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# Nonlinear Schrödinger Equation in Colombeau Algebra

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**Abstract** In the present work we give a new proof of the existence and uniqueness of generalized solution of the nonlinear Schrödinger problem with singular initial data, in the framework of Colombeau algebra. Our work is based on the theory of generalized fixed point and the generalized semigroups theory. We have shown the link between this generalized solution and the classical solution via the concept of the association.

## 1 Introduction

Colombeau's theory of generalized functions has been developed in connection with nonlinear problems, see [3, 13, 15, 18, 21, 22, 23, 24, 25]. The algebra $\mathcal{G}^s$ of Colombeau is differential which contains the space $\mathcal{D}^{'}$ of distributions. Furthermore, nonlinear operations more general than the multiplication make sense in the algebra $\mathcal{G}^s$. Therefore the algebra $\mathcal{G}^s$ is a very convenient one to find and study the solutions of nonlinear differential equations with singular initial data. The theory of Colombeau generalized functions provides new solutions of partial differential equations, these new solutions can be divided into two categories see [16]. The first there are classical functions or distributions which are solutions (in one of the new senses provided by this theory) of partial differential equations without solution in the sense of distributions, e. g. see [1, 5]. The second there are also new objects (such as the square of the Dirac delta distribution,...) which can be solutions of equations. In [16] the fundamental concept of regularized derivatives was studied and results on global solvability, in the framework of this theory, of the Cauchy problem for large classes of regularized partial differential equations have been given. In particular, the well-known Schrödingar problem type with initial data is a distribution with regularized derivatives become solvable in the Colombeau algebra. It is then interesting to show the relation between Colombeau generalized solutions and distributional solutions if they exist. Within the framework of Colombeau's algebra, work on generalized fixed point theory remains somewhat rare, except for the work of J.A.Marti see [8] noting that this author who was gived for the first time the generalized of the fixed point Banach theorem in Colombeau algebra. Our paper is inspired by his work in order to give a new proof of the existence and uniqueness of the schrödinger equation with singular data and potential, $u(t,x) = i(\Delta + v(x))u(t,x)$ which has been extensively studied in the last 20 years by many authors. However the study of this equation within the framework of the algebra of Colombeau remains a little rare. This paper deals, in the framework of the simplified Colombeau algebra, with a class of differential operators non solvable in distributions theory. The generalized is lies in the core of a Colombeau into then self [8], this tow notions are used in solving a class of generalized evolutions problem like a class of heat equatios with singular coefficient and data. singular coeffficients of an evolution problem are regularized can be nets of smooth functions depending on a small parameter $\varepsilon$. Regularized evolution problem is then solve some partial differential equations using an appropriate net of semigroups. In ordre to demonstrate the existence of a moderate solution of the Schrödinger equation we using the generalized point, by this way, a net of solutions obtained represents a generalized solution. We will use differant various of Colombeau type generalized function algebra. They contrain embedded distributions and there consider the self of all smooth functions as a subalgebra. we refer to [1],[4], and [11].

This paper is organized as follows. In section 2, we recall the theory of Colombeau. In Section 3 we recall the theory of fixed point in Colombeau algebra, we define the generalized semigroup which called the Colombeau semigroup in secion 4. Section 5 intersted to study the genrelized evolution problem, more precisely the Schrödinger nonlinear problem. finally in section 6 we studied the association between the classical and generalized solutions, we ended this article with an example of illustration given in the context of probability measures.

## 2 Preliminairies

Here, we list some notations and formulas to be used later. The elements of Colombeau algebras $\mathcal{G}^s$ are equivalence classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter $\varepsilon$. Therefore, for any set $X$, the family of sequences $(u_\varepsilon)_{\varepsilon\in(0,1)}$ of elements of a set X will be denoted by $X^{(0,1)}$, such sequences will also be called nets and simply written as $u_\varepsilon$. see [12]. Let $\Omega$ be an open subset of $\mathbb{R}^n$. The basic objects of the theory are families $(u_\varepsilon)_{\varepsilon\in(0;1)}$ of smooth functions $u_\varepsilon \in \mathcal{C}^\infty(\Omega)$ for $0 < \varepsilon < 1$. We single out the following subalgebras of:
Moderate families, denoted by $\mathcal{E}_M(\Omega)$, are defined by the property

$$\forall K \Subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \exists p \geq 0 : \sup_{x\in K} \|\partial^\alpha u_\varepsilon(x)\| = O_{\varepsilon\to 0}(\varepsilon^{-p}). \tag{2.1}$$

Null families, denoted by $\mathcal{E}_M(\Omega)$, are defned by the property

$$\forall K \Subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \forall q \geq 0 : \sup_{x\in K} \|\partial^\alpha u_\varepsilon(x)\| = O_{\varepsilon\to 0}(\varepsilon^{q}). \tag{2.2}$$

These moderate families satisfy a locally uniform polynomial estimate as $\varepsilon \to 0$, together with all derivatives, while null functionals vanish faster than any power of $\varepsilon$ in the same situation. Null families from a differential ideal in the collection of moderate families. The Colombeau algebra is the factor algebra

$$\mathcal{G}^s(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega). \tag{2.3}$$

This algebra coincides with the special Colombeau algebra in [12], where the notation $\mathcal{G}^s(\Omega)$ has been employed. It was called the simplified Colombeau algebra in [5]. The Colombeau algebra on a closed half space $\mathbb{R}^n \times [0,1)$ is defined in a similary way. Restriction of an element $u \in \mathcal{G}^s(\mathbb{R}^n \times [0,1))$ to the line $\{t = 0\}$ is defined on representatives by

$$u_{|\{t=0\}} = \textit{Class of}\, (u_\varepsilon(\cdot, 0))_{\varepsilon\in(0,1)}. \tag{2.4}$$

Similary, restrictions of the elements of $\mathcal{G}^s(\Omega)$ to open subsets of $\Omega$ are defined on representatives. One can see that $\Omega \to \mathcal{G}^s(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$. The space of compactly supported distributions is imbedded in $\mathcal{G}^s(\Omega)$ by convolution

$$i : \mathcal{E}^{'}(\Omega) \to \mathcal{G}^s(\Omega), \quad i(\omega) = \textit{ class of}\, (\omega * (\phi_\varepsilon)/\Omega)_{\varepsilon\in(0,1)}, \tag{2.5}$$

where

$$\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon}) \tag{2.6}$$

is obtained by scaling a fixed test function $\mathcal{S}(\mathbb{R}^n)$ of integral one with all moments vanishing. By the sheaf property, this can be extended in a unique way to an imbedding of the space of distributions $\mathcal{D}'(\Omega)$. One of the main features of the Colombeau construction is the fact that this imbedding renders $\mathcal{C}^\infty(\Omega)$ a faithful subalgebra. In fact, given $f \in \mathcal{C}^\infty(\Omega)$, one can define a corresponding element of $\mathcal{G}^s(\Omega)$ by the constant imbedding

