R$. Then $< a >, < b >, < c >$ are ideals of R, and $< a >< b >< c >\subseteq D - D^2$. Hence $< a >< b >\subseteq D$ or $< b >< c >\subseteq \delta(D)$
or $< a >< c >\subseteq \delta(D)$ which means $ab \in D$ or $bc \in \delta(D)$ or $ac \in \delta(D)$. Thus D is an almost 2-absorbing δ-primary ideal of R. □

Proposition 3.7. *Let δ be an expansion of ideal such that $\delta(I)/P = \delta(I/P)$, for every ideal I of R satisfying $P \subseteq I$. If D is an almost 2-absorbing δ-primary ideal of R with $Q \subseteq D$, for any proper ideal Q of R, then D/Q is an almost 2-absorbing δ-primary ideal of R/Q.*

Proof. Let $(a + Q)(b + Q)(c + Q) \in D/Q - (D/Q)^2$ and $(a + Q)(b + Q) \notin D/Q$, where $(a + Q), (b + Q), (c + Q) \in R/Q$. Then $abc + Q \in D/Q - (D/Q)^2$ and $ab + Q \notin D/Q$. Hence $abc \in D - D^2$ and $ab \notin D$. As D is an almost 2-absorbing δ-primary ideal of R, we get $bc \in \delta(D)$ or $ac \in \delta(D)$. It implies that $bc + Q \in \delta(D)/Q$ or $ac + Q \in \delta(D)/Q$, so we conclude that $(b + Q)(c + Q) \in \delta(D)/Q = \delta(D/Q)$ or $(a + Q)(c + Q) \in \delta(D)/Q = \delta(D/Q)$. Hence D/Q is an almost 2-absorbing δ-primary ideal of R/Q. □

We prove a characterization for almost 2-absorbing δ-primary ideals.

Theorem 3.8. *Let δ be an expansion of ideal such that $\delta(I)/P = \delta(I/P)$, for every ideal I of R satisfying $P \subseteq I$. A proper ideal D of R is almost 2-absorbing δ-primary if and only if D/D^2 is a weakly 2-absorbing δ-primary ideal of R/D^2.*

Proof. First suppose that D is an almost 2-absorbing δ-primary ideal of R.
Let $0 + D^2 \neq (x + D^2)(y + D^2)(z + D^2) \in D/D^2$
and $(x + D^2)(y + D^2) \notin \delta(D/D^2) = \delta(D)/D^2$,
where $(x+D^2), (y+D^2), (z+D^2) \in R/D^2$. Then $xyz \in D-D^2$, but D is an almost 2-absorbing δ-primary ideal of R and $xy \notin \delta(D)$, so $yz \in D$ or $xz \in \delta(D)$. Then $(y + D^2)(z + D^2) \in D/D^2$ or $(x + D^2)(z + D^2) \in \delta(D/D^2) = \delta(D)/D^2$. Thus D/D^2 is a weakly 2-absorbing δ-primary ideal of R/D^2.

Conversely, suppose that D/D^2 is a weakly 2-absorbing δ-primary ideal of R/D^2. Let $p, q, r \in R$ be such that $pqr \in D - D^2$. Then $pqr + D^2 \in D/D^2$ and so $pqr + D^2 \neq D^2$, it follows that $0 + D^2 \neq (p + D^2)(q + D^2)(r + D^2) \in D/D^2$. So either $(p + D^2)(q + D^2) \in D/D^2$ or $(q + D^2)(r + D^2) \in \delta(D/D^2) = \delta(D)/D^2$ or
$(p + D^2)(r + D^2) \in \delta(D/D^2) = \delta(D)/D^2$ which implies that either $pq \in D$ or $qr \in \delta(D)$ or $pr \in \delta(D)$. Therefore D is an almost 2-absorbing δ-primary ideal of R. □

Now we introduce this concept more generally namely, n-almost n-absorbing δ-primary ideal.

Definition 3.9. A proper ideal D of R such that $a_1a_2a_3 \dots a_{n+1} \in D$ and $a_1a_2a_3 \dots a_{n+1} \notin D^n$ is called n-almost n-absorbing δ-primary if the product of n members of $\{a_1, a_2, a_3, \dots, a_{n+1}\}$ is in $\delta(D)$, for some $a_1, a_2, a_3, \dots, a_{n+1} \in R$.

