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NOVEL RESULTS ON BOUNDS FOR THE RADIO NUMBER OF CYCLIC AND CHAIN SILICATES

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Abstract A radio labeling technique in graph theory is used to maximize the number of channels in a pre-established spectrum bandwidth. A radio labeling of a connected graph $G = (V, E)$ with diameter d is an injection $\varphi : V(G) \to N$ such that $|\varphi(x) - \varphi(y)| + d(x, y) \geq 1 + d \; \forall x, y \in V(G)$. The maximum number assigned to any vertex of G under the mapping φ is called the radio number of φ, denoted by $rn(\varphi)$. The minimum value of $rn(\varphi)$ that is taken over all radio labeling φ of G is called the radio number of G, which is denoted by $rn(G)$. As the distance between two proteins in a protein-protein interaction network and the effects of radio frequency radiation on proteins play a vital role in the study of DNA damage, in this paper, the radio numbers for chemical structures such as chain oxides, chain silicates, cyclic oxides and cyclic silicates are fully investigated.

1 Introduction

The connection between chemistry and graph theory has been fruitful and active for more than the past 150 years. However, many early chemists have used unconsciously graph-theoretical concepts without realizing it. In chemistry, graph theory is significant because of the notion of isomerism that is rationalized by the chemical structure theory [1]. The foremost application of graph theory in chemistry was the representation of individual molecules by graphs [2]. They constructed the chemical graph by fixing the atoms of the molecule as vertices and the valence bonds between a pair of atoms as an edge. Johnson [3] presented a graph-theoretical method to represent structural changes in chemical compounds by a labeled chemical graph. In the computer analysis of chemical compounds, chemical structures are usually represented as graph structured data. These chemical graphs are used to develop efficient algorithms for a number of graph theory problems, especially in the fields of telecommunication engineering [4, 5] and computational fluid dynamics [6, 7].

In 2001, Chartrand et al. [8] was motivated by the application of graph labeling in radio telecommunication [9] and presented a graph labeling technique called radio labeling. It is used to maximize the number of channels for frequency modulation (FM) radio stations in a predefined bandwidth. The foremost job here is to allocate FM radio channels between radio transmitters in the pre-defined geographical area in such a way that there is no co-channel interference between them. The key constraints for the co-channel interference in FM radio stations are the frequency differences and distance between the transmitters. Using these two constraints, Chartrand et al. [8] defined radio labeling as follows: A radio labeling of a connected graph $G = (V, E)$ with diameter d is an injection $\varphi : V(G) \to N$ such that $|\varphi(x) - \varphi(y)| + d(x, y) \geq 1 + d \; \forall x, y \in V(G)$. The maximum number assigned to any vertex of G under the mapping φ is called the radio number of φ, denoted by $rn(\varphi)$. The minimum value of $rn(\varphi)$ taken over all radio labeling φ of G is called the radio number of G, denoted by $rn(G)$. Due to the extensive use of telecommunications in the modern era, the humans are facing a lot of health risks related with exposure to radio frequencies. Many recent studies confirmed the effects of radio frequency on DNA damage [10] proteins in epithelial cells [11], proliferation [12], etc. In order study the protein-protein interaction networks [13], the distance between the proteins plays a vital role. Thus, the distance between the proteins and the effect of radio frequencies between two cells put together to study the radio labeling problem for this real-life problem. Fotakis et al. [14] showed that even for graphs with diameter 2, the problem is NP-hard. Bharati et al. [15] provided a lower bound for the radio number of any simple connected graph in terms of eccentricities. Liu et al. [16, 17] investigated the radio number for paths, trees and cycles. Laxman [18] attained the lower bound for general trees. Yenoke [19]] found the radio number certain graphs with extended wheels. Niranjan et al. [20] determined the radio number for corona of cycles and paths. Devsi [21] investigated the same problem for middle graph of paths. Recently, Yenoke et al. [22], has completely studied the radio labeling problem for nano tree dendrimers.

In chemistry, the radio frequency and radiolabeling methods are used to study the protein absorption in single component or competitive absorption of a blood or plasma. As of the procedures presently available, radiolabeling is perhaps the most sensitive and accurate method for calculating the quantity of protein adsorbed [23, 24]. Due to the planar property, uniform growth and transmission of information is very fast in chemical graphs (networks), in this research work, we have focused on studying the radio labeling problem for certain chemical graphs such as single-chain oxides and silicates, cyclic oxides and cyclic silicates. Moreover, because of the NP-hardness of this channel assignment problem, we have investigated the upper and lower bounds for the radio number of such chemical graphs.

2 Poly-Oxide and Poly-Silicate structures

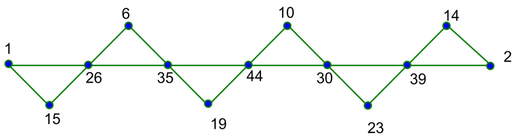
Poly-Oxide and Poly-Silicate structures are widely studied in [25, 26, 27, 28]. By fusing metal carbonates or metal oxides with sand, the silicates are formed. Fundamentally all the silicates comprise $SiO4$ tetrahedron. In chemistry the center node of $SiO4$ tetrahedron signify silicon ion and the corner nodes signifies the oxygen ions. In graph theoretical approach, the center vertex is named as silicon node and the corner vertices are named as oxygen nodes. Different minerals are obtained by continuously fusing oxygen ions of two tetrahedra of different silicates. The different types of silicate structures are formed according to the way of arrangement of these tetrahedra. If they exist, then they are in the form of 1-dimension chains, 2-dimensional sheets or as 3-dimensional frameworks. They are called as pyro silicates, orthosilicates, chain silicates, sheet silicates and cyclic silicates.

2.1 Chain Silicates and Oxides

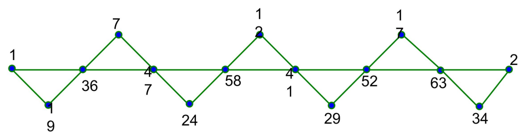
During the polymerization of silicate anions, an oxygen atom is shared with a neighboring tetrahedron. If each of the tetrahedron share two of its oxygen atoms and forming a long chain structure, then such a structure is called a single-chain silicate structure. It is denoted by $SL\,(1,\ \xi)$, where ξ is the number of $_4$ arranged linearly in the chain. It contains $3\,\xi + 1$ vertices and $6\,\xi$ edges. Also, its diameter and radius are ξ and $\left\lceil \frac{\xi}{2} \right\rceil$, respectively. If we delete all the silicon nodes from a single-chain silicate structure, a new structure formed is called a single-chain oxide structure and it is denoted by $OX\,(1,\ \xi)$. It contains $2\,\xi+1$ vertices and $3\,\xi$ edges. Further, the diameter and radius of $OX\,(1,\ \xi)$ are same as $SL\,(1,\ \xi)$.

2.2 Cyclic Silicates and Oxides

In the chain silicate $SL\,(1,\ \xi)$, if the 1^{st} and ξ^{th} tetrahedrons share two of its oxygen atoms, then the structure formed is called cyclic silicates. It is denoted by $SL_c\,(\xi)$. Also, $|V(SL_c\,(\xi)\,)| = 3\xi$ and $|E(SL_c\,(\xi)\,)| = 6\,\xi$. As in chain- oxides, if we remove all the silicon vertices from $SL_c\,(\xi)$, the resulting structure obtained is called cyclic-oxide structure and it's denoted by $OX_c\,(\xi)$. In addition, the diameter for both $SL_c\,(\xi)$ and $OX_c\,(\xi)$ is $\left\lfloor \frac{\xi}{2} \right\rfloor + 1$.



(a)



(b)

Figure 1. .A single chain oxide structures $OX\ (1,6)$ and $OX(1,7)$ and its radio labeling.

3 Main Results

As the silicates are usually stable and well-characterized, in this section, we discuss the upper and lower bounds for the radio number of $OX\ (1,\ \xi)$ and $SL\ (1,\ \xi)$ separately. In addition, we have estimated the upper bounds for $SL_c\ (\xi)$ and $OX_c\ (\xi)$ separately.

