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A New Approach of Fixed Point in Parametric b-Metric Space

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Abstract The fixed point theorems can be obtained using appropriate contraction conditions which are strong enough to generate fixed points. This paper aims to establish some new fixed point results using Rus and Hardy-Roger's contractions in parametric b-Metric space. An example is also furnished to demonstrate the utility of our newly proven results.

1 Introduction

Fixed point theory has been a key of focus after the introduction of Banach Contraction Principle. In 1905, metric space was introduced by Frechet [1]. Since then, that research in Mathematics and applied sciences rely heavily on metric spaces. Czerwik [2] proposed the theory of b-metric spaces. Fixed point theory for different contractive mappings has been studied by numerous mathematicians. Alghamdi. et al. [3] contributed by concluding some coupled fixed point theorems on b-Metric-like space. Metric space has been explored by researchers time to time and as a result we can find number of variants of metric space in the literature [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In 2014, Hussain et al. [17] introduced parametric metric space. Again in 2015, Hussain et al. [18] Parametric b-Metric Space as an extension of metric space. Our objective is to show fixed point theorems for expansive mappings in parametric b-metric space where we used Rus [19] and Hardy-Roger's [20] contractions to show the existence of fixed point theorems.

Definition 1.1. [18] Assume X be a non-empty set, $\sigma \geq 1$ be a real number and $d : X \times X \times (0, \infty) \to (0, \infty)$ be a function which is Parametric b-Metric on X if

(a) $d(x, y, t) = 0 \; \forall \; t > 0 \; y \iff x = y$

(b) $d(x, y, t) = d(y, x, t) \; \forall \; t > 0$

(c) $d(x, y, t) \leq \sigma[d(x, z, t) + d(z, y, t)] \; \forall \; x, y, z \in X, \; t > 0, \sigma \geq 1.$

The pair (X, d) is a parametric b-metric space with a parameter $\sigma \geq 1$.

Remark 1.2. Obviously, for $\sigma = 1$ parametric b-metric space reduces to parametric metric space [17].

Definition 1.3. Let (X, d) be a parametric b-metric space with a parameter $\sigma \geq 1$. A sequence $\{x_n\}$ is said to be (for all $t > 0$)

(1) convergent to $x \in X$ if $\lim_{n\to\infty} d(x_n, x, t) = 0$.

(2) Cauchy if $\lim_{n,m\to\infty} d(x_n, x_m, t) = 0$.

Definition 1.4. A parametric b-metric space is said to be complete if every Cauchy sequence is a convergent in the space.

2 Main Results

Theorem 2.1. *Consider a parametric b-metric space (X, d) where g and T are continuous self-mappings. Consider T to be a one to one mapping. If the mapping T, g fulfill the below condition:*

$$\begin{aligned} d(Tgx, Tgy, t) \leq \alpha d(Tx, Ty, t) + \beta\{d(Tx, Tgx, t) + d(Ty, Tgy, t)\} \\ + \Upsilon \{d(Tx, Tgy, t) + d(Ty, Tgx, t)\} \end{aligned} \tag{2.1}$$

