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New results on MS-Lipschitz summing operators

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Abstract This paper focuses on the study of MS-Lipschitz p-summing operators, which were initially defined by the authors in [1]. Our objective is to establish relationships between T and its linearizations, namely $\widehat{T}$ and $\widetilde{T}$. Additionally, we extend our investigation by introducing a new definition in the category of Lipschitz mappings defined on metric spaces, known as MS-Cohen Lipschitz p-summing. We provide several results and characterizations for this new concept.

1 Introduction and preliminaries

The theory of p-summing operators has undergone several stages of development. It originated in the 1950s with Grothendieck's pioneering work [9], where he introduced the concept of 1-summing operators. In 1967 [12], Pietsch made a significant contribution by defining p-summing operators for all positive values of p. His most notable result in this theory is the Pietsch Factorization Theorem, which provides an integral characterization. Since then, this theory has witnessed substantial advancements. Researchers have expanded this theory in various directions, including the sublinear, multilinear, and more recently, the Lipschitz case. Let X be a pointed metric space and E be a Banach space. We denote by $Lip_0(X, E)$ the Banach space of all Lipschitz functions $f : X \longrightarrow E$ which vanish at 0 under the Lipschitz norm given by

$$Lip\,(f) = \sup\left\{\frac{\|f\,(x) - f\,(y)\|}{d\,(x, y)} : x, y \in X, x \neq y\right\}.$$

If $E = \mathbb{R}$, we simply denote by $Lip_0(X)\ (= X^{\#})$. We consider the evaluation functionals $\delta_x \in Lip_0(X)^*$ for $x \in X$ such that $\delta_x\,(f) = f\,(x)$ for $f \in Lip_0(X)$. The Lipschitz-free space $\mathcal{F}\,(X)$ is the closed space generated by these evaluation functionals. For the general theory of free Banach spaces, see [7, 8, 11, 18, 19]. We have $X^{\#} = \mathcal{F}\,(X)^*$ holds isometrically via the application

$$Q_X\,(f)\,(m) = m\,(f)\,,\ \text{for every } f \in X^{\#} \text{ and } m \in \mathcal{F}\,(X)\,.$$

Every Lipschitz mapping $T : X \to E$ induces a unique linear operator $\widehat{T} : \mathcal{F}\,(X) \longrightarrow E$ such that

$$\widehat{T} \circ \delta_X = T,$$

where $\delta_X : X \to \mathcal{F}\,(X)$ is the canonical embedding so that $\delta_X\,(x) = \delta_x$ for $x \in X$. In this case, the identification

$$Lip_0\,(X, E) = \mathcal{B}\,(\mathcal{F}\,(X)\,, E)\,, \tag{1.1}$$

holds isometrically. It is well-known that if a Lipschitz map $T : X \to E$ is p-summing, its linearization $\widehat{T}$ is not necessarily p-summing. In [16], the author has introduced the concept of strictly Lipschitz p-summing for which the relation between T and its linearization $\widehat{T}$ for the concept of p-summing is well established. Now, let X and Y be two pointed metric spaces, we can associate to every Lipschitz operator $T : X \to Y$ another linear operator $\widetilde{T} : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ such that

$$\widetilde{T} \circ \delta_X = \delta_Y \circ T.$$

Note that (see [8, p. 124]) it is not difficult to check that

$$\widetilde{T}^* = T^{\#}, \tag{1.2}$$

where $T^{\#} : Y^{\#} \to X^{\#}$ is the linear map, celled Lipschitz adjoint, defined by $T^{\#}(g) = g \circ T$. The objective of this study is to examine the relationship between T and its linearization operators $\widetilde{T}$ and $T^{\#}$ within various summability concepts. In a previous work [1], the authors introduced the notion of M-strictly Lipschitz p-summing (MS-Lipschitz p-summing) operators defined on pointed metric spaces and established a significant connection between T and its linearization $\widetilde{T}$. This paper focuses on presenting novel findings concerning strictly Lipschitz p-summing and MS-Lipschitz p-summing operators. We further extend this idea by introducing a new definition, termed MS-Cohen Lipschitz p-summing, within the category of Lipschitz mappings defined on metric spaces. Additionally, we provide results and explore the relationships between T and its linearizations $\widetilde{T}$ and $T^{\#}$ based on this new concept.

The structure of the paper is as follows:

Section 1 provides a brief review of the standard notations that will be employed throughout the paper. Section 2 is dedicated to the examination of various characterizations of MS-Lipschitz p-summing operators defined between metric spaces. In Section 3, we introduce the definition of MS-Lipschitz p-nuclear operators. These operators exhibit remarkable properties, particularly their associations with linearization operators. Furthermore, we present a Pietsch Domination Theorem that applies specifically to this class of operators.

Now, we recall briefly some basic notations and terminology which we need in the sequel. Throughout this paper, the letters E, F will denote Banach spaces and X, Y will denote metric spaces with a distinguished point (pointed metric spaces) which we denote by 0. Let X be a pointed metric space, E be a Banach space and $Lip_0(X, E)$ be the Banach space of all Lipschitz functions (Lipschitz operators) $T : X \to E$ such that $T(0) = 0$ with pointwise addition and Lipschitz norm. Note that for any $T \in Lip_0(X, E)$, there exists a unique linear map (linearization of T) $\widehat{T} : \mathcal{F}(X) \longrightarrow E$ such that $\widehat{T} \circ \delta_X = T$ and $\left\|\widehat{T}\right\| = Lip(T)$, i.e., the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ \delta_X \downarrow & \nearrow \widehat{T} & \\ \mathcal{F}(X) & & \end{array}$$