In order to obtain the lower bounds, we need the concept of eccentricity of vertices in a graph. Let G be a connected graph and let v be a vertex of G. The *eccentricity* $e(v)$ of a vertex v in a connected graph $G = (V, E)$ is the farthest vertex from v to any other vertex in the graph. That is, $e\ (v) = \max\{d\ (u, v)\ \forall u \in V\ (G)\}$. The *diameter of* G, denoted by *diam(G)* is the maximum eccentricity of the vertices of G. Also, the minimum eccentricity of the vertices of G is called the *radius of* G, denoted by $rad\ (G)$.

3.1 Bounds for Chain Oxide Structures

In this subsection, we have determined the upper and lower bounds for the radio number of chain oxide structure.

Theorem 3.1. Radio number of single chain oxide structure satisfies

$$rn(OX(1,\xi) \leq \begin{cases} \frac{3\xi^2}{2} - \frac{5\xi}{2} + 5, & \xi \text{ is even} \\ \xi^2 - 3\xi + \left\lceil \frac{\xi}{2} \right\rceil + \left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1 \right) \left(2 \left\lceil \frac{\xi}{2} \right\rceil + 3 \right) + 9, & \xi \text{ is odd} \end{cases}.$$

*Proof***.** First we name the vertices of the row line or horizontal line (path of length ξ) as $u_1,\ u_2 \ldots u_{\xi+1}$, then naming the vertices above and below of the centre line from left to right as $v_1,\ v_2 \ldots v_{\lfloor \frac{\xi}{2} \rfloor}$ and $w_1,\ w_2 \ldots\ w_{\lceil \frac{\xi}{2} \rceil}$ respectively.

Case 1. ξ is even.

Define an injection $\varphi : V(OX\ (1,\xi)) \to N$ as follows: $\varphi\ (u_1) = 1,\ \varphi\ (u_{\xi+1}) = 2,\ \varphi\ (v_i) = \xi + (i-1)\ (\xi - 2)\ ,\ i = 1, 2 \ldots \frac{\xi}{2},\ \varphi\ (w_i) = \frac{\xi^2}{2} - \xi + 3 + (i-1)\ (\xi-2)\ ,\ i = 1, 2 \ldots \frac{\xi}{2},$ $\varphi\ (u_{i+1}) = \xi^2 - 3\xi + 8 + (i-1)\ (\xi + 3)\ ,\ i = 1, 2 \ldots \frac{\xi}{2},\ \varphi\left(u_{\frac{\xi}{2}+i}\right) = \xi^2 - 3\xi + \frac{\xi}{2} + 9 + (i-1)\ (\xi+3)\ ,\ i = 1, 2 \ldots \frac{\xi}{2} - 1$. See Figure 1(a).

Next, we claim that $|\varphi\ (x) - \varphi(y)| + d\ (x, y) \geq 1 + \xi\ \forall x, y \in V(OX\ (1,\xi))$.

Let $x, y \in V(OX\ (1,\xi))$.

Case 1.1. *Suppose x and y are of the form v_l and v_m, then $\varphi\ (x) = \xi + (l-1)\ (\xi - 2)$ and $\varphi\ (y) = \ \xi + (m-1)\ (\xi-2)\ ,\ 1 \leq l \neq m \leq \frac{\xi}{2}$. Also, $d\ (x,y) \geq 3$. Hence, $|\varphi\ (x) - \varphi(y)| + d\ (x, y) \geq |(l-m)\ (\xi - 2)| + 3 \geq \xi + 1,\ \ l \neq m$.*

Case 1.2. If $x = w_l$ and $y = w_m$ then $d\ (x, y) \geq 3$ and $|\varphi\ (x) - \varphi\ (y)|$

$= \left|\left(\frac{\xi^2}{2} - \xi + 3 + (l-1)\ (\xi-2)\right) - \left(\frac{\xi^2}{2} - \xi + 3 + (m-1)\ (\xi-2)\right)\right| = |(l-m)\ (\xi - 2)|,$

$where\ 1 \leq l \neq m \leq \frac{\xi}{2}$. Since $l \neq m,$ we get, $|\varphi\ (x) - \varphi\ (y)| + d\ (x, y) \geq \xi + 1$.

Case 1.3. Assume that $x = u_{l+1}$ and $y = u_{m+1},\ 1 \leq l \neq m \leq \frac{\xi}{2}$. Then $d\ (x, y) \geq 1$ and $\varphi\ (u_{l+1}) = \xi^2 - 3\xi + 8 + (l-1)\ (\xi+3), \varphi\ (u_{m+1}) = \xi^2 - 3\xi + 8 + (m-1)\ (\xi + 3)$.

Therefore, $|\varphi\ (x) - \varphi(y)| + d\ (x, y) \geq |(l-m)\ (\xi+3)| + 1 > \xi + 1\ ,\ \ l \neq m$.

Case 1.4. Take x and y in the row line such that $x = u_{\frac{\xi}{2}+l}$ and $y = u_{\frac{\xi}{2}+m}$.

$1 \leq l \neq m \leq \frac{\xi}{2} - 1$, respectively. Here, $\varphi\left(u_{\frac{\xi}{2}+l}\right) = \xi^2 - 3\xi + \frac{\xi}{2} + 9 + (l-1)\ (\xi+3)$, $\varphi\left(u_{\frac{\xi}{2}+m}\right) = \xi^2 - 3\xi + \frac{\xi}{2} + 9 + (m-1)\ (\xi + 3)$ and $d\ (x, y) \geq 1$. So, $|\varphi\ (x) - \varphi(y)| + d\ (x, y) \geq |(l-m)\ (\xi+3)| + 1 > \xi + 1\ ,\ \ l \neq m$.

Case 1.5. Suppose $x = v_l$ and $y = w_m$, then $\varphi(x) = \xi + (l-1)(\xi-2)$ and $\varphi(y) = \frac{\xi^2}{2} - \xi + 3 + (m-1)(\xi-2)$, $1 \le l, m \le \frac{\xi}{2}$. Also, $d(x,y) \ge 2$. Hence, $|\varphi(x) - \varphi(y)| + d(x,y) \ge \left|(\xi + (l-1)(\xi-2)) - \left(\frac{\xi^2}{2} - \xi + 3 + (m-1)(\xi-2)\right)\right| + 2 > \xi + 1$.
Case 1.6. If x and y are mapped to $\xi + (l-1)(\xi-2)$ and $\xi^2 - 3\xi + 8 + (m-1)(\xi+3)$, $1 \le l, m \le \frac{\xi}{2}$, then $d(x,y) \ge 1$, where $x = v_l$ and $y = u_{m+1}$. Consequently, $|\varphi(x) - \varphi(y)| + d(x,y) > \xi + 1$.
Case 1.7. Let $x = v_l$ and $y = u_{\frac{\xi}{2}+m}$, $1 \le l \le \frac{\xi}{2}$, $1 \le m \le \frac{\xi}{2} - 1$. Then $\varphi(v_l)$ and $\varphi\left(u_{\frac{\xi}{2}+m}\right)$ are $\xi + (l-1)(\xi-2)$ and $\xi^2 - 3\xi + \frac{\xi}{2} + 9 + (m-1)(\xi+3)$ respectively. Also, $d\left(v_l, u_{\frac{\xi}{2}+m}\right) \ge 1$. So, $|\varphi(x) - \varphi(y)| + d(x,y) > \xi + 1$.
Case 1.8. Pick $x = w_l$ and $y = u_{m+1}$, $1 \le l, m \le \frac{\xi}{2}$. Then $\varphi(x) = (\frac{\xi^2}{2} - \xi + 3 + (l-1)(\xi-2)$, $\varphi(y) = \xi^2 - 3\xi + 8 + (m-1)(\xi+3)$ and $d(w_l, u_{m+1}) \ge 1$. Therefore, $|\varphi(x) - \varphi(y)| + d(x,y) > \xi + 1$.
Case 1.9. Let $x = w_l$ and $y = u_{\frac{\xi}{2}+m}$, $1 \le l \le \frac{\xi}{2}$, $1 \le m \le \frac{\xi}{2} - 1$. Then $\varphi(w_l)$ and $\varphi\left(u_{\frac{\xi}{2}+m}\right)$ are mapped to $\frac{\xi^2}{2} - \xi + 3 + (l-1)(\xi-2)$ and $\xi^2 - 3\xi + \frac{\xi}{2} + 9 + (m-1)(\xi+3)$ respectively. In addition, $d\left(w_{l+1}, u_{\frac{\xi}{2}+m}\right) \ge 1$. Hence, $|\varphi(x) - \varphi(y)| + d(x,y) > \xi + 1$.
Case 1.10. Suppose $x = u_{l+1}$ and $y = u_{\frac{\xi}{2}+m}$, $1 \le l \le \frac{\xi}{2}$, $1 \le m \le \frac{\xi}{2} - 1$, then $\varphi(u_l) = \xi^2 - 3\xi + 8 + (l-1)(\xi+3)$ and $\varphi\left(u_{\frac{\xi}{2}+m}\right) = \xi^2 - 3\xi + \frac{\xi}{2} + 9 + (m-1)(\xi+3)$.
If $l = m$, then $\left|\varphi(u_{l+1}) - \varphi\left(u_{\frac{\xi}{2}+m}\right)\right| = \frac{\xi}{2} + 1$ and $d\left(u_{l+1}, u_{\frac{\xi}{2}+m}\right) = \frac{\xi}{2}$.
Also, if