$\forall x, y \in X$ and $\alpha, \beta, \Upsilon \geq 0$ with $\alpha+2\beta+2\Upsilon < \frac{1}{\sigma}$ then g has a unique fixed point in X. Moreover, if (T, g) is a Banach pair, then g and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary element in X. Then, $\{x_n\}$ is a sequence for which $x_{n+1} = gx_n$ for $n = 0, 1, 2, 3, \cdots$. Now, from (2.1), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}, t) &= d(Tgx_{n-1}, Tgx_n, t) \\ &\leq \alpha d(Tx_{n-1}, Tx_n, t) + \beta\{d(Tx_{n-1}, Tgx_{n-1}, t) \\ &\quad + d(Tx_n, Tgx_n, t)\} + \Upsilon\{d(Tx_{n-1}, Tgx_n, t) + d(Tx_n, Tgx_{n-1}, t)\} \\ &= \alpha d(Tx_{n-1}, Tx_n, t) + \beta\{d(Tx_{n-1}, Tx_n, t) \\ &\quad + d(Tx_n, Tx_{n+1}, t)\} + \Upsilon\{d(Tx_{n-1}, Tx_{n+1}, t) + d(Tx_n, Tx_n, t)\} \\ &\leq \alpha d(Tx_{n-1}, Tx_n, t) + \beta\{d(Tx_{n-1}, Tx_n, t) \\ &\quad + d(Tx_n, Tx_{n+1}, t)\} + \sigma \Upsilon \{d(Tx_{n-1}, Tx_n, t) + d(Tx_n, Tx_{n+1}, t)\} \\ &= \alpha d(Tx_{n-1}, Tx_n, t) + \beta d(Tx_{n-1}, Tx_n, t) \\ &\quad + \beta d(Tx_n, Tx_{n+1}, t)\} + \sigma \Upsilon d(Tx_{n-1}, Tx_n, t) + \sigma \Upsilon d(Tx_n, Tx_{n+1}, t)\} \end{aligned}$$

which implies

$$\begin{aligned} (1 - \beta - \sigma\Upsilon)d(Tx_n, Tx_{n+1}, t) \leq (\alpha + \beta + \sigma\Upsilon)d(Tx_{n-1}, Tx_n, t) \\ d(Tx_n, Tx_{n+1}, t) \leq \frac{(\alpha + \beta + \sigma\Upsilon)}{(1 - \beta - \sigma\Upsilon)} d(Tx_{n-1}, Tx_n, t). \end{aligned}$$

Since $\alpha, \beta, \Upsilon > 0$, we let

$$\lambda = \frac{(\alpha + \beta + \sigma\Upsilon)}{(1 - \beta - \sigma\Upsilon)}.$$

Now,

$$\begin{aligned} d(Tx_n, Tx_{n+1}, t) &\leq \lambda d(Tx_{n-1}, Tx_n, t) \\ &\leq \lambda^n d(Tx_0, Tx_1, t) \;\rightarrow\; 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Now, we claim that $\{Tx_n\}$ is a Cauchy sequence in X. Suppose $m, n \in N$ such that $m > n$. Employing the Definition 1.1, we have

$$\begin{aligned} d(Tx_n, Tx_m, t) &\leq \sigma\{d(Tx_n, Tx_{n+1}, t) + d(Tx_{n+1}, Tx_m, t)\} \\ &\leq \sigma d(Tx_n, Tx_{n+1}, t) + \sigma^2 d(Tx_{n+1}, Tx_{n+2}, t) \\ &\quad + d(Tx_{n+2}, Tx_m, t) \\ &\leq \sigma d(Tx_n, Tx_{n+1}, t) + \sigma^2 d(Tx_{n+1}, Tx_{n+2}, t) \\ &\quad + \cdots + \sigma^{m-n-1} d(Tx_{m-1}, Tx_m, t) \\ &\leq \sigma\lambda^n d(Tx_n, Tx_{n+1}, t) + \sigma^2\lambda^{n+1} d(Tx_{n+1}, Tx_{n+2}, t) \\ &\quad + \cdots + \sigma^{m-n-1}\lambda^{m-1} d(Tx_{m-1}, Tx_m, t) \\ &= (1 + \sigma\lambda + (\sigma\lambda)^2 + \cdots)\sigma\alpha^n d(Tx_0, Tx_1, t). \end{aligned}$$

Since $\sigma\lambda < 1$, it follows that

$$\begin{aligned} d(Tx_n, Tx_m, t) &\leq \frac{\sigma\lambda^n}{1-\sigma\lambda} d(Tx_0, Tx_1, t) \\ \implies \|d(x_n, x_m, t)\| &\leq \frac{\sigma\lambda^n}{1-\sigma\lambda} \|d(x_0, x_1, t)\|. \end{aligned}$$

Since $0 < \sigma\lambda < 1$, then $\lambda^n \to 0$ as $n \to \infty$. So, $\|d(x_n, x_m, t)\| \to 0$.