The following result gives a characterization for a n-almost n-absorbing δ-primary ideal.

Theorem 3.10. *For an ideal D of R, the following statements are equivalent.*

(1) D is a n-almost n-absorbing δ-primary ideal of R.

(2) For every $a_1, a_2, a_3, \dots, a_n \in R$ with $a_1a_2a_3 \dots a_n \notin \delta(D)$, $(D : a_1a_2a_3 \dots a_n) \subseteq (\delta(D) : a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_n)$ for some $i \in \{1, 2, 3, \dots, n\}$ or $(D : a_1a_2a_3 \dots a_n) \subseteq (D^n : a_1a_2a_3 \dots a_n)$.

Proof. (1) $\Rightarrow$ (2): Let $a_1, a_2, a_3, \dots, a_n \in R$ be such that $a_1a_2a_3 \dots a_n \notin \delta(D)$,
Suppose that $b \in (D : a_1a_2a_3 \dots a_n)$, so $a_1a_2a_3 \dots a_nb \in D$. If $a_1a_2a_3 \dots a_nb \notin D^n$ then by (1), $a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_nb \in \delta(D)$ for some $i \in \{1, 2, 3, \dots, n\}$.
So $b \in (\delta(D) : a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_n)$. Hence
$(D : a_1a_2a_3 \dots a_n) \subseteq (\delta(D) : a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_n)$ for some $i \in \{1, 2, 3, \dots, n\}$. If $a_1a_2a_3 \dots a_nb \in D^n$ then $b \in (D^n : a_1a_2a_3 \dots a_n)$.
Thus $(D : a_1a_2a_3 \dots a_n) \subseteq (D^n : a_1a_2a_3 \dots a_n)$. Therefore we conclude that
$(D : a_1a_2a_3 \dots a_n) \subseteq (\delta(D) : a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_n)$
for some $i \in \{1, 2, 3, \dots, n\}$ or $(D : a_1a_2a_3 \dots a_n) \subseteq (D^n : a_1a_2a_3 \dots a_n)$.
(2) $\Rightarrow$ (1): Suppose that $a_1a_2a_3 \dots a_{n+1} \in D$ and $a_1a_2a_3 \dots a_{n+1} \notin D^n$. If the product of n members of $a_1, a_2, a_3, \dots, a_{n+1} \in R$ is in $\delta(D)$ then the proof is clear. So without loss of generality we assume that $a_1a_2a_3 \dots a_n \notin \delta(D)$. Then $a_1a_2a_3 \dots a_n \notin D$. Hence by (2),
$(D : a_1a_2a_3 \dots a_n) \subseteq (\delta(D) : a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_n)$
for some $i \in \{1, 2, 3, \dots, n\}$ or $(D : a_1a_2a_3 \dots a_n) \subseteq (D^n : a_1a_2a_3 \dots a_n)$.
As $a_1a_2a_3 \dots a_{n+1} \in D$, so we get $a_{n+1} \in (D : a_1a_2a_3 \dots a_n)$.
If $(D : a_1a_2a_3 \dots a_n) \subseteq (D^n : a_1a_2a_3 \dots a_n)$, then
$a_{n+1} \in (D : a_1a_2a_3 \dots a_n) \subseteq (D^n : a_1a_2a_3 \dots a_n)$ implies that $a_1a_2a_3 \dots a_na_{n+1} \in D^n$, a contradiction. Hence $(D : a_1a_2a_3 \dots a_n) \subseteq (\delta(D) : a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_n)$
for some $i \in \{1, 2, 3, \dots, n\}$, then
$a_{n+1} \in (D : a_1a_2a_3 \dots a_n) \subseteq (\delta(D) : a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_n)$
for some $i \in \{1, 2, 3, \dots, n\}$. So we get $a_{n+1} \in (\delta(D) : a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_n)$
for some $i \in \{1, 2, 3, \dots, n\}$ implies that $a_1a_2a_3 \dots a_{i-1}a_{i+1} \dots a_na_{n+1} \in \delta(D)$
for some $i \in \{1, 2, 3, \dots, n\}$. Therefore D is a n-almost n-absorbing δ-primary ideal . □