where δ_X is the canonical embedding so that $\langle \delta_X(x), f\rangle = \delta_{(x,0)}(f) = f(x)$ for $f \in X^{\#}$. See [2, 6, 10, 15] for more details about the properties of Lipschitz operators. If X is a Banach space and $T : X \to E$ is a linear operator, then the corresponding linear operator $\widehat{T}$ is given by

$$\widehat{T} = T \circ \beta_X,$$

where $\beta_X : \mathcal{F}(X) \to X$ is a linear quotient map which verifies $\beta_X \circ \delta_X = id_X$ and $\|\beta_X\| \leq 1$, see [8, p. 124] for more details about the operator β_X. Let X, Y be two metric spaces. Let $T : X \to Y$ be a Lipschitz operator, then there is a unique linear operator $\widetilde{T}$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow \delta_X & & \downarrow \delta_Y \\ \mathcal{F}(X) & \xrightarrow{\widetilde{T}} & \mathcal{F}(Y) \end{array}$$

i.e., $\widetilde{T} \circ \delta_X = \delta_Y \circ T$. The Lipschitz adjoint map $T^{\#} : Y^{\#} \to X^{\#}$ of T is defined as follows

$$T^{\#}(g)(x) = g(T(x)), \text{ for every } g \in Y^{\#} \text{ and } x \in X.$$

Let X be a metric space and E be a Banach space, by $X \boxtimes E$ we denote the Lipschitz tensor product of X and E. This is the vector space spanned by the linear functional $\delta_{(x,y)} \boxtimes e$ on $Lip_0(X, E^*)$ defined by

$$\delta_{(x,y)} \boxtimes e(f) = \langle f(x) - f(y), e\rangle.$$

If $m \in \mathcal{F}(X)$ such that $m = \sum_{i=1}^{n} \delta_{(x_i,y_i)}$, we have

$$\langle m, f\rangle = \sum_{i=1}^{n} \left(f(x_i) - f(y_i)\right).$$

See [2] for more details about the properties of the space $X \boxtimes E$. Now, let E be a Banach space, then B_E denotes its closed unit ball and E^* its (topological) dual. Consider $1 \leq p \leq \infty$ and $n \in \mathbb{N}^*$. We denote by $\ell_p^n(E)$ the Banach space of all sequences $(x_i)_{i=1}^n$ in E with the norm

$$\left\|(x_i)_i\right\|_{\ell_p^n(E)} = (\textstyle\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}},$$

and by $\ell_p^{n,w}(E)$ the Banach space of all sequences $(x_i)_{i=1}^n$ in E with the norm

$$\left\|(x_i)_i\right\|_{\ell_p^{n,w}(E)} = \sup_{x^* \in B_{E^*}} (\textstyle\sum_{i=1}^{n} |\langle x_i, x^*\rangle|^p)^{\frac{1}{p}}.$$

If $E = \mathbb{K}$, we simply write ℓ_p^n and $\ell_p^{n,w}$. In particular, if $(m_i)_{i=1}^{n_1} \in \mathcal{F}(X)$ such that $m_i = \sum_{j=1}^{n_2} \delta_{\left(x_i^j, y_i^j\right)}$, $(1 \leq i \leq n_1)$ we have

$$\begin{aligned}\left\|(m_i)_i\right\|_{\ell_p^{n_1,w}(\mathcal{F}(X))} &= \sup_{f \in B_{X^\#}} (\sum_{i=1}^{n_1} |\langle m_i, f\rangle|^p)^{\frac{1}{p}} \\ &= \sup_{f \in B_{X^\#}} (\sum_{i=1}^{n_1} \left|\sum_{j=1}^{n_2} f(x_i^j) - f(y_i^j)\right|^p)^{\frac{1}{p}}.\end{aligned}$$

Let us recall the following concepts:

- The linear operator $L : E \to F$ is p-summing if there exists a constant $C > 0$ such that, for any $(x_i)_{i=1}^n \subset E$, we have

$$(\sum_{i=1}^{n} \|L(x_i)\|^p)^{\frac{1}{p}} \leq C \left\|(x_i)_i\right\|_{\ell_p^{n,w}(E)}. \tag{1.3}$$

The class of p-summing linear operators from E into F, which is denoted by $\Pi_p(E, F)$, is a Banach space for the norm $\pi_p(u)$, i.e., the smallest constant C such that the inequality (1.3) holds.

- The linear operator L is (Cohen) strongly p-summing if there exists a constant $C > 0$ such that, for any $(x_i)_{i=1}^n \subset E$, and any $(y_i^*)_{i=1}^n \subset F^*$, we have

$$\sum_{i=1}^{n} |\langle L(x_i), y_i^*\rangle| \leq C \left\|(x_i)_i\right\|_{\ell_p^n(E)} \left\|(y_i^*)_i\right\|_{\ell_{p^*}^{n,w}(F^*)}. \tag{1.4}$$

The class of Cohen strongly p-summing operators from E into F, which is denoted by $\mathcal{D}_p(E, F)$, is a Banach space for the norm $d_p(L)$, i.e., the smallest constant C such that the inequality (1.4) holds.

- The linear operator L is Cohen p-nuclear if there exists a constant $C > 0$ such that, for any $(x_i)_{i=1}^n \subset E$, and any $(y_i^*)_{i=1}^n \subset F^*$, we have

$$\sum_{i=1}^{n} |\langle L(x_i), y_i^*\rangle| \leq C \left\|(x_i)_i\right\|_{\ell_p^{n,w}(E)} \left\|(y_i^*)_i\right\|_{\ell_{p^*}^{n,w}(F^*)}. \tag{1.5}$$

The class of Cohen p-nuclear linear operators from E into F, which is denoted by $\mathcal{N}_p(E, F)$, is a Banach space for the norm $n_p(L)$, i.e., the smallest constant C such that the inequality (1.5) holds.