$l = m+1$, then $\left|\varphi(u_{l+1}) - \varphi\left(u_{\frac{\xi}{2}+m}\right)\right| = \left|-1 - \frac{\xi}{2} + \xi + 3\right| = \frac{\xi}{2} + 2$ and $d\left(u_{l+1}, u_{\frac{\xi}{2}+m}\right) = \frac{\xi}{2} - 1$.

Otherwise, $\left|\varphi(u_{l+1}) - \varphi\left(u_{\frac{\xi}{2}+m}\right)\right| \ge \xi$.
Hence for all the possibilities in this case, $\left|\varphi(u_{l+1}) - \varphi\left(u_{\frac{\xi}{2}+m}\right)\right| + d\left(u_{l+1}, u_{\frac{\xi}{2}+m}\right) > \xi + 1$.
Otherwise, $\left|\varphi(u_{l+1}) - \varphi\left(u_{\frac{\xi}{2}+m}\right)\right| \ge \xi$.
Hence for all the possibilities in this case, $\left|\varphi(u_{l+1}) - \varphi\left(u_{\frac{\xi}{2}+m}\right)\right| + d\left(u_{l+1}, u_{\frac{\xi}{2}+m}\right) > \xi + 1$.
Case 1.11. Suppose $x = u_1$ and $y = u_{\xi+1}$, then $|\varphi(x) - \varphi(y)| = 1$ and $d(x,y) = \xi$. Again, if $x \in \{u_1, u_{\xi+1}\}$ and y is any other vertex in $OX(1,\xi)$, it is easy to verify the condition $|\varphi(x) - \varphi(y)| + d(x,y) > \xi + 1$ is true. Further, the vertex $u_{\frac{\xi}{2}+1}$ attains the maximum value $\xi^2 - 3\xi + 8 + \left(\frac{\xi}{2} - 1\right)(\xi+3) = \frac{3\xi^2}{2} - \frac{5\xi}{2} + 5$. Hence $rn(\varphi) = \frac{3\xi^2}{2} - \frac{5\xi}{2} + 5$. Therefore, we have attained the result $rn(OX(1,\xi)) \le \frac{3\xi^2}{2} - \frac{5\xi}{2} + 5$, ξ is even.
Case 2. ξ is odd.
Define a 1-1 mapping $\varphi : V(OX(1,\xi)) \to N$ as follows: $\varphi(u_1) = 1$, $\varphi(u_{\xi+1}) = 2$, $\varphi(v_i) = \xi + (i-1)(\xi-2)$, $i = 1, 2 \ldots \left\lfloor \frac{\xi}{2} \right\rfloor$, $\varphi(w_i) = \left\lfloor \frac{\xi}{2} \right\rfloor (\xi-2) + 4 + (i-1)(\xi-2)$, $i = 1, 2 \ldots \left\lceil \frac{\xi}{2} \right\rceil$, $\varphi(u_{i+1}) = \xi^2 - 3\xi + 8 + (i-1)\left(2\left\lceil \frac{\xi}{2} \right\rceil + 3\right)$, $i = 1, 2 \ldots \left\lfloor \frac{\xi}{2} \right\rfloor$, $\varphi\left(u_{\lceil \frac{\xi}{2} \rceil + i}\right) = \xi^2 - 3\xi + \left\lceil \frac{\xi}{2} \right\rceil + 9 + (i-1)\left(2\left\lceil \frac{\xi}{2} \right\rceil + 3\right)$, $i = 1, 2 \ldots \left\lfloor \frac{\xi}{2} \right\rfloor$. Refer Figure 1(b).
Here the vertex u_ξ attains the maximum value $\xi^2 - 3\xi + \left\lceil \frac{\xi}{2} \right\rceil + \left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1\right)\left(2\left\lceil \frac{\xi}{2} \right\rceil + 3\right) + 9$ which is the radio number of φ. Since the remaining part of the proof is similar to case 1, we omit the proof.

To investigate the lower bound, we use the following result which was proved by Bharati et al. [15].

Theorem 3.2. (As Theorem 2 in [15]].): Let G be a simple connected graph of order n. Let $\alpha_0, \alpha_1 \ldots \alpha_k$ be the number of vertices having eccentricities $e_0, e_1 \ldots e_k$, where $diam(G) = e_0 >$

$e_1 > \cdots > e_k = rad(G)$.Then

$$r_n(G) \geq \begin{cases} n - 2(d-e_k) + \sum_{i=1}^{k} 2(d-e_i)\alpha_i, & if\ \alpha_k > 1 \\ n - (d-e_k) - (d-e_{k-1}) + \sum_{i=1}^{k} 2(d-e_i)\alpha_i, & if\ \alpha_k = 1 \end{cases}.$$

Lemma 3.1. Let $e_0, e_1 \ldots e_{\frac{\xi}{2}}$ be the eccentricities of the vertices of $OX(1,\xi)$, where ξ is an even natural number. If $e_k > e_{k+1},\ k = 0,\ 1,\ 2\ldots \frac{\xi}{2} - 1$, then the number of vertices with eccentricity e_k is $4,\ 0 \leq k \leq \frac{\xi}{2} - 1$ and $e_{\frac{\xi}{2}}$ is 1.

*Proof***.** It is easy to verify that only the vertices u_1 and w_1 are diametrically opposite to $u_{\xi+1}$ and $v_{\lfloor \frac{\xi}{2} \rfloor}$ in $OX(1,\xi)$. Therefore, the number of vertices with eccentricity e_0 is 4. Similarly, the four vertices $u_2,\ v_1, u_\xi$ and $w_{\lfloor \frac{\xi}{2} \rfloor}$ are having the eccentricity e_1. If we proceed like this, we are able to identify the vertices having eccentricity up to $e_{\frac{\xi}{2}-1}$ as 4. In addition, the middle vertex $u_{\frac{\xi}{2}+1}$of the horizontal line alone is of eccentricity $e_{\frac{\xi}{2}} = rad(OX(1,\xi))$. Hence, we conclude that the number of vertices with eccentricity e_k is $4,\ 0 \leq k \leq \frac{\xi}{2} - 1$ and $e_{\frac{\xi}{2}}$ is 1.

Theorem 3.3. If $e_0 > e_1 > \cdots > e_{\frac{\xi}{2}}$ be the $\frac{\xi}{2} + 1$ eccentricities of the vertices of $OX(1,\xi)$, where ξ is an even natural number, then $r_n(OX(1,\xi),) \geq 2(\xi+1) + 8\sum_{i=1}^{\frac{\xi}{2}-1}(\xi - i)$.

Proof. The eccentricities of vertices are given by $e_0 = \xi, e_1 = \xi - 1\ \ldots\ e_{\frac{\xi}{2}-1} = \frac{\xi}{2} + 1,\ e_{\frac{\xi}{2}} = \frac{\xi}{2}$. Using Lemma 3.1 in Theorem 3.2, we have $\alpha_i = 4,\ i = 0, 1 \ldots \frac{\xi}{2} - 1$ and $\alpha_{\frac{\xi}{2}} = 1$. Since, $\alpha_k = \alpha_{\frac{\xi}{2}} = 1$, we must apply the second part of the result in Theorem 4.2 and obtained, $r_n(G) \geq 2\xi + 1 - \left(\xi - \frac{\xi}{2}\right) - (\xi - \frac{\xi}{2} - 1) + \sum_{i=1}^{\frac{\xi}{2}} 2(\xi - i)\alpha_i = 2(\xi+1) + 8\sum_{i=1}^{\frac{\xi}{2}-1}(\xi - i)$.