$$\sigma(f) = \textit{ class of}\, [(\varepsilon, x) \to f(x)].$$

Then the important equality $i(f) = \sigma(f)$ holds in $\mathcal{G}^s(\Omega)$. If $u \in \mathcal{G}^s(\Omega)$ and $f$ is a smooth function which is of at most polynomial growth at infinity, together with all its derivatives, the

superposition $f(u)$ is a well-defined element of $\mathcal{G}^s(\Omega)$.
A net$(r_\varepsilon)_{\varepsilon\in(0,1]}$ of complex numbers is called a slow scale net if

$$\|r_\varepsilon^p\| = O_{\varepsilon\to 0}(\varepsilon^{-1}), \tag{2.7}$$

for every $p \geq 0$. We refer to [10] for a detailed discussion of slow scale nets. Finally, an element $u \in \mathcal{G}^s(\Omega)$ is called of total slow scale type, if for some representative, $\|\partial^\alpha u_\varepsilon\|_{L^\infty(K)}$ forms a slow scale net for every $K \subset \Omega$ and $\alpha \in \mathbb{N}_0^n$. We end this section by recalling the association relation on the Colombeau algebra $\mathcal{G}^s(\Omega)$. It identifies elements of $u \in \mathcal{G}^s(\Omega)$ if they coincide in the weak limit. That is, $u, v \in \mathcal{G}^s(\Omega)$ are called associated,

$$u \approx v, \quad \text{if} \quad \lim_{\varepsilon\to 0}\int (u_\varepsilon(x) - v_\varepsilon(x))\psi(x)dx = 0. \tag{2.8}$$

## 3 Fixed point

We start this section by the ring of generalized real numbers. Let

$$\mathcal{E}_M(\mathbb{R}) := \Big\{(x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)} /\ \exists m \in \mathbb{N}, \|x_\varepsilon\| = O_{\varepsilon\to 0}(\varepsilon^{-m})\Big\} \tag{3.1}$$

the set of all family of moderate real sequences, and

$$\mathcal{N}(\mathbb{R}) := \Big\{(x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)} /\ \forall m \in \mathbb{N}, \|x_\varepsilon\| = O_{\varepsilon\to 0}(\varepsilon^{m})\Big\} \tag{3.2}$$

is the set of all family of negligeable real sequences. We have
- $\mathcal{E}_M(\mathbb{R})$ is an algebra with usual multiplication,
- $\mathcal{N}(\mathbb{R})$is an ideal in $\mathcal{E}_M(\mathbb{R})$.

Then the ring of real generalized sequences is given as the factor algebra

$$\tilde{\mathbb{R}} = \mathcal{E}_M(\mathbb{R})/\mathcal{N}(\mathbb{R}). \tag{3.3}$$

### 3.1 Locally convex and complete spaces

Let $X$ be a vector space with a seminorms familly $(p_i)_{i\in I}$. If $\tau_i$ is the toplogy defined by the only seminorm $p_i$. If $\tau$ is the super bound of topology $\tau_i$.

**Definition 3.1.** The spece provided with this topology $\tau$ is called a locally convex space A basis of 0-neighbourhood is the set of all "balls" of the seminorms $(p_i)_{i\in I}$

$$B(i;r) = \Big\{x \in X \ \ /\ p_i(x) < r\Big\} \qquad \forall i \in I \quad \text{and} \quad : \ r > 0.$$

Then, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence iff

$$(\forall\varepsilon > 0)(\forall i \in I)(\exists n_0 \in \mathbb{N})(\forall n, p \in \mathbb{N} \quad \text{if} \quad n \geq n_0 \quad \Rightarrow \quad p_i(x_{n+p} - x_n) < \varepsilon \ \ ).$$

**Definition 3.2.** Let $X$ be a locally convex space with a seminorms familly $(p_i)_{i\in I}$
we define the space of moderate functions based on the locally convex space $X$ by

$$\mathcal{E}_M(X) := \Big\{(x_\varepsilon)_\varepsilon \in (X)^{(0,1)}/\forall i \in I,\ \exists m \in \mathbb{N}, \quad p_i(x_\varepsilon) = O_{\varepsilon\to 0}(\varepsilon^{-m})\Big\}.$$

And the space of null functions based on the locally convex space $X$ is given by

$$\mathcal{N}(X) := \Big\{(x_\varepsilon)_\varepsilon \in (X)^{(0,1)}/\forall i \in I,\ \forall m \in \mathbb{N}, \quad p_i(x_\varepsilon) = O_{\varepsilon\to 0}(\varepsilon^{m})\Big\}.$$

We have the as in the case of the construction of the simplified that
- $\mathcal{E}_M(X)$ is an algebran,
- $\mathcal{N}(X)$ ideal in $\mathcal{E}_M(X)$.

The algebra corresponds to these spaces is given by the quotient

$$\tilde{X} = \mathcal{E}_M(X)/\mathcal{N}(X).$$

First, we are looking if it is possible to define a map $A : \tilde{X} \to \tilde{X}$ by given a family $(A_\varepsilon)_{\varepsilon\in(0,1)}$ of maps $A_\varepsilon : X \to X$, where $A_\varepsilon$ is a linear and continuous operator. The general requirement is given in the following

**Lemma 3.3.** *[8] Let $(A_\varepsilon)_{\varepsilon\in(0,1)}$ be a family of maps $A_\varepsilon : X \to X$. Suppose that for each $(x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$ and $(y_\varepsilon)_\varepsilon \in \mathcal{N}(X)$*
***i)*** *$(A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$;*
***ii)*** *$(A_\varepsilon(x_\varepsilon + y_\varepsilon))_\varepsilon - (A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{N}(X)$.*
*Then*

$$A : \tilde{X} \to \tilde{X}$$
$$x = [x_\varepsilon] \mapsto Ax = [A_\varepsilon x_\varepsilon]$$

*is well defined.*

### 3.2 Contractions in locally convex and complete spaces

**Definition 3.4.** A map $A_\varepsilon : X \to X$ is called a contraction if for all $i \in I$ there exists $k_i < 1$ such that

$$\forall (x_\varepsilon, y_\varepsilon) \in X \times X, \; p_i(A_\varepsilon x_\varepsilon - A_\varepsilon y_\varepsilon) \leq k_i \, p_i(x_\varepsilon - y_\varepsilon).$$

**Theorem 3.5.** *Any contraction $A_\varepsilon : X \to X$ has a fixed point. Moreover, if $X$ is Hausdoff, this fixed point is unique.*

### 3.3 Contraction operator in $\tilde{X}$

**Definition 3.6.** :[8] The following hypotheses permit to well define a map $A : \tilde{X} \to \tilde{X}$ and to call it a contraction.
**a)** for each $(x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$, $(A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$.
**b)** Each $A_\varepsilon$ is a contraction in $(X, \tau_\varepsilon)$ endowed with the family $Q_\varepsilon = (q_{\varepsilon,i})_{i\in I}$ and the corresponding contraction constants are denoted by $l_{\varepsilon,i} < 1$.
**c)** For each $i \in I$ and $\varepsilon \in (0, 1], \exists a_{\varepsilon,i} > 0$ and $b_{\varepsilon,i} > 0$, such that

$$a_{\varepsilon,i} \, p_i \leq q_{\varepsilon,i} \leq b_{\varepsilon,i} \, p_i.$$

**d)** For each $i \in I$ , $\forall \varepsilon \in (0, 1]$, $(\frac{b_{\varepsilon,i}}{a_{\varepsilon,i}})_\varepsilon$ and $(\frac{1}{1-l_{\varepsilon,i}})_\varepsilon \in \|\mathcal{E}_M(\mathbb{R})\|$.