Badawi and B. Fahid [3], introduced expansion of ideals $\delta_\times$ in a product of rings. Let $R_1, R_2, \dots, R_n$, where $n \geq 2$, be commutative rings with $1 \neq 0$. Assume that $\delta_1, \delta_2, \dots, \delta_n$ are expansion of ideals of $R_1, R_2, \dots, R_n$ respectively.
Let $R = R_1 \times R_2 \times \dots, \times R_n$. Define a function $\delta_\times : Id(R) \to Id(R)$ such that $\delta_\times(I_1 \times I_2 \times \dots \times I_n) = \delta_1(I_1) \times \delta_2(I_2) \times \dots \times \delta_n(I_n)$ for every $I_i \in Id(R_i)$, where $1 \leq i \leq n$. Clearly, $\delta_\times$ is an expansion of ideals of R. Note that every ideals of R is of the form $I_1 \times I_2 \times \dots \times I_n$, where each I_i is an ideal of R_i, for $1 \leq i \leq n$.

Theorem 3.11. *Let R_1 and R_2 be commutative rings with identity. Let $R = R_1 \times R_2$ and δ_1, δ_2 and $\delta_\times$ be expansion of ideals of R_1, R_2 and R respectively such that for every $i \in \{1, 2\}$, if $I_i \neq R_i$ and $\delta_i(I_i) \neq R_i$. Then*
(i) P is an almost 2-absorbing δ_1-primary ideal of R_1 if and only if $P \times R_2$ is an almost

2-absorbing $\delta_\times$-primary ideal of $R_1 \times R_2$.
(ii) P is an almost 2-absorbing δ_2-primary ideal of R_2 if and only if $(R_1 \times P)$ is an almost 2-absorbing $\delta_\times$-primary ideal of $R_1 \times R_2$.

Proof. (i): Suppose that P is an almost 2-absorbing δ_1-primary ideal in R_1. As P is a proper ideal in R_1, we get $P \times R_2$ is a proper ideal in $R_1 \times R_2$. Now let $(a, x), (b, y), (c, z) \in R_1 \times R_2$ be such that $(a, x)(b, y)(c, z) \in (P \times R_2) - (P \times R_2)^2$, where $a, b, c \in R_1$ and $x, y, z \in R_2$. Since $(P \times R_2) - (P \times R_2)^2 = (P - P^2) \times R_2$, so we can write $(abc, xyz) \in (P - P^2) \times R_2$ then we get $abc \in P - P^2$, as P is an almost 2-absorbing δ_1-primary ideal in R_1 which implies that either $ab \in P$ or $bc \in \delta_1(P)$ or $ac \in \delta_1(P)$. Hence either $(ab, xy) = (a, x)(b, y) \in P \times R_2$ or $(bc, yz) = (b, y)(c, z) \in \delta_1(P) \times \delta_2(R_2) = \delta_\times(P \times R_2)$ or
$(ac, xz) = (a, x)(c, z) \in (\delta_1(P) \times \delta_2(R_2)) = \delta_\times(P \times R_2)$. Therefore $P \times R_2$ is an almost 2-absorbing $\delta_\times$-primary ideal of $R_1 \times R_2$.
Conversely, Let $P \times R_2$ be an almost 2-absorbing $\delta_\times$-primary ideal of $R_1 \times R_2$. Let $abc \in P - P^2$. So $(a, 1_{R_2})(b, 1_{R_2})(c, 1_{R_2}) \in (P - P^2) \times R_2 = (P \times R_2) - (P \times R_2)^2$, where $a, b, c \in R_1$. As $P \times R_2$ is an almost 2-absorbing $\delta_\times$-primary ideal in $R_1 \times R_2$ then either $(a, 1_{R_2})(b, 1_{R_2}) \in P \times R_2$ or $(b, 1_{R_2})(c, 1_{R_2}) \in \delta_\times(P \times R_2) = \delta_1(P) \times \delta_2(R_2)$ or
$(a, 1_{R_2})(c, 1_{R_2}) \in \delta_\times(P \times R_2) = \delta_1(P) \times \delta_2(1_{R_2})$. Hence either $ab \in P$ or $bc \in \delta_1(P)$ or $ac \in \delta_1(P)$. Therefore P is an almost 2-absorbing δ_1-primary ideal in R_1.
(ii) Can be proved by using technique as in (i). □

4 ϕ-2-absorbing δ-primary ideals

Now we define a ϕ-2-absorbing δ-primary ideal in R.