Lemma 3.2. Let $e_0, e_1 \ldots e_{\lfloor \frac{\xi}{2} \rfloor}$ be the eccentricities of the vertices of $OX(1,\xi)$, where ξ is an odd natural number. If $e_k > e_{k+1},\ k = 0,\ 1,\ 2\ldots \left\lfloor \frac{\xi}{2} \right\rfloor$, then the number of vertices with eccentricity e_k is $4,\ 0 \leq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor - 1$ and $e_{\lfloor \frac{\xi}{2} \rfloor}$ is 3.

*Proof***.** It is easy to realize that the vertices in the sets $\left\{u_1, w_1, u_{\xi+1}, w_{\lceil \frac{\xi}{2} \rceil}\right\}$, $\left\{u_2,\ v_1, u_\xi, w_{\lfloor \frac{\xi}{2} \rfloor}\right\}$ $\ldots$ $\left\{u_{\lfloor \frac{\xi}{2} \rfloor},\ w_{\lceil \frac{\xi}{4} \rceil}, u_{\lceil \frac{\xi}{2} \rceil+2}, w_{\lfloor \frac{\xi}{4} \rfloor+1}\right\}$ having eccentricities $e_0, e_1 \ldots e_{\lfloor \frac{\xi}{2} \rfloor - 1}$ and $\left\{u_{\lceil \frac{\xi}{2} \rceil},\ u_{\lceil \frac{\xi}{2} \rceil+1}, v_{\lceil \frac{\xi}{4} \rceil}\right\}$ having eccentricity $e_{\lfloor \frac{\xi}{2} \rfloor}$. Hence, the number of vertices with eccentricity e_k is $4,\ 0 \leq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor - 1$ and $e_{\lfloor \frac{\xi}{2} \rfloor}$ is 3.

Theorem 3.4. If $e_0 > e_1 > \cdots > e_{\lfloor \frac{\xi}{2} \rfloor}$ be the $\left\lceil \frac{\xi}{2} \right\rceil$ eccentricities of the vertices of $OX(1,\xi)$, where ξ is an odd natural number, then $r_n(OX(1,\xi)) \geq 6\xi - 4\left\lceil \frac{\xi}{2} \right\rceil + 1 + 8\sum_{i=1}^{\lfloor \frac{\xi}{2} \rfloor - 1}(\xi - i)$.

Proof. The eccentricities of vertices are given by $e_0 = \xi,\ e_1 = \xi - 1\ \ldots\ e_{\lfloor \frac{\xi}{2} \rfloor - 1} = \left\lceil \frac{\xi}{2} \right\rceil + 1,\ e_{\lfloor \frac{\xi}{2} \rfloor} = \left\lceil \frac{\xi}{2} \right\rceil$. Using Lemma 3.2 in Theorem 3.2, we have $\alpha_i = 4,\ i = 0, 1 \ldots \left\lfloor \frac{\xi}{2} \right\rfloor - 1$ and $\alpha_{\lfloor \frac{\xi}{2} \rfloor} = 3$. Since, $\alpha_k = \alpha_{\lfloor \frac{\xi}{2} \rfloor} = 3,$ we must apply the first part of the result in Theorem 4.2 gives, $r_n(G) \geq 2\left\lceil \frac{\xi}{2} \right\rceil + 1 + \sum_{i=1}^{\lfloor \frac{\xi}{2} \rfloor - 1} 2(\xi - i)4 + 2\left(\xi - \left\lfloor \frac{\xi}{2} \right\rfloor\right)3 = 6\xi - 4\left\lceil \frac{\xi}{2} \right\rceil + 1 + 8\sum_{i=1}^{\lfloor \frac{\xi}{2} \rfloor - 1}(\xi - i)$.

Combining Theorems 3.1, 3.2 and 3.3 yields the following results.

Theorem 3.5. Let ξ be odd. Then, $6\xi - 4\left\lceil \frac{\xi}{2} \right\rceil + 1 + 8\sum_{i=1}^{\lfloor \frac{\xi}{2} \rfloor - 1}(\xi - i) \leq r_n(OX(1,\xi)) \leq \xi^2 - 3\xi + \left\lceil \frac{\xi}{2} \right\rceil + \left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1\right)\left(2\left\lceil \frac{\xi}{2} \right\rceil + 3\right) + 9.$

Theorem 3.6. Let ξ be even. Then, $2(\xi+1) + 8\sum_{i=1}^{\frac{\xi}{2}-1}(\xi - i) \leq r_n(OX(1,\xi)) \leq \frac{3\xi^2}{2} - \frac{5\xi}{2} + 5.$

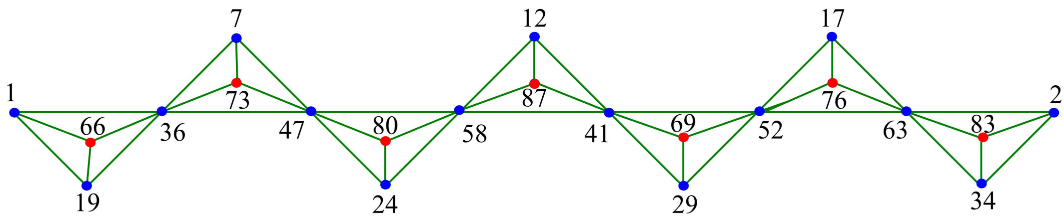


Figure 2. A radio labeling of a single-chain silicate structure $SL\left(1,7\right)$ which attains the bound.

3.2 Bounds for Chain Silicate Structures

In this subsection, we have estimated the upper and lower bounds for the radio number of chain silicate structures.

Theorem 3.7. For any odd natural number ξ, the radio number of single chain Silicate structure satisfies $rn(SL(1,\xi) \leq \xi^2 - 2\xi + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right) + 12.$

Proof. We name the silicon vertices in single chain silicate $SL(1,\xi)$ as $z_1, z_2 \ldots z_\xi$ from left to right and the remaining vertices are named as in $OX\left(1,\xi\right)$. Next, we define an injection $\varphi : V(SL\left(1,\xi\right)) \rightarrow N$ as follows: $\varphi\left(z_i\right) = \xi^2 - 3\xi + \left\lceil\frac{\xi}{2}\right\rceil + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(i-1\right) + 12,\ i = 1, 2 \ldots \left\lceil\frac{\xi}{2}\right\rceil,\ \varphi\left(z_{\left\lceil\frac{\xi}{2}\right\rceil + i}\right) = \xi^2 - 2\xi + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(i-1\right) + 12,\ i = 1, 2 \ldots \left\lfloor\frac{\xi}{2}\right\rfloor$. The rest of the vertices are labelled as in Case 1 of Theorem 3.1.

Claim: $\left|\varphi\left(x\right) - \varphi(y)\right| + d\left(x,y\right) \geq 1 + \xi\ \forall x, y \in V(SL\left(1,\xi\right))$.

Let $x, y \in V(SL\left(1,\xi\right))$.

Case 1. Suppose $x = z_l$ and $y = z_m$, then $\varphi\left(x\right) = \xi^2 - 3\xi + \left\lceil\frac{\xi}{2}\right\rceil + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(l-1\right) + 12$ and $\varphi\left(y\right) = \xi^2 - 3\xi + \left\lceil\frac{\xi}{2}\right\rceil + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(m-1\right) + 12,\ 1 \leq l \neq m \leq \left\lceil\frac{\xi}{2}\right\rceil$. Also, $d\left(x,y\right) \geq 2$. Hence, $\left|\varphi\left(x\right) - \varphi(y)\right| + d\left(x,y\right) \geq \left|\xi\left(l-m\right)\right| + 2 > \xi + 1$, Since $l \neq m$.