2 M-strictly Lipschitz p-summing operators

Let X be a pointed metric space and E be a Banach space. The concept of Lipschitz tensor product, denoted by $X \boxtimes E$, was introduced by Cabrera-Padilla et al. [2]. An element u in $X \boxtimes E$ can be represented as $u = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \boxtimes e_k$ and can be viewed as a linear functional on $Lip_0(X, E^*)$. The action of this linear functional is defined by

$$\sum_{k=1}^{l} \delta_{(x_k,y_k)} \boxtimes e_k (f) = \sum_{k=1}^{l} (f(x_k) - f(y_k)) e_k \text{ for every } f \in Lip_0(X, E^*).$$

The relationship between $X \boxtimes E$ and $\mathcal{F}(X) \otimes E$ is straightforward, where $X \boxtimes E$ is a vector subspace of $\mathcal{F}(X) \otimes E$. Given an element $u \in X \boxtimes E$, we can define the set A_u as the set of all representations of u in $\mathcal{F}(X) \otimes E$, that is,

$$A_u := \left\{ \left((m_i)_{i=1}^n, (e_i)_{i=1}^n \right) : n \in \mathbb{N}^*,\ m_i \in \mathcal{F}(X), e_i \in E : u = \sum_{i=1}^{n} m_i \otimes e_i \right\}. \tag{2.1}$$

Let α be a tensor norm defined on Banach spaces. According to [16, Theorem 3.1], there exists a corresponding Lipschitz cross-norm, denoted as α^L, which is defined on the Lipschitz tensor product $X \boxtimes E$ as follows:

$$\alpha^L(\sum_{k=1}^{l} \delta_{(x_k,y_k)} \boxtimes e_k) = \alpha(\sum_{k=1}^{l} \delta_{(x_k,y_k)} \otimes e_k). \tag{2.2}$$

where $\sum_{k=1}^{l} \delta_{(x_k,y_k)} \otimes e_k \in \mathcal{F}(X) \otimes E$. Before presenting the following definition, it is necessary to recall the norms of Chevet-Saphar d_p and g_p [3, 13, 14, 17], which are defined on two Banach spaces E and F

$$d_p(u) = \inf\left\{ \left\|(x_i)_i\right\|_{\ell_{p^*}^{n,w}(E)} \left\|(y_i)_i\right\|_{\ell_p^n(F)} \right\} \text{ and } g_p(u) = \inf\left\{ \left\|(x_i)_i\right\|_{\ell_{p^*}^{n}(E)} \left\|(y_i)_i\right\|_{\ell_p^{n,w}(F)} \right\}$$

where the infimum is taken over all representations of u in the form of $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$. It is worth noting that we can utilize the Chevet-Saphar norms to provide equivalent definitions for (1.3) and (1.4) (see [14, p. 140]). Specifically, the linear operator $L : E \to F$ is said to be p-summing if there exists a constant $C > 0$ such that for any $u = \sum_{i=1}^{n} x_i \otimes y_i^* \in E \otimes F^*$, the following inequality holds:

$$\left| \sum_{i=1}^{n} \langle L(x_i), y_i^* \rangle \right| \leq C d_p(u).$$

If we replace $d_p(u)$ with $g_p(u)$, we obtain the definition of strongly p-summing. In [16], the Lipschitz cross-norm d_p^L is defined as follows:

$$d_p^L(u) = d_p(\sum_{k=1}^{l} \delta_{(x_k,y_k)} \otimes s_k) = \inf_{\left((m_i)_{i=1}^n, (e_i)_{i=1}^n\right) \in A_u} \left\{ \|(m_i)_i\|_{\ell_p^{n,w}(\mathcal{F}(X))} \left\|(e_i)_i\right\|_{\ell_{p^*}^{n}(E)} \right\}.$$

Most definitions of summability for Lipschitz mappings are typically defined from a metric space to a Banach space. However, in the following definition of MS-Lipschitz p-summing, we consider Lipschitz operators defined on metric spaces. This new perspective allows us to establish a meaningful relationship between T and its linearization $\widetilde{T}$. Before proceeding, let us recall the following definition introduced in [16].

Definition 2.1. [16] Let $1 \leq p \leq \infty$. Let X be a metric space and E be a Banach space. A Lipschitz operator $T : X \to E$ is said to be *strictly Lipschitz p-summing* if there exists a positive constant C such that for every $x_k, y_k \in X$ and $e_k^* \in E^*$ $(1 \leq k \leq l)$ we have

$$\left| \sum_{k=1}^{l} \langle T(x_k) - T(y_k), e_k^* \rangle \right| \leq C d_p^L(u), \tag{2.3}$$

where $u=\sum_{k=1}^{l}\delta_{(x_k,y_k)}\boxtimes e_k^*\in X\boxtimes E^*$.

Building upon the aforementioned idea, we have introduced the concept of MS-Lipschitz p-summing operators [1]. In contrast to considering elements from the dual space of E, we now focus on elements from its Lipschitz space $E^{\#}$.

Definition 2.2. [1] Let $1\leq p\leq\infty$. Let X and Y be two metric spaces. A Lipschitz operator $T:X\rightarrow Y$ is said to be *MS-Lipschitz p-summing* if there exists a positive constant C such that for every $x_k,y_k\in X$ and $g_k\in Y^{\#}$ $(1\leq k\leq l)$ we have

$$\left|\sum_{k=1}^{l}g_k\left(T\left(x_k\right)\right)-g_k\left(T\left(y_k\right)\right)\right|\leq Cd_p^L(u), \tag{2.4}$$

where $u=\sum_{k=1}^{l}\delta_{(x_k,y_k)}\boxtimes g_k\in X\boxtimes Y^{\#}$. We denote by $\Pi_p^{MSL}\left(X,Y\right)$ the set of all MS-Lipschitz p-summing operators from X into Y and by $\pi_p^{MSL}\left(T\right)$ the smallest constant C satisfying (2.4). If Y is a Banach space, we note that $\Pi_p^{MSL}\left(X,Y\right)$ does not have the structure of a vector space.