Definition 4.1. Let δ be an expansion of ideal of R. Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function such that $\phi(I) \subseteq I$, for every I of R. A proper ideal D of R is called ϕ-2-absorbing δ-primary if $pqr \in D - \phi(D)$ implies either $pq \in D$ or $qr \in \delta(D)$ or $pr \in \delta(D)$, for $p, q, r \in R$.

Definition 4.2. Let δ be an expansion of ideal of R. Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function such that $\phi(I) \subseteq I$, for every I of R. A proper ideal D of R is called ω-2-absorbing δ-primary if $pqr \in D - \bigcap_{n=1}^{\infty} D^n$ implies either $pq \in D$ or $qr \in \delta(D)$ or $pr \in \delta(D)$, for $p, q, r \in R$.

Theorem 4.3. *For a proper ideal D of R.*
Consider the following statements hold:
(i) If D is a 2-absorbing δ-primary, then D is a weakly 2-absorbing δ-primary.
(ii)If D is a weakly 2-absorbing δ-primary, then D is a ω-2-absorbing δ-primary.
(iii)If D is a ω-2-absorbing δ-primary, then D is a n-almost 2-absorbing δ-primary.
(iv) If D is a n-almost 2-absorbing δ-primary, then D is an almost 2-absorbing δ-primary.

Proof. (i) Proof is obvious.
(ii) Suppose that D is not a ω-2-absorbing δ-primary ideal of R. Then there exist $p, q, r \in R$ such that $pqr \in D - \bigcap_{n=1}^{\infty} D^n$ and $pq \notin D$ or $qr \notin \delta(D)$ or $pr \notin \delta(D)$. Since D is weakly 2-absorbing δ-primary, it follows that $pq \in D$ or $qr \in \delta(D)$ or or $pr \in \delta(D)$, a contradiction. Hence $pqr = 0$ this contradicts to $pqr \notin \bigcap_{n=1}^{\infty} D^n$. Hence D is a ω-2-absorbing δ-primary ideal of R.
(iii) Suppose that D is ω-2-absorbing δ-primary and $(n \geq 2)$. Let $abc \in D - D^n$ for some $a, b, c \in R$.Then $abc \in D - \bigcap_{n=1}^{\infty} D^n$ for some $a, b, c \in R$, since D is ω-2-absorbing δ-primary it follows that either $ab \in D$ or $bc \in \delta(D)$ or $ac \in \delta(D)$. Hence D is a n-almost 2-absorbing δ-primary ideal, $(n \geq 2)$.
(iv) The last implication is obvious for $n = 2$. □

The following theorem gives a characterization of a ω-2-absorbing δ-primary ideal in R.

Theorem 4.4. *Let D be a proper ideal of R. Then D is a ω-2-absorbing δ-primary if and only if D is a n-almost 2-absorbing δ-primary for every $n \geq 2$.*

Proof. Let D be a n-almost 2-absorbing δ-primary for every $n \geq 2$. Suppose that
$pqr \in D - \bigcap_{n=1}^{\infty} D^n$ for some $p, q, r \in R$, then $pqr \in D - D^m$ for some $m \geq 2$ but for every $n \geq 2$, D is n-almost 2-absorbing δ-primary, we get either $pq \in D$ or $qr \in \delta(D)$ or $pr \in \delta(D)$. Hence D is a ω-2-absorbing δ-primary.
The converse follows from Theorem 4.3(iii). □

Next we show that the radical of a ϕ-2-absorbing δ-primary ideal of L is again a ϕ-2-absorbing δ-primary ideal.

