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Nonlinear boundary periodic equation with arbitrary growth nonlinearity and data measure: Mathematical analysis and Numerical simulation by LBM method

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Abstract This paper aims to study the existence and uniqueness of weak periodic solutions for certain non-linear elliptic equations with measured data, and arbitrary growth with respect to the solution. As the data are not regular and the growths are arbitrary, a new approach is needed to analyse these types of equations. The main operator of our equation is strongly non-linear, thus classical numerical methods such as finite elements, and finite differences cannot be used here. Therefore, the lattice Boltzmann method (LBM) is proposed. This method is fully discretized for our equation. Finally, several numerical examples are given showing the robustness of the proposed algorithm.

1 Introduction

PDEs are widely used to simulate extremely complicated physical systems, structural dynamics and fluid mechanics, as well as theories of gravitation, electromagnetism (Maxwell's equations) or finance. They are used extensively in science. In fact, they play a crucial role in areas such as weather forecasting, image synthesis and aeronautical simulation.

Dirichlet and Neumann boundary conditions are two of the many boundary conditions frequently encountered when solving partial differential equations [1]. Although, in certain cases, these conditions are periodic, such as in the modelling of molecular mechanics and molecular dynamics, they are not always the same.

Many techniques can be used to determine whether the existence of periodic solutions for a specific problem is ensured. Notably, the fixed point theorem, the topological degree, the bifurcation theory and the Lyapunov function method are one of the techniques used to study the existence of such solutions. For more details, we refer the readers to see [5], [9], [11].

The aim of this paper is to study the existence of weak solutions for a class of nonlinear elliptic equations subject to periodic boundary conditions. The model equation is given by the following sense

$$\begin{cases} u(t) - (|u'|^{p-2}u')'(t) + j\,(t, u(t)) = f & \text{in } (0,T) \\ u(0) = u(T), \quad u'(0) = u'(T) \end{cases} \tag{1.1}$$

where j is a measurable and continuous function with respect to u and periodic with respect to time t with a period T. The periodicity of j enables its extension to a continuous periodic function on $\mathbb{R}$, as $j(t, r + kT) = j(t, r)$ for all $r \in (0,T)$ and $k \in \mathbb{Z}$. The function f is a nonnegative bounded Radon measure on the interval $]0, T[$.

Before proceeding to the main objective of this work, a few well-known results are recalled, in which the techniques used are relatively close to the one used here.

The semilinear case of equation (1.1) corresponds to the case when $p = 2$, which can be written as:

$$\begin{cases} u(t) - u''(t) + j\,(t, u(t)) = f \quad \text{in } (0, T) \\ u(0) = u(T), \quad u'(0) = u'(T) \end{cases} \tag{1.2}$$

Numerous researches have been already carried out on this specific topic. Thereafter, some important results are presented, which help us to situate the main objective of this work:

- When $p = 2$, one distinguishes between the regular and non-regular case of f:
  - case where $f \in L^2(0, T)$ was treated in [15], where the authors proved the existence of a periodic solution $u \in H^2(0, T)$ via lower and upper solution technique.
  - case where $f \in \mathcal{M}_B^+(0, T)$ and $p = 2$ was considered in [2], where the existence of a periodic solution $u \in W^{1,1}_{per}(0, T)$ was proved.
- Case where $p \neq 2$ and $j \equiv 0$ was treated in [13], in which the model problem was written as

$$\begin{cases} u(t) - (|u'|^{p-2}u')'(t) = f \quad \text{in } (0, T) \\ u(0) = u(T), \quad u'(0) = u'(T) \end{cases} \tag{1.3}$$

  The authors conducted an analysis of the given problem, and proved the existence and uniqueness of a periodic solution when f belongs to $L^{p'}(0, T)$.

The most significant novelty of this paper is the fact that $1 < p < \infty$ and f is irregular. The aim of this work is to study the existence and uniqueness of weak periodic solutions when f is only a radon measure, and the nonlinearity j has an arbitrary growth with respect to the solution u.

Another important aspect we deal with in this work is the numerical simulation of the proposed model. A number of numerical approximation techniques for ODEs and PDEs exist, including finite difference, finite element or finite volume, ect.... However, the proposed problem is unique as it deals only with Radon data measures and involves a strongly non-linear principal operator, which makes conventional numerical methods unsuitable in this case. Therefore, the Lattice Boltzmann Method (LBM) was chosen, which is a relatively new method compared to conventional approaches. The LBM method is derived from the kinetic theory of gases developed by Boltzmann which was first proposed by [12].
The Lattice Boltzmann method is widely applied to fluid dynamics [7, 16], and constitutes of a mesoscopic approach to simulate macroscopic phenomena governed by partial differential equation problems. Among its features are, the simplicity of the computational procedure, the efficiency of the computer code implementation and its high accuracy.