Thus, the sequence $\{Tx_n\}$ is a Cauchy sequence in X. Since, X is complete parametric b-metric space, then there exist $z \in X$ such that $\lim_{n\to\infty} Tx_n = z$. Since T is subsequentially convergent, x_n has a convergent subsequence x_m such that $\lim_{m\to\infty} x_m = u$ (say). Since T is continuous, $\lim_{m\to\infty} Tx_m = Tu$, we have $z = Tu$. Since g is continuous, $\lim_{m\to\infty} gx_m = gu$. Again, as T is continuous, $\lim_{m\to\infty} Tgx_m = Tgu$. Therefore, $\lim_{m\to\infty} Tx_{m+1} = Tgu$. By using the Definition 1.1, we get

$$\begin{aligned} d(Tgu, Tu, t) &\leq \sigma\{d(Tgu, Tx_m, t) + d(Tx_m, Tu, t)\} \\ &= \sigma d(Tgu, Tgx_{m-1}, t) + \sigma d(Tx_m, Tu, t) \\ &\leq \sigma\{\alpha d(Tu, Tx_{m-1}, t) + \beta\{d(Tu, Tgu, t) \\ &\quad + d(Tx_{m-1}, Tgx_{m-1}, t)\} + \Upsilon\{d(Tu, Tgx_{m-1}, t) \\ &\quad + d(Tx_{m-1}, Tgu, t)\} + \sigma d(Tx_m, Tu, t) \\ &\leq \sigma\{\alpha d(Tu, Tx_{m-1}, t) + \beta\{d(Tu, Tgu, t) \\ &\quad + d(Tx_{m-1}, Tx_m, t)\} + \Upsilon\{d(Tu, Tx_m, t) \\ &\quad + d(Tx_{m-1}, Tgu, t)\}\} + \sigma d(Tx_m, Tu, t) \\ &\leq \sigma^2\alpha d(Tu, Tx_m, t) + \sigma^2\alpha d(Tx_m, Tx_{m-1}, t) \\ &\quad + \sigma\beta d(Tu, Tgu, t) + \sigma\beta d(Tx_{m-1}, Tx_m, t) \\ &\quad + (\sigma\Upsilon + \sigma) d(Tu, Tx_m, t) + \sigma^2\Upsilon d(Tx_{m-1}, Tu, t) \\ &\quad + \sigma^2\Upsilon d(Tu, Tgu, t) \end{aligned}$$

$$\begin{aligned} (1 - \sigma\beta - \sigma^2\Upsilon) d(Tgu, Tu, t) &\leq (\sigma^2\alpha + \sigma\Upsilon + \sigma) d(Tu, Tx_m, t) \\ &\quad + (\sigma^2\alpha + \sigma\beta) d(Tx_m, Tx_{m-1}, t) \\ &\quad + \sigma^2\Upsilon d(Tx_{m-1}, Tu, t) \end{aligned}$$

$$\begin{aligned} d(Tgu, Tu, t) &\leq \frac{(1 - \sigma\beta - \sigma^2\Upsilon)}{(\sigma^2\alpha + \sigma\Upsilon + \sigma)} d(Tu, Tx_m, t) \\ &\quad + \frac{(1 - \sigma\beta - \sigma^2\Upsilon)}{(\sigma^2\alpha + \sigma\beta)} d(Tx_m, Tx_{m-1}, t) \\ &\quad + \frac{(1 - \sigma\beta - \sigma^2\Upsilon)}{\sigma^2\Upsilon} d(Tx_{m-1}, Tu, t). \end{aligned}$$

Since $\lim_{m\to\infty} Tx_m = Tu$ then, we obtain

$$d(Tu, Tgu, t) = 0.$$

Hence, $Tu = Tgu$. But T is a one to one mapping so $u = gu$. The fixed point of g is u. If w is another fixed point of g, we can prove the uniqueness of w. We can then say that $w = gw$.