Case 2. If $x = z_{\left\lceil\frac{\xi}{2}\right\rceil + l}$ and $y = z_{\left\lceil\frac{\xi}{2}\right\rceil + m}$, then $d\left(z_{\left\lceil\frac{\xi}{2}\right\rceil + l}, z_{\left\lceil\frac{\xi}{2}\right\rceil + m}\right) \geq 2$ and $\varphi\left(z_{\left\lceil\frac{\xi}{2}\right\rceil + l}\right) = \xi^2 - 2\xi + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(l-1\right) + 12,\ \varphi\left(z_{\left\lceil\frac{\xi}{2}\right\rceil + m}\right) = \xi^2 - 2\xi + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(m-1\right) + 12,\ 1 \leq l \neq m \leq \left\lfloor\frac{\xi}{2}\right\rfloor$. Since $l \neq m$, the condition becomes $\left|\varphi\left(x\right) - \varphi\left(y\right)\right| + d\left(x,y\right) \geq \left|\xi\left(l-m\right)\right| + 2 > \xi + 1$.

Case 3. Assume that $x = z_l$ and $y = z_{\left\lceil\frac{\xi}{2}\right\rceil + m}$, then
$\varphi\left(z_l\right) = \xi^2 - 3\xi + \left\lceil\frac{\xi}{2}\right\rceil + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(l-1\right) + 12,\ 1 \leq l \neq m \leq \left\lfloor\frac{\xi}{2}\right\rfloor$ and $\varphi\left(y\right) = \varphi\left(z_{\left\lceil\frac{\xi}{2}\right\rceil + m}\right) = \xi^2 - 2\xi + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(m-1\right) + 12,\ 1 \leq l \neq m \leq \left\lfloor\frac{\xi}{2}\right\rfloor$.

Case 3.1. If $l = m$, then $d\left(z_l, z_{\left\lceil\frac{\xi}{2}\right\rceil + l}\right) = \left\lceil\frac{\xi}{2}\right\rceil + 1$ and so] $\left|\varphi\left(x\right) - \varphi\left(y\right)\right| + d\left(x,y\right) \geq \left|-\xi + \left\lceil\frac{\xi}{2}\right\rceil\right| + \left\lceil\frac{\xi}{2}\right\rceil + 1 \geq \xi + 1$.So, $d\left(x,y\right) \geq 2$. Hence, $\left|\varphi\left(x\right) - \varphi(y)\right| + d\left(x,y\right) \geq \left|\xi\left(l-m\right)\right| + 2 > \xi + 1$, Since $l \neq m$.

Case 3.2. If $l = m + 1$, then $d\left(z_{m+1}, z_{\left\lceil\frac{\xi}{2}\right\rceil + m}\right) = \left\lceil\frac{\xi}{2}\right\rceil$ and $\left|\varphi\left(z_{m+1}\right) - \varphi\left(z_{\left\lceil\frac{\xi}{2}\right\rceil + m}\right)\right| = \left|-\xi + \left\lceil\frac{\xi}{2}\right\rceil + \xi\right| = \left\lceil\frac{\xi}{2}\right\rceil$. Therefore, $\left|\varphi\left(x\right) - \varphi(y)\right| + d\left(x,y\right) \geq 2\left\lceil\frac{\xi}{2}\right\rceil > \xi + 1$. Otherwise, $\left|\varphi\left(x\right) - \varphi(y)\right| \geq \xi$ which trivially verifies the condition.

Case 4. Suppose that $x = v_{\left\lceil\frac{\xi}{2}\right\rceil + l}$ and $y = z_{\left\lceil\frac{\xi}{2}\right\rceil + m}$, $1 \leq l, m \leq \left\lfloor\frac{\xi}{2}\right\rfloor$, then $\varphi\left(x\right) = \xi^2 - 3\xi + \left\lceil\frac{\xi}{2}\right\rceil + 9 + \left(i-1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right)$ and $\varphi\left(y\right) = \xi^2 - 2\xi + \left(\left\lfloor\frac{\xi}{2}\right\rfloor - 1\right)\left(2\left\lceil\frac{\xi}{2}\right\rceil + 3\right) + \xi\left(m-1\right) + 12.$

If $l = \left\lfloor \frac{\xi}{2} \right\rfloor$ and $m = 1$, then $d\left(v_\xi, z_{\left\lceil \frac{\xi}{2} \right\rceil+1}\right) = \left\lfloor \frac{\xi}{2} \right\rfloor$ and $\left|\varphi\left(v_\xi\right) - \varphi\left(z_{\left\lceil \frac{\xi}{2} \right\rceil+1}\right)\right| = \left\lceil \frac{\xi}{2} \right\rceil + 2$. Otherwise, $|\varphi(x) - \varphi(y)| > \xi$. Hence, $|\varphi(x) - \varphi(y)| + d(x, y) \geq \xi + 1$.

Case 5. If $x = z_l$ and $y = u_{\left\lceil \frac{\xi}{2} \right\rceil+m}$, then $\varphi(z_l) = \xi^2 - 3\xi + \left\lceil \frac{\xi}{2} \right\rceil + \left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1\right)\left(2\left\lceil \frac{\xi}{2} \right\rceil + 3\right) + \xi(l-1) + 12$, $1 \leq l, m \leq \left\lfloor \frac{\xi}{2} \right\rfloor$ and $\varphi\left(u_{\left\lceil \frac{\xi}{2} \right\rceil+m}\right) = \xi^2 - 3\xi + \left\lceil \frac{\xi}{2} \right\rceil + 9 + (m-1)\left(2\left\lceil \frac{\xi}{2} \right\rceil + 3\right)$, $1 \leq l, m \leq \left\lfloor \frac{\xi}{2} \right\rfloor$. If $l = 1$ and $m = \left\lfloor \frac{\xi}{2} \right\rfloor$, then $d(x, y) = \xi$ and $|\varphi(x) - \varphi(y)| = 3$. Otherwise, $|\varphi(x) - \varphi(y)| > \xi + 1$. So, $|\varphi(x) - \varphi(y)| + d(x, y) \geq \xi + 1$.

Case 6. Presume that $x \in \{ z_k \backslash k = 1, 2 \ldots \xi\}$ and y is another other vertex in $SL(1, \xi)$ except the cases 4 and 5, then $|\varphi(x) - \varphi(y)| > \xi$ and hence we get $|\varphi(x) - \varphi(y)| + d(x, y) \geq \xi + 1$. The rest of the cases can be discussed and verified as in Theorem 3. 1. Since, the vertex z_ξ was labelled with the maximum number $\xi^2 - 2\xi + \left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1\right)\left(2\left\lceil \frac{\xi}{2} \right\rceil + 3\right) + \xi\left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1\right) + 12$ and hence, $rn(SL(1, \xi)) \leq \xi^2 - 2\xi + \left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1\right)\left(2\left\lceil \frac{\xi}{2} \right\rceil + 3\right) + \xi\left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1\right) + 12$, ξ is odd.

Lemma 3.3. Let $e_0, e_1 \ldots e_{\left\lfloor \frac{\xi}{2} \right\rfloor}$ be the eccentricities of the vertices of $SL(1, \xi)$, where ξ is an odd natural number. If $e_k > e_{k+1}$, $k = 0,\ 1,\ 2 \ldots \left\lfloor \frac{\xi}{2} \right\rfloor$, then the number of vertices with eccentricity e_k is 6, $0 \leq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor - 1$ and $e_{\left\lfloor \frac{\xi}{2} \right\rfloor}$ is 4.

Proof. We have noticed that the vertices in the sets $\left\{u_1, w_1, z_1, u_{\xi+1}, w_{\left\lceil \frac{\xi}{2} \right\rceil}, z_\xi\right\}$, $\left\{u_2, v_1, z_2, u_\xi, w_{\left\lfloor \frac{\xi}{2} \right\rfloor}, z_{\xi-1}\right\} \ldots \left\{u_{\left\lfloor \frac{\xi}{2} \right\rfloor}, w_{\left\lceil \frac{\xi}{4} \right\rceil}, z_{\left\lfloor \frac{\xi}{2} \right\rfloor}, u_{\left\lceil \frac{\xi}{2} \right\rceil+2}, w_{\left\lfloor \frac{\xi}{4} \right\rfloor+1}, z_{\left\lfloor \frac{\xi}{2} \right\rfloor+1}\right\}$ having eccentricities $e_0, e_1 \ldots e_{\left\lfloor \frac{\xi}{2} \right\rfloor-1}$ and $\left\{u_{\left\lceil \frac{\xi}{2} \right\rceil}, u_{\left\lceil \frac{\xi}{2} \right\rceil+1}, v_{\left\lceil \frac{\xi}{4} \right\rceil}, z_{\left\lfloor \frac{\xi}{2} \right\rfloor}\right\}$ having eccentricity $e_{\left\lfloor \frac{\xi}{2} \right\rfloor}$. Therefore, the number of vertices in each set shows that the number of vertices with eccentricity e_k as 6, $0 \leq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor - 1$ and $e_{\left\lfloor \frac{\xi}{2} \right\rfloor}$ is 4.