**Theorem 3.7.** *:[8] Any contraction $A : \tilde{X} \to \tilde{X}$ has a fixed point in $\tilde{X}$.*

## 4 Generalized semigroup

**Definition 4.1.** [19] $\mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}_c(X))$ is the space of nets $(S_\varepsilon)_\varepsilon$ of strongly continuous mappings
$S_\varepsilon : \mathbb{R}_+ \to \mathcal{L}_c(X), \;\; \varepsilon \in (0, 1)$ with the property that for every $T > 0$ there exists $a \in \mathbb{R}$ such that,

$$\sup_{t\in[0,T)} \|S_\varepsilon(t)\| = O_{\varepsilon\to 0}(\varepsilon^a). \tag{4.1}$$

$\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}_c(X))$ is the space of nets $(N_\varepsilon)_\varepsilon$ of strongly continuous mappings $N_\varepsilon : \mathbb{R}_+ \to \mathcal{L}_c(X)$, $\varepsilon \in (0, 1)$ with the properties
For every $b \in \mathbb{R}$ and $T > 0$,

$$\sup_{t\in[0,T)} \|N_\varepsilon(t)\| = O_{\varepsilon\to 0}(\varepsilon^b). \tag{4.2}$$

There exist $t_0 > 0$ and $a \in \mathbb{R}$ such that,

$$\sup_{t<t_0} \|\frac{N_\varepsilon(t)}{t}\| = O_{\varepsilon\to 0}(\varepsilon^a). \tag{4.3}$$

There exists a net $(H_\varepsilon)_\varepsilon$ in $\mathcal{L}_c(X)$ and $\varepsilon_0 \in (0, 1)$ such that,

$$\lim_{t\to 0} \frac{N_\varepsilon(t)}{t} = H_\varepsilon x, \quad x \in X, \quad \varepsilon < \varepsilon_0. \tag{4.4}$$

For every $b > 0$,

$$\|H_\varepsilon\| = O_{\varepsilon\to 0}(\varepsilon^b). \tag{4.5}$$

**Remark 4.2.** Let us remark that,because of (4.1), it is enough that (4.2) holds for all $x \in \mathcal{D}$ where $\mathcal{D}$ is a dense subspace of $X$.

**Proposition 4.3.** *[19] $\mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}_c(X))$ is algebra with respect to composition and $\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}_c(X))$ is an ideal of $\mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}_c(X))$.*

Now we define Colombeau type algebra as the factor algebra

$$\mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X)) = \mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}(X))/\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}(X)). \tag{4.6}$$

Elements of $\mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X))$ will be denoted by $S = [S_\varepsilon]$, where $(S_\varepsilon)_\varepsilon$ is a representatvie of the above class.

**Definition 4.4.** [19] $S \in \mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X))$ is a called a Colombeau $C_0$-Semigroup if it has a representative $(S_\varepsilon)_\varepsilon$ such that, for some $\varepsilon_0 > 0$, $S_\varepsilon$ is a $C_0$-Semigroup, for every $\varepsilon < \varepsilon_0$.

In the sequel we will use only representatives $(S_\varepsilon)_\varepsilon$ of a Colombeau $C_0$-semigroup $S$ which are $C_0$-semigroups, for $\varepsilon$ small enough.

**Proposition 4.5.** *[19] Let $(S_\varepsilon)_\varepsilon$ and $(\tilde{S}_\varepsilon)_\varepsilon$ be representatives of a Colombeau $C_0$-semigroup $S$, with the infinitesimal generators $A_\varepsilon$, $\varepsilon < \varepsilon_0$, and $\tilde{A}_\varepsilon$, $\varepsilon < \tilde{\varepsilon}_0$, respectively, where $\varepsilon_0$ and $\tilde{\varepsilon}_0$ correspond ( in the sense of Definition 4.4) to $(S_\varepsilon)_\varepsilon$ and $(\tilde{S}_\varepsilon)_\varepsilon$, respectively. Then, $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \bar{\varepsilon} = \min\{\varepsilon_0, \tilde{\varepsilon}_0\}$ and $A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to an element of $\mathcal{L}(X)$, denoted again by $A_\varepsilon - \tilde{A}_\varepsilon$. Moreover, for every $a \in \mathbb{R}$,*

$$\|A_\varepsilon - \tilde{A}_\varepsilon\| = O_{\varepsilon\to 0}(\varepsilon^a). \tag{4.7}$$

Now we define the infinitesimal generator of a Colombeau $C_0$-semigroup $S$.
Denote by $\mathcal{A}$ the set of pairs $((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon)$ where $A_\varepsilon$ is a closed linear operator on $X$ with the dense domain $D(A_\varepsilon) \subset X$, for every $\varepsilon \in (0, 1)$. We introduce an equivalence relation in $A$,

$$((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon) \sim ((\tilde{A}_\varepsilon)_\varepsilon, (D(\tilde{A}_\varepsilon))_\varepsilon),$$

if there exist $\varepsilon_0 \in (0, 1)$ such that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \varepsilon_0$, and for every $a \in \mathbb{R}$ there exist $C > 0$ and $\varepsilon_a \le \varepsilon_0$ such that, for $x \in D(A_\varepsilon)$, $\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| \le C\varepsilon^a\|x\|, x \in D(A_\varepsilon), \varepsilon \le \varepsilon_a$. Since $A_\varepsilon$ has a dense domain in $X$, $R_\varepsilon := A_\varepsilon - \tilde{A}_\varepsilon$ ca, be extended to be an operator in $\mathcal{L}_c(X)$ satisfying $\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| = O_{\varepsilon\to 0}(\varepsilon^a)$, for every $a \in \mathbb{R}$. Such an operator $R_\varepsilon$ is called the zero operator. We denote by $A$ the corresponding element of the quotient space $\mathcal{A}/\sim$. Due to Propostion 4.3, the following definition makes sense.

**Definition 4.6.** $A \in \mathcal{A}/\sim$ is the infinitesimal generator of a Colombeau $C_0$-semigroup $S$ if there exists a representative $(A_\varepsilon)_\varepsilon$ of $A$ such that $A_\varepsilon$ is the infinitesimal generator of $S_\varepsilon$, for $\varepsilon$ small enough.