Consider

$$\begin{aligned} d(Tu,Tw,t) &= d(Tgu,Tgw,t) \\ &\leq \alpha d(Tu,Tw,t)+\beta\{d(Tu,Tgu,t)+d(Tw,Tgw,t)\} \\ &\quad + \Upsilon\,\{d(Tu,Tgw,t)+d(Tw,Tgu,t)\} \\ &= \alpha d(Tu,Tw,t)+\beta\{d(Tu,Tu,t)+d(Tw,Tw,t)\} \\ &\quad + \Upsilon\,\{d(Tu,Tw,t)+d(Tw,Tu,t)\} \\ &\leq (\alpha+2\beta+2\Upsilon)d(Tu,Tw,t). \end{aligned}$$

Since, $(\alpha+2\beta+2\Upsilon)<1$, which implies a contradiction. Hence $u=w$ is a unique fixed point of g. Since (T,g) is a Banach pair, then $Tgu=gTu$. Hence $Tu=gTu$. So, Tu is another fixed point of g. Hence $u=gu=Tu$ is a unique common fixed point of g and T in X. □

Now, we state and proof and analogue of Hardy-Rogers type fixed point theorem as given under:

Theorem 2.2. *Consider a parametric b-metric space* (X,d) *with* $\sigma\geq 1$ *and* $\alpha\sigma^2<1$ *and the pair of continuous self-mappings* g *and* T *satisfies the following condition*

$$\begin{aligned} d(Tgx,Tgy,t) \leq \alpha_1 d(Tx,Ty,t)+\alpha_2 d(Tx,Tgx,t)+\alpha_3 d(Ty,Tgy,t) \\ +\alpha_4 d(Tx,Tgy,t)+\alpha_5 d(Ty,Tgx,t) \end{aligned} \tag{2.2}$$

$\forall x,y\in X$ *and* $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\geq 0$ *with* $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5<\alpha$*, where* $\alpha[0,1)$. *Consider* T *to be one to one mapping. Then* g *has a unique fixed point in* X. *Moreover, if* (T,g) *is a Banach pair, then* g *and* T *have a unique common fixed point in* X.

Proof. Let x_0 be an arbitrary element in X. Then, $\{x_n\}$ is a sequence for which $x_{n+1}=gx_n$ for $n=0,1,2,3,\cdots$. Now, from (2.2), we have

$$\begin{aligned} d(Tx_n,Tx_{n+1},t) &= d(Tgx_{n-1},Tgx_n,t) \\ &\leq \alpha_1 d(Tx_{n-1},Tx_n,t)+\alpha_2 d(Tx_{n-1},Tgx_{n-1},t)+ \\ &\quad \alpha_3 d(Tx_n,Tgx_n,t)+\alpha_4 d(Tx_{n-1},Tgx_n,t)+ \\ &\quad \alpha_5 d(Tx_n,Tgx_{n-1},t) \\ &= \alpha_1 d(Tx_{n-1},Tx_n,t)+\alpha_2 d(Tx_{n-1},Tx_n,t)+ \\ &\quad \alpha_3 d(Tx_n,Tx_{n+1},t)+\alpha_4 d(Tx_{n-1},Tx_{n+1},t)+ \\ &\quad \alpha_5 d(Tx_n,Tx_n,t) \\ &\leq \alpha_1 d(Tx_{n-1},Tx_n,t)+\alpha_2 d(Tx_{n-1},Tx_n,t)+ \\ &\quad \alpha_3 d(Tx_n,Tx_{n+1},t)+\sigma\alpha_4[d(Tx_{n-1},Tx_n,t)+ \\ &\quad d(Tx_n,Tx_{n+1},t)] \\ &\leq \alpha_1 d(Tx_{n-1},Tx_n,t)+\alpha_2 d(Tx_{n-1},Tx_n,t)+ \\ &\quad \alpha_3 d(Tx_n,Tx_{n+1},t)+\sigma\alpha_4 d(Tx_{n-1},Tx_n,t)+ \\ &\quad \sigma\alpha_4 d(Tx_n,Tx_{n+1},t) \end{aligned}$$