Proposition 2.3. *Let $1\leq p\leq\infty$. Every MS-Lipschitz p-summing operator from a pointed metric space X into a Banach space E is strictly Lipschitz p-summing.*

Proof. Let $T:X\rightarrow E$ be a MS-Lipschitz p-summing operator. Let $x_k,y_k\in X$ and $e_k^*\in E^*$ $\left(1\leq k\leq l\right)$, then

$$\begin{aligned}\left|\sum_{k=1}^{l}\left\langle T\left(x_k\right)-T\left(y_k\right),e_k^*\right\rangle\right| &= \left|\sum_{k=1}^{l}e_k^*\left(T\left(x_k\right)\right)-e_k^*\left(T\left(y_k\right)\right)\right| \\ &\leq Cd_p^L(u),\end{aligned}$$

where $u=\sum_{k=1}^{l}\delta_{(x_k,y_k)}\boxtimes e_k^*\in X\boxtimes E^{\#}$. So, as $X\boxtimes E^*\subset X\boxtimes E^{\#}$, the definition of $d_p^L(u)$ on $X\boxtimes E^{\#}$ is smaller than on $X\boxtimes E^*$, consequently, the condition (2.3) is verified. ■

Remark 2.4. In the context where E and F are Banach spaces, it is well-known that the definitions of strictly Lipschitz p-summing, Lipschitz p-summing, and p-summing coincide for linear operators from E to F (see [16, Proposition 3.8]). Furthermore, in our specific case, the definition of MS-Lipschitz p-summing implies p-summing; however, the converse is not true, as illustrated in the following example: Consider the identity operator $id_E:E\rightarrow E$. It can be easily demonstrated that $\widetilde{id_E}=id_{\mathcal{F}(E)}$, indicating that the following diagram is commutative

$$\begin{array}{ccc} E & \overset{id_E}{\longrightarrow} & E \\ \downarrow\delta_E & & \downarrow\delta_E \\ \mathcal{F}\left(E\right) & \overset{id_{\mathcal{F}(E)}}{\longrightarrow} & \mathcal{F}\left(E\right)\end{array}$$

If E is a finite-dimensional space, then id_E is indeed p-summing, and consequently, it is also strictly Lipschitz p-summing. However, $id_{\mathcal{F}(E)}$ cannot be p-summing since $\mathcal{F}\left(E\right)$ is not finite-dimensional. Therefore, id_E is not MS-Lipschitz p-summing.

The following statement presents the main result of this section.

Theorem 2.5. *Let $1\leq p\leq\infty$. Let X and Y be two metric spaces. Let $T:X\rightarrow Y$ be a Lipschitz operator. The following properties are equivalent.*
1) T is MS-Lipschitz p-summing.
2) $\widetilde{T}:\mathcal{F}\left(X\right)\rightarrow\mathcal{F}\left(Y\right)$ is p-summing.
3) $\delta_Y\circ T:X\rightarrow\mathcal{F}\left(Y\right)$ is strictly Lipschitz p-summing.
4) There is a constant $C>0$ such that for every $\left(x_i^j\right)_{i=1}^{n_1},\left(y_i^j\right)_{i=1}^{n_1}$ in $X;\left(1\leq j\leq n_2\right)$ and

$n_1, n_2 \in \mathbb{N}^*$, *we have*

$$(\sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \delta_Y \circ T(x_i^j) - \delta_Y \circ T(y_i^j) \right\|^p)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^\#}} (\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} f(x_i^j) - f(y_i^j) \right|^p)^{\frac{1}{p}}. \qquad (2.5)$$

Proof. 1) $\Leftrightarrow$ 2) See [1, Proposition 2.4.].
2) $\Rightarrow$ 3) : Suppose that $\widetilde{T}$ is p-summing. Then

$$\begin{aligned}
\sum_{i=1}^{n_1}(\left\| \sum_{j=1}^{n_2} \delta_Y \circ T(x_i^j) - \delta_Y \circ T(y_i^j) \right\|^p)^{\frac{1}{p}} &= (\sum_{i=1}^{n_1} \left\| \widetilde{T}(m_i) \right\|^p)^{\frac{1}{p}} \\
&\leq \pi_p \left(\widetilde{T}\right) \sup_{f \in B_{X^\#}} (\sum_{i=1}^{n_1} |f(m_i)|^p)^{\frac{1}{p}} \\
&\leq \pi_p \left(\widetilde{T}\right) \sup_{f \in B_{X^\#}} (\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} f(x_i^j) - f(y_i^j) \right|^p)^{\frac{1}{p}}
\end{aligned}$$

3) $\Rightarrow$ 2) : We have

$$\begin{aligned}
(\sum_{i=1}^{n_1} \left\| \widetilde{T}(m_i) \right\|^p)^{\frac{1}{p}} &= (\sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \delta_Y \circ T\left(x_i^j\right) - \delta_Y \circ T\left(y_i^j\right) \right\|^p)^{\frac{1}{p}} \\
&\leq C \sup_{f \in B_{X^\#}} (\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} f(x_i^j) - f(y_i^j) \right|^p)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^\#}} (\sum_{i=1}^{n_1} |f(m_i)|^p)^{\frac{1}{p}}
\end{aligned}$$

Then $\widetilde{T}$ is p-summing and by [5, Theorem 2.12] we obtain the result.
3) $\Leftrightarrow$ 4) It is immediate. ■

By setting $n_2 = 1$ in formula (2.5) and considering the isometric property of δ_Y, we arrive at the precise formulation of Lipschitz p-summing as originally defined by Farmer [6], indeed

$$\begin{aligned}
(\sum_{i=1}^{n_1} \|\delta_Y \circ T(x_i) - \delta_Y \circ T(y_i)\|^p)^{\frac{1}{p}} &= (\sum_{i=1}^{n_1} d\left(T(x_i), T(y_i)\right)^p)^{\frac{1}{p}} \\
&\leq C \sup_{f \in B_{X^\#}} (\sum_{i=1}^{n_1} |f(x_i) - f(y_i)|^p)^{\frac{1}{p}}.
\end{aligned}$$

Corollary 2.6. *Let $T : X \to Y$ be a Lipschitz operator between metric spaces. The following properties are equivalent.*
1) T is MS-Lipschitz p-summing.
2) The Lipschitz adjoint $T^\# : Y^\# \to X^\#$ is strongly p^*-summing.