**Remark 4.7.** [19] Let the assymptions of Definition 4.1 hold. Moreover,assume a stronger assumption than (4.1). There exist $M > 0$, $a \in \mathbb{R}$ and $\varepsilon_0 \in (0, 1)$ such that,

$$\|S_\varepsilon(t)\| \le M\varepsilon^a e^{\alpha_\varepsilon t}, \quad \varepsilon < \varepsilon_0, \quad t \ge 0, \tag{4.8}$$

where $0 < \alpha_\varepsilon < \alpha$, for some $\alpha > 0$.

Then we obtain the corresponding subalgebra of $\mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X))$. For this subalgebra we can formulate the Hille-Yosida theorem in a usual way.
For the whole algebra of Colombeau $C_0$-semigroups, $\mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X))$ the formulation of the Hille-Yosida-Type theorem is an open problem.

## 5 Nonlinear Generalized Schrödinger Equation

In order to study the existence and uniqueness of Colombeau generalized solutions of Cauchy problems with partial regularized derivatives, one introduces the algebra of generalized functions suitable to this context. We denote by $D_{L^\infty}(\Omega)$ the algebra of restrictions to $\Omega$ of smooth functions defined on $\mathbb{R}^n$ with all derivatives are boundeds. With the same method of construction of the simplified algebra of Colombeau, we define the simplified algebra of global generalized functions, which must be compatible with the study of the Schrödinger equation denoted $\mathcal{G}_{s,g}(\Omega)$ by the quotient algebra

$$\mathcal{G}_{s,g}(\Omega) = \frac{\mathcal{E}_{Ms,g}(\Omega)}{\mathcal{N}_{s,g}(\Omega)}, \tag{5.1}$$

where $\mathcal{E}_{Ms,g}(\Omega)$ is the vector space of a family of functions $(u_\varepsilon)_\varepsilon$ belong to $(\mathcal{E}_{s,g}(\Omega))^{(0,1)}$ (with $\mathcal{E}_{s,g}(\Omega) = \mathcal{C}^0([0,\infty); L^2(\mathbb{R})) \cap \mathcal{C}^1([0,\infty); \mathcal{D}_{L^\infty}(\mathbb{R}))$ ) with the property that $\forall T_1 \in (0,T)$ $\exists N > 0$ such that

$$\max\left\{ \sup_{t\in[0,T]} \| u_\varepsilon(t,.) \|_{L^\infty} \ ; \ \sup_{t\in(0,T_1)} \| \partial_t u_\varepsilon(t,.) \|_{L^2} \right\} = O_{\varepsilon\to 0}(\varepsilon^{-N}).$$

Similarly $\mathcal{N}_{s,g}(\Omega)$, $T > 0$ , is the vector space of a family of functions $(u_\varepsilon)_\varepsilon$ such that $u_\varepsilon \in \mathcal{C}^0([0,\infty); L^2(\mathbb{R})) \cap \mathcal{C}^1([0,\infty); \mathcal{D}_{L^\infty}(\mathbb{R}))$ for which the following property holds for every $q > 0$,

$$\max\left\{ \sup_{t\in[0,T]} \| u_\varepsilon(t,.) \|_{L^\infty} \ ; \ \sup_{t\in(0,T_1)} \| \partial_t u_\varepsilon(t,.) \|_{L^2} \right\} = O_{\varepsilon\to 0}(\varepsilon^{q}).$$

We define also a new class of generalized functions noted by $\mathcal{G}_{L^\infty}(\mathbb{R}^n)$ as the factor set $\mathcal{E}_{L^\infty}(\mathbb{R}^n)/\mathcal{N}_{L^\infty}(\mathbb{R}^n)$, where $\mathcal{E}_{L^\infty}(\mathbb{R}^n)$ is the space of a family of functions $(v_\varepsilon)_\varepsilon$, with $v_\varepsilon \in \mathcal{D}_{L^\infty}$, $\varepsilon \in (0,1)$, satisfying the property: $\exists N > 0$ such that $\| v_\varepsilon \|_{\mathcal{D}_{L^\infty}} = O_{\varepsilon\to 0}(\varepsilon^{-N})$. $\mathcal{N}_{L^\infty}(\mathbb{R}^n)$ is the space of a family of functions $(v_\varepsilon)_\varepsilon$, with $v_\varepsilon \in \mathcal{D}_{L^\infty}$, $\varepsilon \in (0,1)$, satisfying the property: $\forall q > 0$ such that $\| v_\varepsilon \|_{\mathcal{D}_{L^\infty}} = O_{\varepsilon\to 0}(\varepsilon^{q})$. $\mathcal{G}_{L^\infty}(\mathbb{R})$ is an algebra with multiplication. Multiplication of elements from $\mathcal{G}_{L^\infty}(\mathbb{R})$ and $\mathcal{G}_{s,g}(\Omega)$ gives an element from $\mathcal{G}_{s,g}(\Omega)$ such as $V \in \mathcal{G}_{L^\infty}(\mathbb{R})$ and $u \in \mathcal{G}_{s,g}(\Omega)$ ppearing in the equation. which allows nonlinear operations.
Let us consider the nonlinear generalized Schrödinger problem with singular initial data,

$$\begin{cases} \partial_t u(t,x) + Au(t,x) + F(u(t,x)) = 0 \quad for \quad t > 0, \\ u(0,x) = u_0(x), \end{cases} \tag{5.2}$$

where $u$ is in $\mathcal{G}_{s,g}(\Omega)$, $\Omega = (0,T] \times \mathbb{R}$, $T > 0$ and $F(u) = i\,k\,|u|^2 u,\ and\ A = i\Delta$.

### 5.1 Existence and Uniqueness of the Generalized Solution

We can write the regularization of the initial value problem (5.2) in the space $\mathcal{D}_{L^\infty}(\Omega)$ as follows:

$$\begin{cases} \partial_t u_\varepsilon(t,x) + A_\varepsilon u_\varepsilon(t,x) + F(u_\varepsilon(t,x)) = 0\ for\ t > 0, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x) = \delta_\varepsilon(x). \end{cases} \tag{5.3}$$

Without loss of generality we can teak $F$ as follows $F(t,.) = v(.)u(t,.)$, where in quantum mechanics $v$ means the potential that represents the environment in which the particle exists. Multiplication of elements from $\mathcal{G}_{L^\infty}(\mathbb{R})$ and $\mathcal{G}_{s,g}(\Omega)$, $\Omega = [0,\infty) \times \mathbb{R}$, gives an element from $\mathcal{G}_{s,g}(\Omega)$which allows nonlinear operations appearing in the equation. Let us consider the follwing probem

$$\begin{cases} \partial_t u(t,x) = i(\Delta + v(x))u(t,x), \\ u(0,x) = \delta(x), \\ v(x) = \delta(x), \end{cases} \tag{5.4}$$

where $\delta$ is the embedding of the dirac measure in $\mathcal{G}_{s,g}(\Omega)$ with the representative is given by:

$$v_\varepsilon(\cdot) = \delta_\varepsilon(\cdot) = \mid ln\varepsilon \mid^c \psi(x \mid ln\varepsilon \mid^c \cdot), \quad 0 < c < 1, \ \ x \in \mathbb{R}, \tag{5.5}$$

where $\psi$ is a test function such that $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $\int \psi(x)dx = 1, \quad \psi(x) \geq 0.$
For the initial data we use

$$u_\varepsilon(0,t) = u_{0\varepsilon}(x) = \mid ln\varepsilon \mid^a \psi(x \mid ln\varepsilon \mid^a), \quad a > 0. \tag{5.6}$$