which implies that

$$\begin{aligned} (1-\alpha_3-\sigma\alpha_4)d(Tx_n,Tx_{n+1},t) &\leq (\alpha_1+\alpha_2+\alpha_4)\,d(Tx_{n-1},Tx_n,t) \\ d(Tx_n,Tx_{n+1},t) &\leq \frac{\alpha_1+\alpha_2+\sigma\alpha_4}{1-\alpha_3-\sigma\alpha_4}d(Tx_{n-1},Tx_n,t). \end{aligned}$$

Since $\sigma\alpha<1$, we let

$$\alpha=\frac{\alpha_1+\alpha_2+\sigma\alpha_4}{(1-\alpha_3-\sigma\alpha_4)}$$

Now,

$$d(Tx_n, Tx_{n+1}, t) \leq \alpha d(Tx_{n-1}, Tx_n, t) \leq \alpha^n d(Tx_0, Tx_1, t).$$

Now, to prove $\{Tx_n\}$ is a Cauchy sequence in X. Suppose $m, n \in N$ such that $m > n$. Employing the Definition 1.1, we have

$$\begin{aligned}
d(Tx_n, Tx_m, t) &\leq \sigma[d(Tx_n, Tx_{n+1}, t) + d(Tx_{n+1}, Tx_m, t)] \\
&\leq \sigma d(Tx_n, Tx_{n+1}, t) + \sigma^2 d(Tx_{n+1}, Tx_{n+2}, t) \\
&\quad + \sigma^2 d(Tx_{n+2}, Tx_m, t) \\
&\leq \sigma d(Tx_n, Tx_{n+1}, t) + \sigma^2 d(Tx_{n+1}, Tx_{n+2}, t) \\
&\quad + \cdots + \sigma^{m-n-1} d(Tx_{m-1}, Tx_m, t) \\
&\leq \sigma\alpha^n d(Tx_n, Tx_{n+1}, t) + \sigma^2\alpha^{n+1} d(Tx_{n+1}, Tx_{n+2}, t) \\
&\quad + \cdots + \sigma^{m-n-1}\alpha^{m-1} d(Tx_{m-1}, Tx_m, t) \\
&= (1 + \sigma\alpha + (\sigma\alpha)^2 + \cdots)\sigma\alpha^n d(Tx_0, Tx_1, t).
\end{aligned}$$

Since $\sigma\alpha < 1$, it follows that

$$\begin{aligned}
d(Tx_n, Tx_m, t) &\leq \frac{\sigma\alpha^n}{1 - \sigma\alpha} d(Tx_0, Tx_1, t) \\
\implies \|d(x_n, x_m, t)\| &\leq \frac{\sigma\alpha^n}{1 - \sigma\alpha} \|d(x_0, x_1, t)\|.
\end{aligned}$$

Since $0 < \alpha < 1$, then $\alpha^n \to 0$ as $n \to \infty$. So, $\|d(x_n, x_m, t)\| \to 0$.