Proof. According to (1.2), the dual operator of $\widetilde{T}$ is $T^\#$. To establish this equivalence, we can easily utilize the result mentioned in [4, Theorem 2.2.2]. ■

Proposition 2.7. *Let $T : X \to Y$ be a Lipschitz mapping between pointed metric spaces such that X or Y is finite, then T is MS-Lipschitz p-summing.*

Proof. Suppose that X is finite. By [18, Example 2.3.6], $\mathcal{F}(X)$ is finite dimensional, then $\widetilde{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is p-summing. Consequently $T : X \to Y$ is MS-Lipschitz p-summing. ■

The Pietsch domination theorem is an intriguing characterization that is satisfied by the class of MS-Lipschitz p-nuclear operators.

Theorem 2.8. *Let X and Y be two pointed metric spaces. Let $T : X \to Y$ be a Lipschitz operator. The following properties are equivalent.*
1) T is MS-Lipschitz p-summing.
2) There exist a constant $C > 0$, a Radon probability μ on $B_{X^{\#}}$ such that for every $(x^j)_{j=1}^n, (y^j)_{j=1}^n \subset X$, we have

$$\left\| \sum_{j=1}^{n} \delta_Y \circ T(x^j) - \delta_Y \circ T(y^j) \right\| \leq C(\int_{B_{X^{\#}}} \left| \sum_{j=1}^{n} f(x^j) - f(y^j) \right|^p d\mu(f))^{\frac{1}{p}}. \tag{2.6}$$

In this case, we have

$$\pi_p^{MSL}(T) = \inf \{ C : \text{ verifying (2.6)} \}.$$

Proof. $1) \Rightarrow 2)$: Since T is MS-Lipschitz p-summing, then $\widetilde{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is p-summing. By Pietsch domination theorem for p-summing linear operator [5, Theorem 2.12], we have

$$\begin{aligned} \left\| \widetilde{T} \left(\sum_{j=1}^{n} \delta_{(x^j, y^j)} \right) \right\| &\leq \pi_p \left(\widetilde{T} \right) (\int_{B_{X^{\#}}} \left| \left\langle \sum_{j=1}^{n} \delta_{(x^j, y^j)}, f \right\rangle \right|^p d\mu(f))^{\frac{1}{p}} \\ &\leq \pi_p \left(\widetilde{T} \right) (\int_{B_{X^{\#}}} \left| \sum_{j=1}^{n} f(x^j) - f(y^j) \right|^p d\mu(f))^{\frac{1}{p}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left\| \widetilde{T} \left(\sum_{j=1}^{n} \delta_{(x^j, y^j)} \right) \right\| &= \left\| \sum_{j=1}^{n} \widetilde{T} \left(\delta_{(x^j, y^j)} \right) \right\| \\ &= \left\| \sum_{j=1}^{n} \widetilde{T} \circ \delta_X \left(x^j \right) - \widetilde{T} \circ \delta_X \left(y^j \right) \right\| \\ &= \left\| \sum_{j=1}^{n} \delta_Y \circ T(x^j) - \delta_Y \circ T(y^j) \right\| \end{aligned}$$

Therefore, we have obtained the desired result.
$2) \Rightarrow 1)$: Similarly, we can apply the same argument. ■

We conclude this section with a result concerning Lipschitz operators that have a finite image. The following Lemma establishes a relationship between the free space of $T(X)$ and the image $\widetilde{T}(\mathcal{F}(X))$.

Lemma 2.9. *Let X and Y be pointed metric spaces. Consider a Lipschitz operator $T : X \to Y$, where $T(X)$ is a closed subset of Y. Then, we have the following*

$$\widetilde{T}(\mathcal{F}(X)) = \mathcal{F}(T(X)).$$

Proof. By [18, Theorem 2.2.6] we have

$$\begin{aligned} \mathcal{F}(T(X)) &= \overline{span} \left\{ \delta_{T(x)} : x \in X \right\} \\ &= \overline{span} \left\{ \delta_Y(T(x)) : x \in X \right\} \\ &= \overline{span} \left\{ \widetilde{T}(\delta_x) : x \in X \right\} \\ &= \widetilde{T}(\mathcal{F}(X)). \quad ■ \end{aligned}$$

Corollary 2.10. *Let X and Y be pointed metric spaces. Suppose that $T : X \to Y$ is a Lipschitz operator such that $T(X)$ is a finite set. Then, the linearization $\widetilde{T}$ has finite rank. As a consequence, every finite Lipschitz operator is MS-Lipschitz p-summing.*

3 Cohen MS-Lipschitz p-nuclear operators

In [4], Cohen introduced the concepts of strongly p-summing and p-nuclear operators in the context of linear operators. Since then, many authors have explored and extended these notions in various directions, including multilinear, sublinear, and Lipschitz cases. Building upon this line of thought, we will further extend these concepts using a similar approach to the one presented in the previous section.