Imbedding of the lapacien operatvr in the $\mathcal{G}_{s,g}$ is given by the following equalty

$$\Delta u = [(\Delta u_\varepsilon)_\varepsilon].$$

**Construction of semigroup:**
In the sequel, we shall use the $L^p - L^q$ estimates for the Schrödinger semigroup from [6]. Recall that the semigroup $s(t)$ which has the infintesimal generator $i\Delta$ possesses an explicit formula givn by

$$s(t)u(x) = f(t,.) * u(x),$$

where $f(t,x) = (4\pi\, i\, t)^{-1/2} e^{i\, x^2/(4\, t)}$.
In the sequel, we shall use the following lemma

**Lemma 5.1.** *[6] If $1 \leq p \leq q \leq \infty$, $1/p + 1/q = 1$, then $e^{-it\Delta}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ for $f \neq 0$ and*

$$\| e^{-it\Delta} \| \leq (4\pi|t|)^{-(1/p-1/2)} \quad \textit{for} \quad t \neq 0.$$

For $p = q = 2$ the mapping $s(t) : L^2 \to L^2$ holds for all $t \in \mathbb{R}$, $(e^{it\Delta})_{t\geq 0}$ is the semigroup on $L^2$. And by this estimate and according to the norm of convolution we can prove that the $L^2-$norm of $s(t)$ has a moderate bound, which gives that $s(t) \in \mathcal{E}_{M,s,g}(\mathbb{R})$.

**Theorem 5.2.** *With the previous notations the problem (5.4) has a unique solution in the space $\mathcal{G}_{s,g}(\Omega)$ of Colmbeau.*

*Proof.* By duhamel principle, Eq (5.4) has an integral from

$$u(t,x) = S(t)u(0,x) + \int_{t_0}^t S(t-s)v(x)u(s,x)ds,$$

where $S(t) = [(S_\varepsilon(t))_\varepsilon] = [(s(t))_\varepsilon]$ be Colombeau $C_0$-semigroup with infinitesimal generator $\tilde{\Delta} = -i\Delta$.

The problem reduces to finding a fixed point of the map

$$\phi : \mathcal{G}_{s,g}(\Omega) \to \mathcal{G}_{s,g}(\Omega),$$
$$u \mapsto \phi(u)(t,x) \quad = S(t)u(0,x) + \int_{t_0}^t S(t-s)v(x)u(s,x)ds, \quad \forall t \in \mathbb{R}^+.$$

**a)** We can write the previous equality in terme of representatives as follows:

$$\forall t \in \mathbb{R}^+ \quad \phi_\varepsilon(u_\varepsilon)(t,x) = S_\varepsilon(t)u_{0\varepsilon}(x) + \int_0^t S_\varepsilon(t-s)v_\varepsilon(x)u_\varepsilon(s,x)ds,$$

form what it is clear that

$$\phi_\varepsilon : \mathcal{E}_{s,g}(\Omega) \longrightarrow \mathcal{E}_{s,g}(\Omega),$$

$\mathcal{E}_{s,g}(\Omega)$ is here a topological space where $\tau$ is given by the family of norms $(p_T)_{T\in\mathbb{R}^+}$, such that $p_T(u_\varepsilon) = \sup_{t\in[0,T]} \| u_\varepsilon(t,.) \|_{L^2(\mathbb{R})}$.
Let $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{Ms,g}(\Omega)$ and $(w_\varepsilon)_\varepsilon \in \mathcal{N}_{s,g}(\Omega)$, we have

$$\mid \phi_\varepsilon(u_\varepsilon)(t,x) \mid \leq \mid S_\varepsilon(t)u_{0\varepsilon}(x) \mid + \int_0^t \mid S_\varepsilon(t-s)v_\varepsilon(x)u_\varepsilon(s,x) \mid ds.$$

So,

$$\| \phi_\varepsilon(u_\varepsilon)(t,\cdot) \|_{L^2} \leq \| u_{0\varepsilon} \|_{L^2} + \int_0^t \| v_\varepsilon \|_\infty \| u_\varepsilon(s,\cdot) \|_{L^2} ds.$$

Gronwell lemma gives

$$\| \phi_\varepsilon(u_\varepsilon)(t,\cdot) \|_{L^2} \leq \| u_{0\varepsilon} \|_{L^2} \; e^{t\|v_\varepsilon\|_\infty},$$

$$\sup_{t\in(0,T]} \| \phi_\varepsilon(u_\varepsilon)(t,\cdot) \|_{L^2} \leq \| u_{0\varepsilon} \|_{L^2} \; e^{T\|v_\varepsilon\|_\infty}.$$

And thus,

$$p_T(\phi_\varepsilon(x_\varepsilon)) \leq p_T(u_{0\varepsilon}) \; e^{T\|v_\varepsilon\|_\infty}.$$

Which implies,

$$(\phi_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{E}_{M,s,g}(\mathbb{R}^+ \times \mathbb{R}).$$

**b)** We first have to write (5.4) in term of representatives

$$\begin{cases} \partial_t u_\varepsilon(t,x) = i(\Delta + v_\varepsilon(x))u_\varepsilon(t,x), \\ u_\varepsilon(0,x) = \delta_\varepsilon(x), \\ v_\varepsilon(x) = \delta_\varepsilon(x). \end{cases}$$

It is clear that

$$\phi_\varepsilon : \mathcal{E}_{s,g}(\Omega) \to \mathcal{E}_{s,g}(\Omega).$$

Denote by $(\mathcal{E}_{s,g}(\Omega), \tau_\varepsilon)$ a topological space where $\tau_\varepsilon$ is given by the family of norms $(q_{T,\varepsilon})_{T\in\mathbb{R}^+}$ such that, $\forall y_\varepsilon \in \mathcal{E}_{s,g}(\Omega), \;\; q_{T,\varepsilon}(y_\varepsilon) = \sup_{t\in[0,T]} \{\| y_\varepsilon(t,.) \|_{L^2} \; e^{-t\|v_\varepsilon\|_\infty}\}$, we have

$$\phi_\varepsilon(u_\varepsilon)(t,x) - \phi_\varepsilon(w_\varepsilon)(t,x) = \int_0^t S_\varepsilon(t-s)v_\varepsilon(u_\varepsilon(s,x) - w_\varepsilon(s,x))ds,$$

then

$$\| \phi_\varepsilon(u_\varepsilon)(t,\cdot) - \phi_\varepsilon(w_\varepsilon)(t,\cdot) \|_{L^2} \leq \int_0^t \| v_\varepsilon \|_\infty \| u_\varepsilon(s,\cdot) - w_\varepsilon(s,\cdot) \|_{L^2} \; ds,$$

multiply the two preceding inequalities by we get

$$e^{-t\|v_\varepsilon\|_\infty} \; \| \phi_\varepsilon(u_\varepsilon)(t,\cdot) - \phi_\varepsilon(w_\varepsilon)(t,\cdot) \|_{L^2} \leq e^{-t\|v_\varepsilon\|_\infty} \int_0^t \| v_\varepsilon \|_\infty \| u_\varepsilon(s,\cdot) - w_\varepsilon(s,\cdot) \|_{L^2} \; ds.$$