The sequence $\{Tx_n\}$ is a Cauchy sequence in X. Since, X is complete parametric b-metric space, then there exist $z \in X$ such that $\lim_{n\to\infty} Tx_n = z$. Since T is subsequentially convergent, x_n has a convergent subsequence x_m such that $\lim_{m\to\infty} x_m = u$ (say). Since T is continuous, $\lim_{m\to\infty} Tx_m = Tu$, we have $z = Tu$. Since g is continuous, $\lim_{m\to\infty} gx_m = gu$. Again, as T is continuous, $\lim_{m\to\infty} Tgx_m = Tgu$. Therefore, $\lim_{m\to\infty} Tx_{m+1} = Tgu$. By using the Definition 1.1, we get

$$\begin{aligned}
d(Tgu, Tu, t) &\leq \sigma\{d(Tgu, Tx_m, t) + d(Tx_m, Tu, t)\} \\
&= \sigma d(Tgu, Tgx_{m-1}, t) + \sigma d(Tx_m, Tu, t) \\
&\leq \sigma\{\alpha_1 d(Tu, Tx_{m-1}, t) + \alpha_2 d(Tu, Tgu, t) \\
&\quad + \alpha_3 d(Tx_{m-1}, Tgx_{m-1}, t) + \alpha_4 d(Tu, Tgx_{m-1}, t) \\
&\quad + \alpha_5 d(Tx_{m-1}, Tgu, t)\} + \sigma d(Tx_m, Tu, t) \\
&= \sigma\{\alpha_1 d(Tu, Tx_{m-1}, t) + \alpha_2 d(Tu, Tgu, t) \\
&\quad + \alpha_3 d(Tx_{m-1}, Tx_m, t) + \alpha_4 d(Tu, Tx_m, t) \\
&\quad + \alpha_5 d(Tx_{m-1}, Tgu, t)\} + \sigma d(Tx_m, Tu, t) \\
&\leq \sigma^2\alpha_1 d(Tu, Tx_m, t) + \sigma^2\alpha_1 d(Tx_m, Tx_{m-1}, t) \\
&\quad + \sigma\alpha_2 d(Tu, Tgu, t) + \sigma\alpha_3 d(Tx_{m-1}, Tx_m, t) \\
&\quad + (\sigma + \sigma\alpha_4) d(Tu, Tx_m, t) + \sigma^2\alpha_5 d(Tx_{m-1}, Tu, t) \\
&\quad + \sigma^2\alpha_5 d(Tu, Tgu, t)
\end{aligned}$$

$$\begin{aligned}
(1 - \sigma\alpha_2 - \sigma^2\alpha_5) d(Tgu, Tu, t) &\leq (\sigma^2\alpha_1 + \sigma + \sigma\alpha_4 + \sigma^2\alpha_5) d(Tu, Tx_m, t) \\
&\quad + (\sigma^2\alpha_1 + \sigma\alpha_3 + \sigma^2\alpha_5) d(Tx_{m-1}, Tx_m, t) \\
&\quad + \sigma^2\alpha_5 d(Tx_{m-1}, Tu, t)
\end{aligned}$$

Therefore,

$$\begin{aligned} d(Tgu,Tu,t) &\leq \frac{(\sigma^2\alpha_1+\sigma+\sigma\alpha_4+\sigma^2\alpha_5)}{(1-\sigma\alpha_2-\sigma^2\alpha_5)}d(Tu,Tx_m,t) \\ &+\frac{(\sigma^2\alpha_1+\sigma\alpha_3+\sigma^2\alpha_5)}{(1-\sigma\alpha_2-\sigma^2\alpha_5)}d(Tx_m,Tx_{m-1},t) \\ &+\frac{\sigma^2\alpha_5}{(1-\sigma\alpha_2-\sigma^2\alpha_5)}d(Tx_{m-1},Tu,t). \end{aligned}$$