Theorem 3.8. Let $e_0 > e_1 > \cdots > e_{\left\lfloor \frac{\xi}{2} \right\rfloor}$ be the $\left\lceil \frac{\xi}{2} \right\rceil$ eccentricities of the vertices of single chain silicate structure $SL(1, \xi)$. Then, $r_n(SL(1, \xi)) \geq 6\xi - 4\left\lceil \frac{\xi}{2} \right\rceil + 1 + 8\sum_{i=1}^{\left\lfloor \frac{\xi}{2} \right\rfloor - 1}(\xi - i)$, ξ is odd.

Proof. The eccentricities of vertices are given by $e_0 = \xi$, $e_1 = \xi - 1 \ \ldots \ e_{\left\lfloor \frac{\xi}{2} \right\rfloor-1} = \left\lceil \frac{\xi}{2} \right\rceil + 1$, $e_{\left\lfloor \frac{\xi}{2} \right\rfloor} = \left\lceil \frac{\xi}{2} \right\rceil$. Using Lemma 3.3 and Theorem 3.2, we have $\alpha_i = 6$, $i = 0, 1 \ldots \left\lfloor \frac{\xi}{2} \right\rfloor - 1$ and $\alpha_{\left\lfloor \frac{\xi}{2} \right\rfloor} = 4$. Since, $\alpha_k = 3 > 1$, we get, $r_n(SL(1, \xi)) \geq 3\xi + 1 - 2\left(\xi - \left\lceil \frac{\xi}{2} \right\rceil\right) + \sum_{i=1}^{\left\lfloor \frac{\xi}{2} \right\rfloor} 2(\xi - i)6 + 2\left(\xi - \left\lfloor \frac{\xi}{2} \right\rfloor\right)4 = \xi + 10\left\lceil \frac{\xi}{2} \right\rceil + 1 + 12\sum_{i=1}^{\left\lfloor \frac{\xi}{2} \right\rfloor - 1}(\xi - i)$.

Combining Theorems 3.7 and 3.8, we get the following result:

Theorem 3.9. Let ξ be odd. Then the radio number of single chain silicate structure lies between $6\xi - 4\left\lceil \frac{\xi}{2} \right\rceil + 1 + 8\sum_{i=1}^{\left\lfloor \frac{\xi}{2} \right\rfloor - 1}(\xi - i)$ and $\xi^2 - 2\xi + \left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1\right)\left(2\left\lceil \frac{\xi}{2} \right\rceil + 3\right) + \xi\left(\left\lfloor \frac{\xi}{2} \right\rfloor - 1\right) + 12$.

Theorem 3.10. Let ξ be an even natural number. Then the radio number of single chain silicate structure satisfies $rn(SL(1, \xi) \leq \xi^2 - \xi + \left(\frac{\xi}{2} - 1)\right)(2\xi + 4) + 10$.

Proof. First we label the vertices of $OX(1, \xi)$ as same as of Case 1 in Theorem 3. 1. Then label the rest of the vertices, namely the silicon vertices as $\varphi(z_i) = \xi^2 - 3\xi + \left(\frac{\xi}{2} - 1\right)(\xi + 3) + \frac{\xi}{2} + (\xi + 1)(i - 1) + 9$, $i = 1, 2 \ldots \frac{\xi}{2}$.

$$\varphi\left(z_{\frac{\xi}{2}+i}\right) = \xi^2 - 2\xi + \left(\frac{\xi}{2} - 1\right)(\xi + 3) + (\xi + 1)(i - 1) + 10, \ i = 1, 2 \ldots \frac{\xi}{2}.$$

Since the rest of the proof is Similar to Theorem 3.1 and Theorem 3.2, we omit the proof.

Lemma 3.4. Let ξ be even. If $e_0, e_1 \ldots e_{\frac{\xi}{2}}$ be the eccentricities of the vertices of $SL(1, \xi)$, such that $e_k > e_{k+1}$, $k = 0, 1, 2 \ldots \frac{\xi}{2} - 1$, then the number of vertices with eccentricity e_k is 6, $0 \leq k \leq \frac{\xi}{2} - 1$ and $e_{\frac{\xi}{2}}$ is 1.

Theorem 3.11. If $e_0 > e_1 > \cdots > e_{\frac{\xi}{2}}$ be the $\frac{\xi}{2} + 1$ eccentricities of the vertices of $SL(1, \xi)$, then $r_n\left(OX(1, \xi),\right) \geq 3\xi + 2 + 12\sum_{i=1}^{\frac{\xi}{2}-1}(\xi - i)$, ξ is even.

As the proof of Lemma 3.4 and Theorem 3.6 are the combinations of Lemmas 3.1, 3.3 and Theorems 3.3, 2.4, we left the proof to the reader.

Combining Theorems 3.10 and 3.11, we acquired the following theorem.

Theorem 3.12. Let ξ be even. Then the radio number of $SL(1, \xi)$ lies between $3\xi + 2 + 12\sum_{i=1}^{\frac{\xi}{2}-1}(\xi - i)$ and $\xi^2 - \xi + \left(\frac{\xi}{2} - 1)\right)(2\xi + 4) + 10$.

3.3 Upper Bounds for cyclic Silicate and oxide Structure

Theorem 3.13. Let $SL_c(\xi)$ be the cyclic silicate having the diameter $\left\lfloor \frac{\xi}{2} \right\rfloor + 1$, then for $\xi > 5$ the radio number of $SL_c(\xi)$ satisfies, $rn(SL_c(\xi))) \leq \begin{cases} \frac{3\xi^2}{4} + 5, & \xi \text{ is even} \\ 2\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor \left\lceil \frac{\xi}{2} \right\rceil + 4, & \xi \text{ is odd} \end{cases}$.

Proof. First let us partition the vertex set of $SL_c(\xi)$ into three disjoint sets V_1, V_2 and V_3 such that V_1 contains the silicate vertices, V_2 and V_3 contains the oxide vertices of degree 6 and 3 respectively. Again, we name the vertices in V_1, V_2 and V_3 as $\{w_1, w_2 \ldots w_\xi\}$, $\{u_1, u_2 \ldots u_\xi\}$ and $\{v_1, v_2 \ldots v_\xi\}$ respectively. Next, we assign distinct natural numbers to $V(SL_c(\xi))$ as follows:
$\varphi(u_i) = \left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)(i-1) + 1$, $i = 1, 2 \ldots \left\lceil \frac{\xi}{2} \right\rceil$,
$\varphi\left(u_{\lceil \frac{\xi}{2} \rceil + i}\right) = \left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)(i-1) + 3$, $i = 1, 2 \ldots \left\lfloor \frac{\xi}{2} \right\rfloor$,
$\varphi(v_i) = \left\lfloor \frac{\xi}{2} \right\rfloor \left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right) + \left\lfloor \frac{\xi}{2} \right\rfloor (i-1) + 3$, $i = 1, 2 \ldots \left\lceil \frac{\xi}{2} \right\rceil$,
$\varphi\left(v_{\lceil \frac{\xi}{2} \rceil + i}\right) = \left\lfloor \frac{\xi}{2} \right\rfloor \left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right) + \left\lfloor \frac{\xi}{2} \right\rfloor (i-1) + 4$, $i = 1, 2 \ldots \left\lfloor \frac{\xi}{2} \right\rfloor$, $\varphi(w_i) = 2\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor i + 4$

$i = 1, 2 \ldots \left\lceil \frac{\xi}{2} \right\rceil$, $\varphi\left(w_{\lceil \frac{\xi}{2} \rceil + i}\right) = 2\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor i + 5$, $i = 1, 2 \ldots \left\lfloor \frac{\xi}{2} \right\rfloor$. This mapping is evident through Figure 3. Now, we verify the above mapping φ is a valid radio labeling. Since the diameter of $SL_c(\xi)$ is $\left\lfloor \frac{\xi}{2} \right\rfloor + 1$, we must show that $|\varphi(x) - \varphi(y)| + d(x, y) \geq 2 + \left\lfloor \frac{\xi}{2} \right\rfloor$ $\forall\, x, y \in V(SL_c(\xi))$. Let $x, y \in V(SL_c(\xi))$.