Definition 3.1. Let $1 \leq p \leq \infty$. A Lipschitz operator $T : X \to Y$ is said to be *MS-strongly Lipschitz p-summing* if there exists a positive constant C such that for every $x_k, y_k \in X$ and $g_k \in Y^{\#}$ $(1 \leq k \leq l)$ we have

$$\left|\sum_{k=1}^{l} g_k\left(T(x_k)\right) - g_k\left(T(y_k)\right)\right| \leq C g_p^L(u), \tag{3.1}$$

where $u = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \boxtimes g_k \in X \boxtimes Y^{\#}$. We denote by $\mathcal{D}_p^{MSL}(X,Y)$ the set of all MS-strongly Lipschitz p-summing operators from X into Y and by $d_p^{MSL}(T)$ is the smallest constant C verifying (3.1). Note that $\mathcal{D}_p^{MSL}(X,Y)$ does not have a structure of a vector space.

Theorem 3.2. *Let $1 \leq p \leq \infty$. Let X and Y be two metric spaces. Let $T : X \to Y$ be a Lipschitz operator. The following properties are equivalent.*
1) T is MS-strongly Lipschitz p-summing.
2) There exists a positive constant C such that for every $(x_i)_{i=1}^n, (y_i)_{i=1}^n$ in X and $(g_i)_{i=1}^n$ in $Y^{\#}$; ($n \in \mathbb{N}^*$),

$$\left|\sum_{i=1}^{n} g_i\left(T(x_i)\right) - g_i\left(T(y_i)\right)\right| \leq C\Big(\sum_{i=1}^{n} d\left(x_i,y_i\right)^p\Big)^{\frac{1}{p}} \left\|(g_i)_{i=1}^n\right\|_{\ell_{p^*}^{n,w}(Y^{\#})}. \tag{3.2}$$

3) $\delta_Y \circ T : X \to \mathcal{F}(Y)$ is strongly Lipschitz p-summing.
4) $\widetilde{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is strongly p-summing.

Proof. $1) \Rightarrow 2)$: Let T be a MS-strongly Lipschitz p-summing operator. Let $(x_i)_{i=1}^n, (y_i)_{i=1}^n$ in X and $(g_i)_{i=1}^n \subset Y^{\#}$. We have

$$\left|\sum_{i=1}^{n} g_i\left(T(x_i)\right) - g_i\left(T(y_i)\right)\right| \leq d_p^{MSL}(T)\, g_p^L(u),$$

where $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes g_i \in X \boxtimes Y^{\#}$. Then

$$\begin{aligned}\left|\sum_{i=1}^{n} g_i\left(T(x_i)\right) - g_i\left(T(y_i)\right)\right| &\leq d_p^{MSL}(T)\Big(\sum_{i=1}^{n} \left\|\delta_{(x_i,y_i)}\right\|^p\Big)^{\frac{1}{p}} \left\|(g_i)_{i=1}^n\right\|_{\ell_{p^*}^{n,w}(Y^{\#})} \\ &\leq d_p^{MSL}(T)\Big(\sum_{i=1}^{n} d\left(x_i,y_i\right)^p\Big)^{\frac{1}{p}} \left\|(g_i)_{i=1}^n\right\|_{\ell_{p^*}^{n,w}(Y^{\#})}.\end{aligned}$$

$2) \Rightarrow 3)$: We will prove that $\delta_Y \circ T : X \to \mathcal{F}(Y)$ is strongly Lipschitz p-summing. Let $(x_i)_{i=1}^n, (y_i)_{i=1}^n$ in X and $(g_i)_{i=1}^n \subset Y^{\#}\left(= \mathcal{F}(Y)^*\right)$. Then

$$\begin{aligned}\left|\sum_{i=1}^{n} \langle \delta_Y \circ T(x_i) - \delta_Y \circ T(y_i), g_i\rangle\right| &= \left|\sum_{i=1}^{n} g_i\left(\delta_Y \circ T(x_i)\right) - g_i\left(\delta_Y \circ T(y_i)\right)\right| \\ &= \left|\sum_{i=1}^{n} g_i\left(T(x_i)\right) - g_i\left(T(y_i)\right)\right| \\ &\leq C\Big(\sum_{i=1}^{n} d\left(x_i,y_i\right)^p\Big)^{\frac{1}{p}} \left\|(g_i)_{i=1}^n\right\|_{\ell_{p^*}^{n,w}(Y^{\#})}.\end{aligned}$$

Then $\delta_Y \circ T$ is strongly Lipschitz p-summing.
$3) \Rightarrow 4)$: We know that

$$\widehat{\delta_Y \circ T} = \widetilde{T}.$$

Furthermore, according to [15, Proposition 3.1], the linearization $\widehat{\delta_Y \circ T}$ is strongly p-summing, which implies that $\widetilde{T}$ is also strongly p-summing.
$4) \Rightarrow 1)$: Assuming that $\widetilde{T}$ is strongly p-summing, we can deduce from [14, Proposition 6.12] that $\widetilde{T}$ satisfies the following

$$\left|\sum_{i=1}^{n}\left\langle \widetilde{T}\left(m_i\right), g_i\right\rangle\right| \leq C g_p(u)$$

where $u = \sum_{i=1}^{n} m_i \otimes g_i \in \mathcal{F}(X) \boxtimes Y^{\#}$. If we put $m_i = \delta_{(x_i,y_i)}$, we find

$$\begin{aligned}\left|\sum_{i=1}^{n}\left\langle \widetilde{T}\left(\delta_{(x_i,y_i)}\right), g_i\right\rangle\right| &= \left|\sum_{i=1}^{n} g_i\left(T\left(x_i\right)\right) - g_i\left(T\left(y_i\right)\right)\right| \\ &\leq C g_p(u) = C g_p^L(u)\end{aligned}$$

where $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes g_i \in \mathcal{F}(X) \boxtimes Y^{\#}$. ■

Using the same reasoning as in Corollary 2.6, we can establish the following result.