Since $\lim_{m\to\infty} Tx_m = Tu$. We get

$$d(Tu,Tgu,t)=0.$$

Hence, $Tu = Tgu$. But T is a one to one mapping so $u = gu$. The fixed point of g is u. If w is another fixed point of g, we can prove the uniqueness of w. We can then say that $w = gw$. Consider

$$\begin{aligned} d(Tu,Tw,t) &= d(Tgu,Tgw,t) \\ &\leq \alpha_1 d(Tu,Tw,t)+\alpha_2 d(Tu,Tgu,t)+\alpha_3 d(Tw,Tgw,t) \\ &\quad +\alpha_4 d(Tu,Tgw,t)+\alpha_5 d(Tw,Tgu,t) \\ &= \alpha_1 d(Tu,Tw,t)+\alpha_2 d(Tu,Tu,t)+\alpha_3 d(Tw,Tw,t) \\ &\quad +\alpha_4 d(Tu,Tw,t)+\alpha_5 d(Tw,Tu,t) \\ &\leq (\alpha_1+\alpha_4+\alpha_5)d(Tu,Tw,t). \end{aligned}$$

Since $\alpha_1+\alpha_4+\alpha_5<1$, which implies a contradiction. Hence $u = w$ is a unique fixed point of g. Since (T,g) is a Banach pair, then $Tgu = gTu$. Hence $Tu = gTu$. So, Tu is another fixed point of g. Hence $u = gu = Tu$ is a unique common fixed point of g and T in X. □

Example 2.3. Consider $X = [0,1],\ E = \mathbb{R}^2,\ p \in [1,\infty)$. Let

$$P = \{(x,y,t)\in E : x,y \geq 0\}.$$

Define $d : X \times X \to E$ by

$$d(x,y,t) = t(|\, x-y\,|^p, |\, x-y\,|^p),\ \ \forall x,y\in X,\ t>0.$$

Thus (X,d,t) is a complete parametric b-metric space. Assume $T,g : X \to X$ as $Tx = \frac{1}{4}x^2$ and $gx = \frac{1}{2}x^2$

$$\begin{aligned} d(Tgx,Tgy,t) &= t\left(\left|\frac{1}{16}x^2-\frac{1}{16}y^2\right|^p, \left|\frac{1}{16}x^2-\frac{1}{16}y^2\right|^p\right) \\ &= \left(\frac{1}{16}\right)^p t\left(\left|x^2-y^2\right|^p, \left|x^2-y^2\right|^p\right) \\ &\leq \left(\frac{1}{3}\right)^p\left(\frac{1}{4}\right)^p t\left(\left|x^2-y^2\right|^p, \left|x^2-y^2\right|^p\right). \end{aligned}$$

Again consider

$$\begin{aligned} d(Tx,Ty,t) &= t\left(\left|\frac{1}{4}x^2-\frac{1}{4}y^2\right|^p, \left|\frac{1}{4}x^2-\frac{1}{4}y^2\right|^p\right) \\ &= \left(\frac{1}{4}\right)^p t\left(\left|x^2-y^2\right|^p, \left|x^2-y^2\right|^p\right). \end{aligned}$$

Then,

$$d(Tgx,Tgy,t) \leq \left(\frac{1}{3}\right)^p d(Tx,Ty,t).$$

Let $\left(\frac{1}{3}\right)^p = \alpha_1$. Then we get,

$$\begin{aligned} d(Tgx, Tgy, t) &\leq \alpha_1 d(Tx, Ty, t) \\ &\leq \alpha_1 d(Tx, Ty, t) + \alpha_2 d(Tx, Tgx, t) + \alpha_3 d(Ty, Tgy, t) \\ &\quad + \alpha_4 d(Tx, Tgy, t) + \alpha_5 d(Ty, Tgx, t). \end{aligned}$$

Thus, the mappings T and g satisfy the condition (2.2). Therefore, all the assumptions of Theorem 2.2 is true. Hence 0 is unique common fixed point of X.

3 Conclusion

By introducing additional parameters under contraction conditions, we proved whether some known results in metric space are valid or not for Parametric b-Metric Space in this paper.It is proved that the contractions of Rus and Hardy-Rogger can be utilized to find a fixed point in Parametric b-Metric Space. To satisfy these contraction conditions, a parameter $\sigma \geq 1$ is employed. An example is provided to verify our result.

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