The remainder of this paper is organised as follows. In the second section, the assumptions and hypotheses related to the proposed problem are outlined, and the adapted notion of the weak periodic solution for problem (1.1) is given. Afterwards, the main result on the existence and uniqueness of a weak periodic solution is given. Section 3 is devoted to a result on the existence and uniqueness of weakly periodic solutions when the data f is regular. In section 4, the proof of the main result is presented. Section 5 focuses on the numerical simulation of our general periodic problem, using the Lattice Boltzmann Method (LBM) as a micro-macro solver. In section 6, we have the conclusion and some perspectives.

2 Statement of the theoretical main result

Throughout this paper, we assume the following:

$A_1)$ $f \in \mathcal{M}_B^+(0, T)$ and T-periodic.

$A_2)$ $j : [0, T] \times \mathbb{R} \to [0, +\infty[$ is a measurable function, and $j(., r)$ is T-periodic.

$A_3)$ $\forall t, r \to j(t,r)$ is continuous and non-decreasing.

$A_4)$ $\forall t \in [0,T], \quad max\, \{j(t,r), |r| \le R\} \le c(|R|)$
where $c : [0,+\infty[\to [0,+\infty[$ is non-decreasing.

Considers for $1 \leqslant p \leqslant \infty$ the following space,

$$W^{1,p}_{per}(0,T) = \{u \in W^{1,p}(0,T), \text{ such that } u(0) = u(T)\}$$

which is equipped with the norm that is induced by $W^{1,p}(0,T)$

$$\|u\|_{1,p} = \|u\|_p + \|u'\|_p\,.$$

When $p = 2$, this space is denoted by $H^1_{per}(0,T)$.
Now, we define the concept of periodic Radon measure

Definition 2.1. We denote by $\mathcal{M}^+_B(0,T)$ the set of nonnegative bounded Radon measure on $]0,T[$.
The function $f \in \mathcal{M}^+_B(0,T)$ is said to be T-periodic, if there exists $f_\varepsilon \in C([0,T])^+$ such that, $f_\varepsilon(0) = f_\varepsilon(T)$ and

$$\forall\, \phi \in C([0,T]), < f, \phi >= \lim_{\varepsilon\to 0} \int_0^T f_\varepsilon \phi$$

An example of a 1-periodic Radon measure is $f = \delta_{\frac{1}{2}}$. Since the Lorentzian sequence

$$f_\varepsilon(t) = \frac{1}{\pi\varepsilon}\frac{1}{1+\frac{(t-\frac{1}{2})^2}{\varepsilon^2}} \tag{2.1}$$

is 1-periodic continuous (in the sense that f_ε is defined on $[0,1]$ by (2.1), and its extension outside $]0,1[$ is given by $f_\varepsilon(t+k) = f_\varepsilon(t)$ with $k \in \mathbb{Z}$, and $t \in (0,1)$), one can prove that f_ε is convergent in the sense of measure to $\delta_{\frac{1}{2}}$.

Note that the 1-periodic Lorentzian sequence is used in physics and in engineering, particularly in the study of resonant systems and signal processing. Furthermore, it is widely used in statistics and probability theory as a model for heavy-tailed distributions.

At this stage, we clarify in which sense the notion of a weak periodic solution of problem (1.1) is understood.

Definition 2.2. A function u is said to be a weak periodic solution (1.1), if

$$\begin{cases} u \in W^{1,p}_{per}(0,T) \\ \displaystyle\int_0^T u\phi + \int_0^T (|u'|^{p-2}u')\phi' + \int_0^T j\,(t,u)\,\phi =< f, \phi > \quad \forall \phi \in W^{1,p}_{per}(0,T) \end{cases} \tag{2.2}$$

Remark 2.3. i) For the remainder of this paper, we denote by C every generic and nonnegative constant.

ii) For all $1 \le p \le \infty$, $W^{1,p}_{per}(0,T) \hookrightarrow C([0,T])$ with compact embedding.

iii) $<,>$ denotes the duality product between $\mathcal{M}_B(0,T)$ and $L^\infty(0,T)$.

iv) If $u \in W^{1,p}_{per}(0,T)$, then $u \in L^\infty(0,T)$ and $|u'|^{p-2}u' \in L^{p'}(0,T)$, moreover since j satisfy (A_4) then $j(t,u(t)) \in L^1(0,T)$ consequently all the terms in (2.2) make sense.