Corollary 3.3. *Let $T : X \to Y$ be a Lipschitz operator between metric spaces. The following properties are equivalent.*
1) T is MS-strongly Lipschitz p-summing.
2) The Lipschitz adjoint $T^{\#} : Y^{\#} \to X^{\#}$ is p^*-summing.

The following integral characterization is an adaptation of the linear case. To prove it, we rely on the fact that $T^{\#}$ is p^*-summing or $\widetilde{T}$ is strongly p-summing.

Theorem 3.4. *Let $1 \leq p \leq \infty$.Let X and Y be two metric spaces. Let $T : X \to Y$ be a Lipschitz operator. The following properties are equivalent.*
1) T is MS-strongly Lipschitz p-summing.
2) There exist a constant $C > 0$ and a Radon probability μ on $B_{Lip_0(Y)^*}$ such that for every $x, y \in X$ and $g \in Y^{\#}$, we have

$$|g(T(x)) - g(T(y))| \leq C d(x,y)\left(\int_{B_{Lip_0(Y)^*}} |\langle g, m\rangle|^{p^*} d\mu(m)\right)^{\frac{1}{p^*}}.$$

Let us now recall the definition of the tensor norm w_p on the product of two Banach spaces $E \otimes F$, which has been studied in [14, p. 180]. Let $p \in [1,\infty]$ we have

$$w_p(u) = \inf\left\{\left\|(x_i)_i\right\|_{\ell_p^{n,w}(E)} \left\|(y_i)_i\right\|_{\ell_{p^*}^{n,w}(F)}\right\},$$

where the infimum is taken over all representations of u of the form $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$.

Definition 3.5. Let $1 \leq p \leq \infty$. Let X and Y be two metric spaces. A Lipschitz operator $T : X \to Y$ is said to be Cohen MS-Lipschitz p-nuclear if there is a constant $C > 0$ such that such that for every $x_k, y_k \in X$ and $g_k \in Y^{\#}$ $(1 \leq k \leq l)$ we have

$$\left|\sum_{k=1}^{l} g_k\left(T(x_k)\right) - g_k\left(T(y_k)\right)\right| \leq C w_p^L(u), \tag{3.3}$$

where $u = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \boxtimes g_k \in X \boxtimes Y^{\#}$. We denote by $\mathcal{N}_p^{MSL}(X,Y)$ the set of all Cohen MS-Lipschitz p-nuclear operators from X into Y and by $n_p^{MSL}(T)$ is the smallest constant C

verifying (3.3). Again, we note that $\mathcal{N}_p^{MSL}(X,Y)$ does not have a structure of a vector space.

Theorem 3.6. *Let $T: X \to Y$ be a Lipschitz operator. The following properties are equivalent.*
1) T is Cohen MS-Lipschitz p-nuclear.
2) There is a constant $C>0$ such that for every $\left(x_i^j\right)_{i=1}^{n_1}, \left(y_i^j\right)_{i=1}^{n_1}$ in X, $(g_i)_{i=1}^{n_1} \subset Y^\#$; $(1 \leq j \leq n_2)$ and $n_1, n_2 \in \mathbb{N}^*$, we have

$$\left|\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} g_i\left(T(x_i^j)\right) - g_i\left(T(y_i^j)\right)\right| \leq C \sup_{f\in B_{X^\#}} (\sum_{i=1}^{n_1}\left|\sum_{j=1}^{n_2} f(x_i^j) - f(y_i^j)\right|^p)^{\frac{1}{p}} \left\|(g_i)_{i=1}^{n_1}\right\|_{\ell_{p^*}^{n_1,w}(Y^\#)}. \tag{3.4}$$

3) $\widetilde{T}: \mathcal{F}(X) \to \mathcal{F}(Y)$ is Cohen p-nuclear.

Proof.
1) $\Rightarrow$ 2) : Let T be a Cohen MS-Lipschitz p-nuclear operator. Let $\left(x_i^j\right)_{i=1}^{n_1}, \left(y_i^j\right)_{i=1}^{n_1}$ in X and $(g_i)_{i=1}^{n_1}$ in $Y^\#$; $(1 \leq j \leq n_2)$, we have

$$\left|\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} g_i\left(T(x_i^j)\right) - g_i\left(T(y_i^j)\right)\right| \leq C g_p^L(u),$$

where $u = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \delta_{\left(x_i^j, y_i^j\right)} \boxtimes g_i \in X \boxtimes Y^\#$. Then

$$\begin{aligned}
&\left|\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} g_i\left(T(x_i^j)\right) - g_i\left(T(y_i^j)\right)\right| \\
\leq\ & n_p^{MSL}(T) \sup_{f\in B_{X^\#}} (\sum_{i=1}^{n_1}\left|f(\sum_{j=1}^{n_2} \delta_{\left(x_i^j, y_i^j\right)})\right|^p)^{\frac{1}{p}} \left\|(g_i)_{i=1}^{n_1}\right\|_{\ell_{p^*}^{n_1,w}(Y^\#)} \\
\leq\ & n_p^{MSL}(T) \sup_{f\in B_{X^\#}} (\sum_{i=1}^{n_1}\left|\sum_{j=1}^{n_2} f(x_i^j) - f(y_i^j)\right|^p)^{\frac{1}{p}} \left\|(g_i)_{i=1}^{n_1}\right\|_{\ell_{p^*}^{n_1,w}(Y^\#)}.
\end{aligned}$$

2) $\Rightarrow$ 3) : Let $(m_i)_{i=1}^{n_1} \subset \mathcal{F}(X)$ and $(g_i)_{i=1}^{n_1} \subset Y^\#$ such that

$$m_i = \sum_{j=1}^{n_2} \delta_{\left(x_i^j, y_i^j\right)} \in \mathcal{F}(X), \ \ (1 \leq i \leq n_1).$$