We take the supremum, and by the definition of the seminorm $(q_T)_{T\in\mathbb{R}^+}$ we obtain

$$\sup_{t\in(0,T]} \{e^{-t\|v_\varepsilon\|_\infty} \; \| \phi_\varepsilon(u_\varepsilon)(t,\cdot) - \phi_\varepsilon(w_\varepsilon)(t,\cdot) \|_{L^2}\} \leq q_{T,\varepsilon}(x_\varepsilon - y_\varepsilon)(1 - e^{-T\|v_\varepsilon\|_\infty}).$$

And thus

$$q_{T,\varepsilon}(\phi_\varepsilon(u_\varepsilon) - \phi_\varepsilon(w_\varepsilon)) \leq q_{T,\varepsilon}(x_\varepsilon - y_\varepsilon)(1 - e^{-T\|v_\varepsilon\|_\infty}),$$

Then $\phi_\varepsilon$ is a contraction in $\mathcal{E}_{s,g}(\Omega)$.
**c)** We can write for $\forall T \in \mathbb{R}^+$ and $x_\varepsilon \in \mathcal{E}_{s,g}(\Omega)$

$$\begin{aligned} \sup_{t\in[0,T]} \{| \; y_\varepsilon(t) \; | \;\; e^{-t\|v_\varepsilon\|_\infty}\} &\leq \sup_{t\in[0,T]} \{| \; y_\varepsilon(t) \; | \;\; e^{-t\|v_\varepsilon\|_\infty}\} \\ &\leq \sup_{t\in[0,T]} \; | \; y_\varepsilon(t) \; |, \end{aligned}$$

then

$$e^{-T\|v_\varepsilon\|_\infty} p_T \leq q_{T,\varepsilon} \leq p_T.$$

**d)** Assume now that for each $T \in \mathbb{R}^+$ we have $(e^{T\|v_\varepsilon\|_\infty})_\varepsilon \in \mathcal{E}_M(\mathbb{R})$
and $(1/(1 - e^{-T\|v_\varepsilon\|_\infty}))_\varepsilon \; \in \mathcal{E}_M(\mathbb{R})$, where $\mathcal{E}_M(\mathbb{R})$ is the ring of generalized real numbers associated to the maderate space.

Finally, from the Definition 3.6 of contracting mapping in the Colombeau algebras. The maps

$$\phi : \mathcal{G}_{s,g}(\Omega) \to \mathcal{G}_{s,g}(\Omega)$$
$$u(t,x)) = [u_\varepsilon(t,x)] \mapsto \phi(x)(t) = [\phi_\varepsilon(u_\varepsilon(t,x))],$$

is contraction, from the generalized fixed point Theorem 3.7, $z = [z_\varepsilon]$ is a fixed point of $\phi$, with $z_\varepsilon$ being the unique fixed point of $\phi_\varepsilon$.
Moreover $z$ is the unique solution of (5.4). Indeed, assume that there exist two solutions $u_1$, $u_2$ to problem (5.4). The difference $H_\varepsilon$ given by

$$H_\varepsilon(t,x) = u_{1\varepsilon}(t,x) - u_{2\varepsilon}(t,x),$$

is the solution to the problem

$$\begin{cases} \partial_t H_\varepsilon(t,x) = i(\Delta + v_\varepsilon(x))H_\varepsilon(t,x) + N_\varepsilon(t,x), \\ H_\varepsilon(0,x) = N_{0\varepsilon}(x); \end{cases}$$

where $(N_\varepsilon)_\varepsilon \in \mathcal{N}_{s,g}(\mathbb{R}^+ \times \mathbb{R})$.
Since the family $(H_\varepsilon)_\varepsilon$ belongs to the $\mathcal{E}_{M,s,g}(\Omega)$, it suffices according to the Theorem 1.2.3 in [12] to prove that the zero order derivative of $H_\varepsilon$ has a negligeable bound. Indeed, by Duhamel principle we have

$$H_\varepsilon(t,x) = S_\varepsilon(t)N_{0\varepsilon}(x) + \int_0^t S_\varepsilon(t-s)v_\varepsilon(x)H_\varepsilon(s,x)ds + \quad \textstyle\int_0^t S_\varepsilon(t-s)N_\varepsilon(s,x)ds,$$

so,

$$\begin{aligned} \| H_\varepsilon(t,\cdot) \|_{L^2} &\leq \| N_{0\varepsilon} \|_{L^2} + \int_0^t \| v_\varepsilon \|_\infty \| H_\varepsilon(s,\cdot) \|_{L^2} \, ds + \int_0^t \| N_\varepsilon(s,\cdot) \|_{L^2} \, ds \\ &\leq C\varepsilon^q + \int_0^t \| v_\varepsilon \|_\infty \| H_\varepsilon(s,\cdot) \|_{L^2} \, ds + C\varepsilon^q T \\ &\leq C_1\varepsilon^q + \int_0^t \| v_\varepsilon \|_\infty \| H_\varepsilon(s,\cdot) \|_{L^2} \, ds. \end{aligned}$$

By using Gronwell lemma, we can get

$$\| H_\varepsilon(t,\cdot) \|_{L^2} \leq C_1\varepsilon^q + e^{t\|v_\varepsilon\|_\infty}.$$

So,

$$(H_\varepsilon)_\varepsilon \in \mathcal{N}_{s,g}(\mathbb{R}^+ \times \mathbb{R}).$$

Thus ,the solution is unique in $\mathcal{G}_{s,g}(\mathbb{R}^+ \times \mathbb{R})$. □

## 6 Association

Let $w_1$ be the solution to the problem

$$\begin{cases} \frac{1}{i}\partial_t w_1(t,x) - \triangle w_1(t,x) = 0, \\ w_1(0,x) = \delta(x), \end{cases} \tag{6.1}$$

and $w_2$ is the solution to the problem

$$\begin{cases} \frac{1}{i}\partial_t w_2(t,x) - \triangle w_2(t,x) + v(x)w_{2,\varepsilon}(t,x) = 0, \\ v(x) = \delta(x), \quad w_2(0,x) = 0. \end{cases} \tag{6.2}$$