Case 1. Choose x and y in the set V_2.

Case 1.1. If x and y are any two distinct vertices in V_2 such that $x = u_s$ and $y = u_k$, $1 \leq s \neq k \leq \left\lceil \frac{\xi}{2} \right\rceil$, then under the mapping φ, the labeling of x and y are $\left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)(s-1) + 1$ and $\left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)(k-1) + 1$ respectively. Hence, $|\varphi(x) - \varphi(y)| + d(x, y) \geq \left\lfloor \frac{\xi}{2} \right\rfloor + 2$, since $s \neq k$.

Case 1.2. If $x, y \in V_2$ such that $x = u_{\lceil \frac{\xi}{2} \rceil + s}$ and $y = u_{\lceil \frac{\xi}{2} \rceil + k}$, $1 \leq s \neq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor$, then $|\varphi(x) - \varphi(y)| = \left|\left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)(s-1) + 3 - \left(\left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)(k-1) + 3\right)\right| = \left|(s-k)\left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)\right|$, Since $s \neq k$, we get, $|\varphi(x) - \varphi(y)| + d(x, y) \geq \left\lfloor \frac{\xi}{2} \right\rfloor + 2$.

Case 1.3. Assume that $x = u_s$ and $y = u_{\lceil \frac{\xi}{2} \rceil + k}$. $1 \leq s \leq \left\lceil \frac{\xi}{2} \right\rceil$, $1 \leq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor$. Then $d(x, y) \geq 1$ and $\varphi(u_s) = \left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)(s-1) + 1$ and $\varphi\left(u_{\lceil \frac{\xi}{2} \rceil + k}\right) = \left(\left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)(k-1) + 3\right)$. If $s = k$, then u_s and $u_{\lceil \frac{\xi}{2} \rceil + k}$ assigned with a labeling difference exactly 2. Also, since they are diametrically opposite vertices, the radio labeling condition is satisfied. Otherwise,
$|\varphi(x) - \varphi(y)| + d(x, y) \geq \left|\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)(s-) + 1 - \left(\left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1\right)(k-1) + 3\right)\right| + 1 > \left\lfloor \frac{\xi}{2} \right\rfloor + 2$, since $s \neq k$.

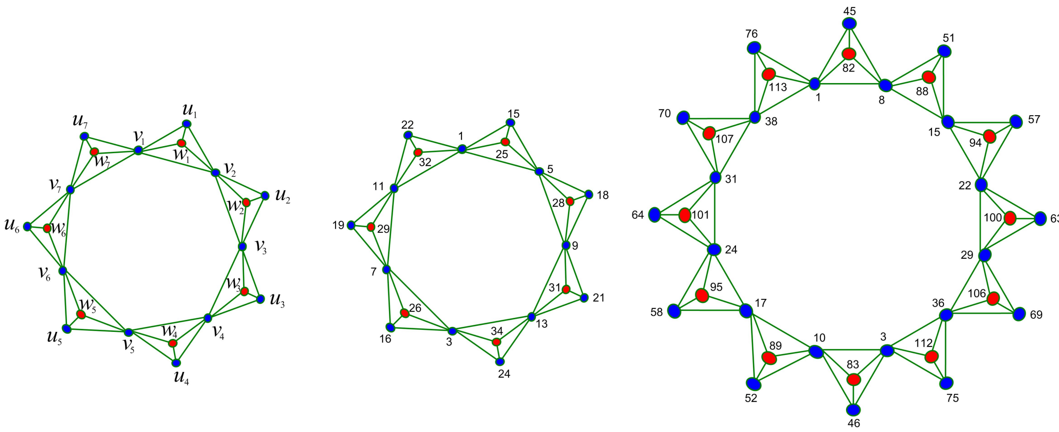


Figure 3. A radio labeling of cyclic silicates $SL_c(\xi)$)for ξ =7 and 12 which attains the bound as in Theorem 3.13 and Theorem 3.14.

Case 2. Let $x, y \in V_3$.

Case 2.1. Guess x and y takes the form v_s and v_k, $1 \leq s \neq k \leq \left\lceil \frac{\xi}{2} \right\rceil$, then $\varphi(v_s) = \left\lfloor \frac{\xi}{2} \right\rfloor \left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{\xi}{2} \right\rfloor (s-1) + 3$, $\varphi(v_k) = \left\lfloor \frac{\xi}{2} \right\rfloor \left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{\xi}{2} \right\rfloor (k-1) + 3$ and $d(v_s, v_k) \geq 2$. Again, since $s \neq k$, we get
$|\varphi(x) - \varphi(y)| + d(x,y) \geq \left| \left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)(s-1) - \left(\left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)(k-1) \right) \right| + 2 \geq \left\lfloor \frac{\xi}{2} \right\rfloor + 2$.

Case 2.2. suppose we take $x = v_{\left\lceil \frac{\xi}{2} \right\rceil + s}$ and $y = v_{\left\lceil \frac{\xi}{2} \right\rceil + k}$, $1 \leq s \neq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor$, then the modulus difference of $\varphi\left(v_{\left\lceil \frac{\xi}{2} \right\rceil + s} \right)$ and $\varphi\left(v_{\left\lceil \frac{\xi}{2} \right\rceil + k} \right) v_{\left\lceil \frac{\xi}{2} \right\rceil + k}$ is at least $\left\lfloor \frac{\xi}{2} \right\rfloor$. Further, the distance between $v_{\left\lceil \frac{\xi}{2} \right\rceil + s}$ and $v_{\left\lceil \frac{\xi}{2} \right\rceil + k}$ is at least two. Hence, the radio labeling condition becomes, $|\varphi(x) - \varphi(y)| + d(x,y) \geq \left\lfloor \frac{\xi}{2} \right\rfloor + 2$.

Case 2.3. Assume that $x = v_s$ and $y = v_{\left\lceil \frac{\xi}{2} \right\rceil + k}$, $1 \leq s \leq \left\lceil \frac{\xi}{2} \right\rceil$, $1 \leq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor$. Then, $d(x,y) \geq 1$ and $|\varphi(x) - \varphi(y)| = \left| \left\lfloor \frac{\xi}{2} \right\rfloor \left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{\xi}{2} \right\rfloor (s-1) + 3 - \left(\left\lfloor \frac{\xi}{2} \right\rfloor \left(\left\lfloor \frac{\xi}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{\xi}{2} \right\rfloor (k-1) + 4 \right) \right|$
If $s = k$, then $d\left(v_s, v_{\left\lceil \frac{\xi}{2} \right\rceil + s} \right) = \left\lfloor \frac{\xi}{2} \right\rfloor + 1$, else $\left| \varphi(v_s) - \varphi\left(v_{\left\lceil \frac{\xi}{2} \right\rceil + s} \right) \right| \geq \left\lfloor \frac{\xi}{2} \right\rfloor$ and $d\left(v_s, v_{\left\lceil \frac{\xi}{2} \right\rceil + s} \right) = 2$. Hence in both the chances, $|\varphi(x) - \varphi(y)| + d(x,y) \geq \left\lfloor \frac{\xi}{2} \right\rfloor + 2$.

Case 3. Choose x and y in V_1.

Case 3.1. If $x = w_s$ and $y = w_k$, $1 \leq s \neq k \leq \left\lceil \frac{\xi}{2} \right\rceil$, then $\varphi(w_s) = 2\left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor s + 4$, $\varphi(w_k) = 2\left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor k + 4$. Also, the distance between w_s and w_k is at least 2. So, the radio labeling condition becomes
, $|\varphi(x) - \varphi(y)| + d(x,y) \geq \left| 2\left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor s + 4 - \left(2\left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor k + 4 \right) \right| + 2 \geq \left\lfloor \frac{\xi}{2} \right\rfloor + 2$, since $s \neq k$.