Lemma 4.5. *Let D be a ϕ-2-absorbing δ-primary ideal of R such that $\sqrt{\phi(D)} = \phi(\sqrt{D})$ and $\sqrt{\delta(D)} = \delta(\sqrt{D})$. Then $\sqrt{D}$ is a ϕ-2-absorbing δ-primary ideal in R.*

Proof. Assume that $pqr \in \sqrt{D} - \phi(\sqrt{D})$ but $pq \notin \sqrt{D}$ for some $p, q, r \in R$. Then there exists a positive integer n such that $(pqr)^n \in D$. If $(pqr)^n \in \phi(D)$, then by hypothesis $pqr \in \sqrt{\phi(D)} = \phi(\sqrt{D})$, a contradiction. So assume that $(pqr)^n \notin \phi(D)$ and $(pq)^n \notin D$. Then we get $(qr)^n \in \delta(D)$ or or $(pr)^n \in \delta(D)$, as D is ϕ-2-absorbing δ-primary.
Hence $qr \in \sqrt{\delta(D)} = \delta(\sqrt{D})$ or $pr \in \sqrt{\delta(D)} = \delta(\sqrt{D})$. Therefore $\sqrt{D}$ is a ϕ-2-absorbing δ-primary ideal in R. □

Lemma 4.6. *Let $\phi_1, \phi_2 : Id(R) \to Id(R) \cup \{\emptyset\}$ be function with $\phi_1(I) \subseteq \phi_2(I)$ for every I of R. If D is ϕ_1-2-absorbing δ-primary, then D is also a ϕ_2-2-absorbing δ-primary.*

Proof. Let $a, b, c \in R$ be such that $abc \in D - \phi_2(D)$ implies $abc \notin \phi_1(D)$. Since D is ϕ_1-2-absorbing δ-primary then we get $ab \in D$ or $bc \in \delta(D)$ or $ac \in \delta(D)$. Thus D is a ϕ_2-2-absorbing δ-primary. □

Lemma 4.7. *Let D be a proper ideal of L. Suppose that $\phi(D)$ is a 2-absorbing δ-primary ideal of R. If D is a ϕ-2-absorbing δ-primary ideal of R, then D is a 2-absorbing δ-primary ideal of R.*

Proof. Assume that $pqr \in D$ for some $p, q, r \in R$ and $pq \notin D$. If $pqr \in \phi(D)$. Since $\phi(D) \subseteq D$ and $pq \notin D$ then $pq \notin \phi(D)$. As $\phi(D)$ is 2-absorbing δ-primary, we get either $qr \in \delta(\phi(D)) \subseteq \delta(D)$ or $pr \in \delta(\phi(D)) \subseteq \delta(D)$. If $pqr \notin \phi(D)$, then as D is a ϕ-2-absorbing δ-primary, we get either $qr \in \delta(D)$ or $pr \in \delta(D)$. Hence D is a 2-absorbing δ-primary ideal of R. □

Definition 4.8. Let D be a ϕ-2-absorbing δ-primary ideal of R and $a, b, c \in R$. If $abc \in \phi(D)$ but $ab \notin D$, $bc \notin \delta(D)$ and $ac \notin \delta(D)$, then (a, b, c) is called a ϕ-δ-triple zero of D.

Lemma 4.9. *If D is a ϕ-2-absorbing δ-primary ideal of R that is not a 2-absorbing δ-primary ideal of R, then D has a ϕ-δ-triple-zero (a, b, c), for some $a, b, c \in R$.*

Proof. Since D is not 2-absorbing δ-primary, then there exist $a, b, c \in R$ such that $abc \notin D$, $ab \notin D$, $bc \notin \delta(D)$ and $ac \notin \delta(D)$. As D is a ϕ-2-absorbing δ-primary ideal of R, if $abc \notin \phi(D)$, then either $ab \in D$ or $bc \in \delta(D)$ or $ac \in \delta(D)$ which is not possible. Hence $abc \in \phi(D)$. Thus D has a ϕ-δ-triple-zero (a, b, c). □

Theorem 4.10. *Let D be a ϕ-2-absorbing δ-primary ideal of R and suppose that (x, y, z) is a ϕ-δ-triple zero of D for some $x, y, z \in R$. Then*
(1) $xyD, yzD, xzD \subseteq \phi(D)$.
(2) $xD^2, yD^2, zD^2 \subseteq \phi(D)$.