**Proposition 6.1.** *The generalized solution $u$ to the problem (5.4) is associated with $w_1 + w_2$.*

*Proof.* Let $w_{1,\varepsilon}$ be the classical solution to the problem

$$\begin{cases} \frac{1}{i}\partial_t w_{1,\varepsilon}(t,x) - \triangle w_{1,\varepsilon}(t,x) = 0, \\ w_{1,\varepsilon}(0,x) = \delta_\varepsilon(x), \end{cases} \tag{6.3}$$

and $w_{2,\varepsilon}$ the classical solution to problem

$$\begin{cases} \frac{1}{i}\partial_t w_{2,\varepsilon}(t,x) - \triangle w_2(t,x) + v_\varepsilon(x)\big(w_{2,\varepsilon}(t,x) + m(t,x)\big) = 0, \\ v_\varepsilon(x) = \delta(x), \quad w_{2,\varepsilon}(0,x) = 0. \end{cases} \tag{6.4}$$

Then

$$\begin{aligned} \frac{1}{i}\partial_t \Big( u_\varepsilon(t,x) \quad &- \quad w_{1,\varepsilon}(t,x) - w_{2,\varepsilon}(t,x)\Big) - \triangle[u_\varepsilon(t,x) - w_{1,\varepsilon}(t,x) - w_{2,\varepsilon}(t,x)] \\ &+ \quad v_\varepsilon(x)[u_\varepsilon(t,x) - w_{2,\varepsilon}(t,x) - m(t,x)] = 0, \end{aligned}$$

and

$$u_\varepsilon(0,x) - w_{1,\varepsilon}(0,x) - w_{2,\varepsilon}(0,x) = 0.$$

The integral solution is

$$\begin{aligned} u_\varepsilon(t,x) - w_{1,\varepsilon}(t,x) - w_{2,\varepsilon}(t,x)) &= \int_0^t \int_{\mathbb{R}} S_\varepsilon(t-\tau, x-y) v_\varepsilon(y) \\ &\qquad \big[u_\varepsilon(\tau,y) - w_{2,\varepsilon}(\tau,y) - m(\tau,y)\big] dy d\tau \\ &= \int_0^t \int_{\mathbb{R}} S_\varepsilon(t-\tau, x-y) v_\varepsilon(y) \\ &\qquad \big[u_\varepsilon(\tau,y) - w_{1,\varepsilon}(\tau,y) - w_{2,\varepsilon}(\tau,y)\big] dy d\tau \\ &+ \int_0^t \int_{\mathbb{R}} S_\varepsilon(t-\tau, x-y) v_\varepsilon(y) \\ &\qquad \big[w_{1,\varepsilon}(\tau,y) - m(\tau,y)\big] dy d\tau. \end{aligned}$$

So,

$$\begin{aligned} \|u_\varepsilon(t,.) - w_{1,\varepsilon}(t,.) - w_{2,\varepsilon}(t,.)\|_{L^\infty} &\leq \int_0^t \|S_\varepsilon(t-\tau, x-.)\|_{L^1} \|v_\varepsilon\|_{L^\infty} \\ &\qquad \|(w_{1,\varepsilon}(\tau,.) - m(\tau,.))\|_{L^\infty} d\tau \\ &+ \int_0^t \|S_\varepsilon(t-\tau, x-.)\|_{L^1} \|v_\varepsilon\|_{L^\infty} \\ &\qquad \|u_\varepsilon(\tau,.) - w_{1,\varepsilon}(\tau,.) - w_{2,\varepsilon}(\tau,.)\|_{L^\infty} d\tau \\ &\leq C\|v_\varepsilon\|_{L^\infty} \Bigg[ \int_0^t \|(w_{1,\varepsilon}(\tau,.) - m(\tau,.))\|_{L^\infty} d\tau \\ &+ \int_0^t \|u_\varepsilon(\tau,.) - w_{1,\varepsilon}(\tau,.) - w_{2,\varepsilon}(\tau,.)\|_{L^\infty} \Bigg] d\tau. \end{aligned}$$

Thank's to the Granwall lemma, we get

$$||u_\varepsilon(t,.) - w_{1,\varepsilon}(t,.) - w_{2,\varepsilon}(t,.)||_{L^\infty} \leq \big[C||v_\varepsilon||_{L^\infty)} \int_0^t ||w_{1,\varepsilon}(\tau,.) - m(\tau,.)||_{L^\infty} d\tau\big] e^{CT||v_\varepsilon||_{L^\infty}},$$

by passing to the limit, we obtain

$$u \approx w_1 + w_2.$$

Which ends the proof of the proposition. □

## 7 Example

As an example of a detailed regularization model we give the definition of Colombeau generalized positive square roots of arbitrary probability measures, which can be used it as initial values in the Cauchy problem analyzed in our example of application to the well known Schrödinger equation. Before giving our illustrative example we present the following

**Lemma 7.1.** *Let $\mu$ be a (Borel) probability measure on $\mathbb{R}^n$. Choose $\phi \in L^1(\mathbb{R}) \cap W^{\infty,\infty}(\mathbb{R}^n)$ to be positive with $\int \phi = 1$ and satisfying $\phi(x) \geq\mid x \mid^{?m_0}$ when $\mid x \mid\geq 1$ with some $m_0 > n$. Set $h_\varepsilon := \mu * \phi_\varepsilon$, then we have $h_\varepsilon$ is positive and setting $\psi_\varepsilon := \sqrt{h_\varepsilon}$, where the net $(\psi_\varepsilon)_{\varepsilon\in]0,1]}$ represents an element $\psi \in \mathcal{G}^s(\mathbb{R}^n)$ such that $\psi^2 \approx \mu$.*

*Proof.* The Borel measure $\mu$ is regular since it is finite and $\mathbb{R}^n$ is locally compact and second countable. Thus $\mu(\mathbb{R}^n) = 1$ implies that we can find a compact subset $A \subseteq \mathbb{R}^n$ such that $\mu(A) \geq 1/2$. A variant of Young's inequality for measures applied to $\partial^\alpha h_\varepsilon = \mu * \partial^\alpha \phi_\varepsilon$ directly implies that the net $(h_\varepsilon)$ is $\mathcal{C}^\infty(\mathbb{R}^n)$-moderate, even with global $L^\infty$-norms, since

$$\begin{aligned} \| \mu * \partial^\alpha \phi_\varepsilon \|_{L^\infty} &\geq \| \mu \|_T \| \partial^\alpha \phi_\varepsilon \|_{L^\infty} \\ &\geq \| \phi \|_{L^\infty} \varepsilon^{-n-|\alpha|}. \end{aligned}$$

(Here $\| \,.\, \|_T$ denotes the total variation norm on the space of finite Radon measures on $\mathbb{R}^n$, which gives $\| \mu \|_T = \mu(\mathbb{R}^n) = 1$ in case of the positive probability measure $\mu$).