Then

$$\begin{aligned}
(\sum_{i=1}^{n_1}\left|\left\langle \widetilde{T}(m_i), g_i\right\rangle\right|^p)^{\frac{1}{p}} &= (\sum_{i=1}^{n_1}\left|\left\langle \sum_{j=1}^{n_2}\left(\delta_Y \circ T\left(x_i^j\right) - \delta_Y \circ T\left(x_i^j\right)\right), g_i\right\rangle\right|^p)^{\frac{1}{p}} \\
&= \sum_{i=1}^{n_1}(\left|\sum_{j=1}^{n_2} g_i\left(T(x_i^j)\right) - g_i\left(T(y_i^j)\right)\right|^p)^{\frac{1}{p}} \\
&\leq C \sup_{f\in B_{X^\#}} (\sum_{i=1}^{n_1}\left|\sum_{j=1}^{n_2} f(x_i^j) - f(y_i^j)\right|^p)^{\frac{1}{p}} \left\|(g_i)_{i=1}^{n_1}\right\|_{\ell_{p^*}^{n_1,w}(Y^\#)} \\
&\leq C \sup_{f\in B_{X^\#}} (\sum_{i=1}^{n_1}\left|f(\sum_{j=1}^{n_2} \delta_{(x_i^j, y_i^j)})\right|^p)^{\frac{1}{p}} \left\|(g_i)_{i=1}^{n_1}\right\|_{\ell_{p^*}^{n_1,w}(Y^\#)} \\
&\leq C \sup_{f\in B_{X^\#}} (\sum_{i=1}^{n_1} |f(m_i)|^p)^{\frac{1}{p}} \left\|(g_i)_{i=1}^{n_1}\right\|_{\ell_{p^*}^{n_1,w}(Y^\#)}.
\end{aligned}$$

Then $\widetilde{T}$ is p-nuclear.
3) $\Rightarrow$ 1) : Suppose that $\widetilde{T}$ is Cohen p-nuclear. Let $x_k, y_k \in X$ and $g_k \in Y^{\#}$ $(1 \leq k \leq l)$ we have

$$\begin{aligned} \left|\sum_{k=1}^{l} g_k\left(T(x_k)\right) - g_k\left(T(y_k)\right)\right| &= \left|\sum_{k=1}^{l} \langle \delta_Y \circ T\left(x_k\right) - \delta_Y \circ T\left(y_k\right), g_k\rangle\right| \\ &= \left|\sum_{k=1}^{l} \left\langle \widehat{T}\left(\delta_{(x_k,y_k)}\right), g_k\right\rangle\right| \\ &\leq n_p\left(\widetilde{T}\right) w_p(u) = n_p\left(\widetilde{T}\right) w_p^L(u), \end{aligned}$$

where $u = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \boxtimes g_k \in X \boxtimes Y^{\#}$. Finally, T is Cohen MS-Lipschitz p-nuclear and we have

$$n_p^{MSL}(T) \leq n_p\left(\widetilde{T}\right). \qquad \blacksquare$$

By utilizing the result presented in [4, Theorem 2.2.4], we can establish the following relationship between T and its Lipschitz adjoint $T^{\#}$.

Corollary 3.7. *Let $T : X \to Y$ be a Lipschitz operator between metric spaces. The following properties are equivalent.*
1) T is Cohen MS-Lipschitz p-nuclear.
2) The Lipschitz adjoint $T^{\#} : Y^{\#} \to X^{\#}$ is Cohen p^*-nuclear.

The following integral characterization is an adaptation of the linear case. Its proof will be omitted.

Theorem 3.8. *Let $1 \leq p \leq \infty$.Let X and Y be two metric spaces. Let $T : X \to Y$ be a Lipschitz operator. The following properties are equivalent.*
1) T is Cohen MS-Lipschitz p-nuclear.
2) There exist a constant $C > 0$, a Radon probability μ on $B_{X^{\#}}$ and $\eta \in B_{Lip_0(Y)^*}$ such that for every $(x^j)_{j=1}^n, (y^j)_{j=1}^n$ in X and *$g \in Y^{\#}$, we have*

$$\begin{aligned} \left|\sum_{j=1}^{n} g\left(T(x^j)\right) - g\left(T(x^j)\right)\right| &\leq C\Big(\int_{B_{X^{\#}}} \left|\sum_{j=1}^{n} f(x^j) - f(y^j)\right|^p d\mu\left(f\right)\Big)^{\frac{1}{p}} \times \\ &\Big(\int_{B_{Lip_0(Y)^*}} \left|\langle g, m\rangle\right|^{p^*} d\eta\left(m\right)\Big)^{\frac{1}{p^*}}. \end{aligned}$$

Theorem 3.9. *Let X, Y and Z be three pointed metric spaces. Let $u : X \to Z$ be a MS-Lipschitz p-summing operator and $v : Z \to Y$ be a MS-strongly Lipschitz p-summing operator. Then $T = v \circ u$ is Cohen MS-Lipschitz p-nuclear.*

Proof. By [8, p. 124] we have

$$\widetilde{T} = \widetilde{v} \circ \widetilde{u}.$$

According to a result due to Cohen [4], the linear operator $\widetilde{u}$ being p-summing and $\widetilde{v}$ being strongly p-summing imply that $\widetilde{T}$ is Cohen p-nuclear. Consequently, T is also MS-Lipschitz p-nuclear. $\blacksquare$

Remark 3.10.
In the linear case, the converse of the previous statement is true. However, in our case, it is unknown whether every Cohen MS-Lipschitz p-nuclear operator can be expressed as the product of an MS-Lipschitz p-summing operator and an MS-strongly p-summing operator.

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