Case 3.2. If we choose x as $w_{\left\lceil \frac{\xi}{2} \right\rceil + s}$ and y as $w_{\left\lceil \frac{\xi}{2} \right\rceil + k}$, $1 \leq s \neq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor$, then $|\varphi(x) - \varphi(y)| \geq \left| 2\left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor s + 5 - \left(2\left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor t + 5 \right) \right| \geq \left\lfloor \frac{\xi}{2} \right\rfloor$, since $s \neq k$. Again, in this case, the minimum distance between $w_{\left\lceil \frac{\xi}{2} \right\rceil + s}$ and $w_{\left\lceil \frac{\xi}{2} \right\rceil + k}$ is 2. Hence, the radio labeling condition is verified.

Case 3.3. Let $x = w_s$ and $y = w_{\left\lceil \frac{\xi}{2} \right\rceil + k}$, $1 \leq s \leq \left\lceil \frac{\xi}{2} \right\rceil$, $1 \leq k \leq \left\lfloor \frac{\xi}{2} \right\rfloor$. Then, $d\left(w_s, w_{\left\lceil \frac{\xi}{2} \right\rceil + k} \right) \geq 1$ and $\left| \varphi(w_s) - \varphi(w_{\left\lceil \frac{\xi}{2} \right\rceil + k}) \right| = \left| 2\left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor s + 4 - \left(2\left(\left\lfloor \frac{\xi}{2} \right\rfloor \right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor t + 5 \right) \right| = \left| \left\lfloor \frac{\xi}{2} \right\rfloor (s-k) - 1 \right|$.
If $s = k$, then $d(x,y) = \left\lfloor \frac{\xi}{2} \right\rfloor + 1$, otherwise, $d\left(w_s, w_{\left\lceil \frac{\xi}{2} \right\rceil + s} \right) = 2$ and $\left| \varphi(w_s) - \varphi\left(w_{\left\lceil \frac{\xi}{2} \right\rceil + s} \right) \right| \geq$

$\left\lfloor \frac{\xi}{2} \right\rfloor$. Hence in both the possibilities, the radio labeling condition is verified.
Case 4. Suppose ξ is odd and $x \in V_2$, $y \in V_3$. Then, the minimum and maximum values of $\varphi(x)$ are 1 and $\left(\left\lfloor \frac{\xi}{2} \right\rfloor +1\right)\left(\left\lceil \frac{\xi}{2} \right\rceil -1\right)+1$ respectively. Also, from the mapping φ, the least value of $\varphi(y)$ is $\left(\left\lfloor \frac{\xi}{2} \right\rfloor +1\right)\left(\left\lceil \frac{\xi}{2} \right\rceil -1\right)+3$. Further, in this case, $d(x,y) = \left\lfloor \frac{\xi}{2} \right\rfloor$ and hence we get, $|\varphi(x) - \varphi(y)| + d(x,y) \geq 2 + \left\lfloor \frac{\xi}{2} \right\rfloor$.
Case 5. Choose $x \in V_2$, $y \in V_3$ and ξ as even. Then, the maximum value of $\varphi(x)$ which is assigned to the vertex u_ξ is $\left(\left\lfloor \frac{\xi}{2} \right\rfloor +1\right)\left(\left\lceil \frac{\xi}{2} \right\rceil -1\right)+1$ and the least value of $\varphi(y)$ is $\left\lfloor \frac{\xi}{2} \right\rfloor\left(\left\lfloor \frac{\xi}{2} \right\rfloor +1\right)+3$. Again, in this case, since $d(x,y) \geq 2$, hence we get,$|\varphi(x) - \varphi(y)| + d(x,y) \geq$
$\left|\left(\left\lfloor \frac{\xi}{2} \right\rfloor +1\right)\left(\left\lceil \frac{\xi}{2} \right\rceil -1\right)+1 - \left(\left\lfloor \frac{\xi}{2} \right\rfloor\left(\left\lfloor \frac{\xi}{2} \right\rfloor +1\right)+3\right)\right| \geq 2 + \left\lceil \frac{\xi}{2} \right\rceil = 2 + \left\lfloor \frac{\xi}{2} \right\rfloor$.
Case 6. If $x \in V_2$ and $y \in V_1$, then, from the assigned labeling pattern, the difference between the maximum $\varphi(x)$ and minimum $\varphi(y)$ values is greater than $\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)^2$. Hence, the radio labeling condition is trivially satisfied.
Case 7. Let $x \in V_3$, $y \in V_1$
Case 7.1. If ξ is even, then, the maximum value of $\varphi(x)$ is labelled to the vertex v_ξ as $\left\lfloor \frac{\xi}{2} \right\rfloor\left(\left\lfloor \frac{\xi}{2} \right\rfloor +1\right) + \left\lfloor \frac{\xi}{2} \right\rfloor\left(\left\lfloor \frac{\xi}{2} \right\rfloor -1\right)+4$ and the minimum value of $\varphi(y)$ is labelled to the vertex w_1 as$2\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor +4$. Hence, the difference between $\varphi(w_1)$and $\varphi(v_\xi)$ is $\left\lfloor \frac{\xi}{2} \right\rfloor$. The condition is now verified in this case because, the distance between them is 2.
Case 7.2. If ξ is odd, then, the maximum value of $\varphi(x)$ labelled to the vertex $v_{\lceil \frac{\xi}{2} \rceil}$ is $\left\lfloor \frac{\xi}{2} \right\rfloor\left(\left\lfloor \frac{\xi}{2} \right\rfloor +1\right) + \left\lfloor \frac{\xi}{2} \right\rfloor\left(\left\lceil \frac{\xi}{2} \right\rceil -1\right)+3$ and the minimum value of $\varphi(y)$ labelled to the vertex w_1 is
$2\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor +4$. So, the difference between $\varphi(w_1)$and $\varphi(v_\xi)$ is 1 and $d(x,y) = \left\lfloor \frac{\xi}{2} \right\rfloor + 1$. Thus, $|\varphi(x) - \varphi(y)| + d(x,y) \geq 2 + \left\lfloor \frac{\xi}{2} \right\rfloor$.

Hence, the mapping φ is a valid radio labeling. Therefore, if ξ is even, then the vertex w_ξ received the maximum number $2\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor \left\lfloor \frac{\xi}{2} \right\rfloor +5 = \frac{3}{4}\xi^2 + 5$ and if ξ is odd, then the vertex $w_{\lceil \frac{\xi}{2} \rceil}$ received the maximum number $2\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)^2 + \left\lfloor \frac{\xi}{2} \right\rfloor \left\lceil \frac{\xi}{2} \right\rceil +4$, which is the required radio number of $SL_c(\xi)$.

Theorem 3.14. Let $OX_c(\xi)$ be the cyclic oxide having diameter $\left\lfloor \frac{\xi}{2} \right\rfloor + 1$, then for the same labeling pattern as in Theorem 3.1, the radio number of $OX_c(\xi)$ satisfies,

$$rn(OX_c(\xi)) \leq \begin{cases} \left\lfloor \frac{\xi}{2} \right\rfloor\left(\left\lfloor \frac{\xi}{2} \right\rfloor +1\right) + \left\lfloor \frac{\xi}{2} \right\rfloor\left(\left\lceil \frac{\xi}{2} \right\rceil -1\right)+3, & \xi \text{ is odd} \\ 2\left(\left\lfloor \frac{\xi}{2} \right\rfloor\right)^2+4, & \xi \text{ is even} \end{cases}, \text{ where } \xi > 5.$$

Proof. As the proof is similar to Theorem 3.13, we omit the proof.

The bounds obtained in this research work provides the maximum usage of bandwidth for the communication networks designed in the form of chain or cyclic silicates. Further, these results help to study the properties of protein-protein interaction networks.

4 Conclusion remarks

This research work pertaining to the chemical graphs has explored the ways to study radio frequency difference between the atoms of the molecules (vertices) by the interesting concept in telecommunication called radio labeling. As, cyclic, single chain oxide and silicate structures are obtained by continuously fusing oxygen ions of two tetrahedra of different silicates, the upper bounds for the radio number for such chemical structures have been successfully presented. Further, the lower bounds for the single chain chemical structures have been proven by calculating the eccentricities of the vertices. This problem is still open to other forms of chemical graphs such as double-chain silicates, silicate sheets, silicate networks oxide networks and copper oxide networks.

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