$$\begin{aligned} h_\varepsilon(x) &= \mu * \phi_\varepsilon(x) = \int_{\mathbb{R}^n} \phi_\varepsilon(x-y) d\mu(y) \\ &\geq \int_A \phi_\varepsilon(x-y) d\mu(y) \\ &\geq \mu(A). \min_{z\in\{x\}-A} \phi_\varepsilon(x) > 0, \end{aligned}$$

and thus and hence that $\psi_\varepsilon = \sqrt{h_\varepsilon} \in \mathcal{C}^\infty(\mathbb{R}^n)\,(0 < \varepsilon \leq 1)$. Moreover, if $K \subseteq \mathbb{R}^n$ is compact then we have the estimate

$$\begin{aligned} \inf_{x\in K} h_\varepsilon(x) &= \inf_{x\in K} \mu * \phi_\varepsilon(x) \\ &\geq \mu(A) \cdot \inf_{x\in K} \min_{z\in\{x\}-A} \phi_\varepsilon(z) \\ &\geq \frac{1}{2} \cdot \min_{z\in K-A} \frac{\phi(z/\varepsilon)}{\varepsilon^n} \\ &\geq \frac{1}{2\varepsilon^n} \min_{|z|\leq r(K)} \phi(z/\varepsilon) \\ &\geq \frac{1}{2\varepsilon^n} \frac{\varepsilon^{m_0}}{r(K)^{m_0}} \\ &= \frac{\varepsilon^{m_0-n}}{2r(K)^{m_0}}, \qquad (\varepsilon < 1/r(K)), \end{aligned}$$

where $r(K) = \max\{\mid x \mid /x \in K - A\}$ If $\alpha \in \mathbb{N}^n$ is arbitrary then $\partial^\alpha \phi_\varepsilon$ is a linear combination of terms of the form $\partial^{\beta_1} h_\varepsilon \cdots \partial^{\beta_k} h_\varepsilon / h_\varepsilon^{l/2}$ with appropriate $\beta_1, \ldots, \beta_k \in \mathbb{N}^n$ and $l \in \mathbb{N}$. Hence $\mathcal{C}^\infty(\mathbb{R}^n)$-moderateness of $(h_\varepsilon)$ together with the lower bounds for $\inf_{x\in K} h_\varepsilon(x)$ obtained above prove $\mathcal{C}^\infty(\mathbb{R}^n)$-moderateness of $\psi_\varepsilon$.
Finally, we have that by construction $\psi^2$ is represented by $h_\varepsilon = \mu * \phi_\varepsilon$ and thus clearly converges to $\mu$ as $\varepsilon \to 0$ in the sense of distributions.

□

**Example 7.2.** Consider the strictly positive square root of $\phi$ represented by $(\sqrt{\phi_\varepsilon})_\varepsilon$, $\varepsilon \in ]0,1]$, where $\phi_\varepsilon$ is a mollifier defined by $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$. (Note that we obtain $h_\varepsilon = \delta * \phi_\varepsilon = \phi_\varepsilon$ in this case.) To simplify technical matters, assume in addition that $\phi$ satisfies for example, any suitably normalized function of the form $x \mapsto (1+ \mid x \mid^2)^{-m/2}$ with $m > 2n$. Then we can consider the class $g \in \mathcal{G}_{s,g}(\mathbb{R}^n)$, given directly by $(\sqrt{\phi_\varepsilon})$, $\varepsilon \in ]0,1]$ , as a square root of $\delta$. We have $g^2 \approx \delta$, but $g \approx 0$ , which is easily seen by action on a test function $\psi$ upon substituting $y = x/\varepsilon$ we can get

$$\int \sqrt{\phi_\varepsilon(x)}\psi(x)dx = \varepsilon^{n/2} \int \sqrt{\phi(y)}\psi(\varepsilon y)dy$$

and applying dominated convergence and then using that $\sqrt{\phi} \in L^1$ . The Cauchy problem (**5.4**) with generalized initial value g right-hand side $f = 0$ and $V = 0$, written out for representatives then reads

$$\partial_t u_\varepsilon = i\Delta u_\varepsilon \qquad u_{\varepsilon/\{t=0\}} = \sqrt{\phi_\varepsilon}, \quad \varepsilon \in (0,1].$$

The solution is given by the action of the strongly continuous unitary group $U_t := e^{it\Delta}$ $(t \in \mathbb{R})$ of operators on $L^2(\mathbb{R}^n)$ in the form $u_\varepsilon(t,x) = (U_t\sqrt{\phi_\varepsilon})(x)$.

Let $t \in \mathbb{R}$ and $\mu_\varepsilon^t$ denote the positive measure on $\mathbb{R}^n$ with density function $\|u_\varepsilon(t,.)\|^2$ for the Lebesgue measure. By unitarity of $U_t$ we obtain

$$\begin{aligned} \mu_\varepsilon^t(\mathbb{R}^n) &= \int_{\mathbb{R}^n} \mid u_\varepsilon(t,x) \mid^2 dx \\ &= \int_{\mathbb{R}^n} \left(U_t\sqrt{\phi_\varepsilon}\right)(x) \cdot \overline{\left(U_t\sqrt{\phi_\varepsilon}\right)(x)}dx \\ &= \int_{\mathbb{R}^n} \mid \sqrt{\phi_\varepsilon(x)} \mid^2 dx = \int_{\mathbb{R}^n} \phi_\varepsilon(x)dx = 1, \end{aligned}$$

hence $\{\mu_\varepsilon^t : t \in \mathbb{R}, \varepsilon \in ]0,1]\}$ is a family probability measures on $\mathbb{R}^n$ and $\|\mu_\varepsilon^t\|_T = 1$, for all $t \in \mathbb{R}$ and $\varepsilon \in ]0,1]$ holds in the Banach space of finite measures.
We claim that for any $t \neq 0$ the net $(\mu_\varepsilon^t)_{\varepsilon \in ]0,1]}$ converges to 0 with respect to the vague topology on finite measures [20]. Since $\sqrt{\phi_\varepsilon} \in L^1(\mathbb{R}^n)$ we obtain from the $L^1$-$L^\infty$-estimate for the Schrödinger propagator [20] that $\|u_\varepsilon(t,.)\|_{L^\infty} \leq \left\|\sqrt{\phi_\varepsilon}\right\|_{L^1}/(4\pi|t|)^{n/2}$ and therefore for any $\psi \in \mathcal{C}(\mathbb{R}^n)$ with compact support,

$$\begin{aligned} \mid \langle \mu_\varepsilon^t, \psi \rangle \mid &\leq \int_{\mathbb{R}^n} \mid u_\varepsilon(t,x) \mid^2 \mid\mid \psi(x) \mid dx \\ &\leq (4\pi|t|)^{-n}\|\psi\|_{L^1} \| \sqrt{\phi_\varepsilon} \|_{L^1} \\ &= (4\pi|t|)^{-n}\|\psi\|_{L^1}\|\sqrt{\phi}\|_{L^1}\varepsilon^{n/2} \to 0 \ as \ (\varepsilon \to 0). \end{aligned}$$

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### Conflict of interest

The authors declare that they have no conflict of interest.

### Data Availability

The data used to support the findings of this study are included in the references within the article.

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