Proof. (1) Suppose that $xyD \not\subseteq \phi(D)$. Then there exists $d \in D$ such that $xyd \notin \phi(D)$.
Then $xyz + xyd = xy(z + d) \in D$ and $xyz + xyd = xy(z + d) \notin \phi(D)$. As $xy \notin D$ and D is ϕ-2-absorbing δ-primary ideal, either $x(z + d) \in \delta(D)$ or $y(z + d) \in \delta(D)$. So we get either $xz \in \delta(D)$ or $yz \in \delta(D)$, which is a contradiction to (x, y, z) is ϕ-δ-triple zero of D. Hence $xyd \in \phi(D)$ and so $xyD \subseteq \phi(D)$. Similarly, we can show that $yzD, xzD \subseteq \phi(D)$.
(2) Suppose that $xD^2 \not\subseteq \phi(D)$. Then there exists $d_1, d_2 \in D$ such that $xd_1d_2 \notin \phi(D)$.
$xyz + xd_1d_2 + xzd_1 + xyd_2 = x(y + d_1)(z + d_2) \in D$ and $x(y + d_1)(z + d_2) \notin \phi(D)$. As D is ϕ-2-absorbing δ-primary ideal, we have either $x(y + d_1) \in D$ or $x(z + d_2) \in \delta(D)$ or $(y + d_1)(z + d_2) \in \delta(D)$. So we get either $xy \in D$ or $xz \in \delta(D)$ or $yz \in \delta(D)$, which is a contradiction to (x, y, z) is ϕ-δ-triple zero of D. Hence $xd_1d_2 \in \phi(D)$ and so $xD^2 \subseteq \phi(D)$. Similarly, we can prove that $yD^2, zD^2 \subseteq \phi(D)$. □

Theorem 4.11. *If D is a ϕ-2-absorbing δ-primary ideal of R which is not a 2-absorbing δ-primary ideal, then $D^3 \subseteq \phi(D)$.*

Proof. Suppose that D is a ϕ-2-absorbing δ-primary ideal of R which is not a 2-absorbing δ-primary ideal of R, then by Lemma 4.9, D has a ϕ-δ-triple-zero (x, y, z), for some $x, y, z \in R$. Suppose that $D^3 \nsubseteq \phi(D)$. Then there exist $d_1, d_2, d_3 \in D$ such that $d_1d_2d_3 \notin \phi(D)$.
Consider
$xyz + d_1d_2d_3 + xzd_2 + xyd_3 + xd_2d_3 + yzd_1 + yd_1d_3 + zd_1d_2 = (x + d_1)(y + d_2)(z + d_3) \in D$ and $(x + d_1)(y + d_2)(z + d_3) \notin \phi(D)$. As D is ϕ-2-absorbing δ-primary ideal, we have either $(x + d_1)(y + d_2) \in D$ or $(x + d_1)(z + d_3) \in \delta(D)$ or $(y + d_2)(z + d_3) \in \delta(D)$. So we get either $xy \in D$ or $xz \in \delta(D)$ or $yz \in \delta(D)$, which is a contradiction to (x, y, z) is ϕ-δ-triple zero of D. Hence $d_1d_2d_3 \in \phi(D)$ and so $D^3 \subseteq \phi(D)$. □

The following proposition gives some conditions for a ϕ-2-absorbing δ-primary ideal of R to be a 2-absorbing δ-primary ideal of R.

Proposition 4.12. *Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function such that $\phi(I) \subseteq I$, for every ideal I of R.*
(1) If D is a ϕ-2-absorbing δ-primary ideal of R such that $D^3 \nsubseteq \phi(D)$, then D is 2-absorbing δ-primary.
(2) If D is a ϕ-2-absorbing δ-primary ideal that is not a 2-absorbing δ-primary ideal of R and $\delta(D^3) = \delta(D)$, then $\delta(D) = \delta(\phi(D))$.

Proof. (1) The proof follows from Remark 4.9 and Theorem 4.11. (2) Since $\phi(D) \subseteq D$, we have $\delta(\phi(D)) \subseteq \delta(D)$. On the otherhand, it follows that from part (1) that $D^3 \subseteq \phi(D)$. Hence $\delta(D) = \delta(D^3) \subseteq \delta(\phi(D))$, so $\delta(D) = \delta(\phi(D))$. □

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