Now, we state the following main result

Theorem 2.4. *Assume that $(A_2) - (A_4)$ holds, then for all $f \in \mathcal{M}^+_B(0,T)$, there exists a weak periodic solution u of* (1.1).

3 An auxiliary existence and uniqueness result

In order to develop the mathematical analysis of the proposed model, we first present a result on the existence and uniqueness of weak periodic solutions when the data f are regular. In this section, we consider $f \in L^2(0,T)$, T-periodic and we obtain the result given below

Theorem 3.1. *Let $f \in L^2(0,T)$, T-periodic and j satisfies (A_2)-(A_4). Then there exists a unique weak periodic solution of problem (1.1)*

$$\begin{cases} u \in W^{1,p}_{per}(0,T) \\ \displaystyle\int_0^T u\phi + \int_0^T (|u'|^{p-2}u')\phi' + \int_0^T j(t,u)\phi = \int_0^T f\phi \quad \forall \phi \in W^{1,p}_{per}(0,T) \end{cases} \tag{3.1}$$

In addition if $f \geq 0$, then $u \geq 0$.

Proof. First, we define on $L^2(0,T)$ the following functional

$$J(u) = \begin{cases} \displaystyle\frac{1}{p}\int_0^T |u'|^p + \int_0^T J_p(t,u) & \textbf{if } u \in W^{1,p}_{per}(0,T) \\ +\infty & \textbf{else} \end{cases}$$

Where $J_p(t,r) = \displaystyle\int_0^r j(t,s)ds$ is the primitive of j with respect to r.

a) **Convexity of J:**
Since

$$u \mapsto \frac{1}{p}\int_0^T |u'|^p,$$

and

$$u \mapsto \int_0^T J_p(u)$$

are convex, then J is convex as the sum of two convex functions.

b) **Lower-semicontinuity of J:**
To prove that the functional J is l.s.c, it is enough to show that $\forall C \in \mathbb{R}$

$$A = [J \leq C] = \{u \in W^{1,p}_{per}(0,T) \text{ such that } J(u) \leq C\}$$

is a closed set.

- Let us consider a sequence $u_n \in W^{1,p}_{per}(0,T)$ such that

$$u_n \to u \in W^{1,p}_{per}(0,T).$$

Since $J(u_n) \leq C$, and u_n is bounded in $W^{1,p}(0,T)$, one must shows that

$$J(u) \leq C.$$

- One can justify the existence of a sub-sequence $u_{n,k}$ such that

$$u_{n,k} \rightharpoonup u \text{ in } W^{1,p}_{per}(0,T),$$

Therefore,

$$u_{nk} \to u \text{ a.e } (0,T).$$

Following the continuity of J_p, we have

$$J_p(u_{nk}(t)) \to J_p(u(t)) \text{ a.e on } (0,T).$$

Since J_p is nonnegative, then by using the Fatou's lemma, one obtains

$$\int_0^T J_p(u) \leq \liminf_{n\to+\infty} \int_0^T J_p(u_{nk}) \leq C.$$

By following the l.s.c of the norm $\|\cdot\|_{W^{1,p}}$, one can deduce that $J(u) \leq C$.
Since J is convex and lower semi-continuous on $L^2(0,T)$, then its sub-differential ∂J is a maximal monotone operator on $L^2(0,T)$ [4]. Further, we continue the proof by computing ∂J.
By definition, for $u \in L^2(0,T)$, one has the following:

$$\partial J(u) = \{w \in L^2(0,T), J(v) - J(u) \geq < w, v-u > \forall v \in L^2(0,T)\}$$

Therefore, it is necessary to find $w \in W^{1,p}_{per}(0,T)$ such that, for $J_p(u) \in L^1(0,T)$ and $\forall v \in W^{1,p}_{per}(0,T)$, the following inequality holds

$$\frac{1}{p}\int_0^T |v_x|^p + \int_0^T J_p(v) - \frac{1}{p}\int_0^T |u'|^p + \int_0^T J_p(u) \geq \int_0^T w.(v-u)$$

By choosing $v = u + z\phi$ for a z close to 0 and a $\phi \in \mathcal{D}(0,T)$, one gets:

$$\frac{1}{p}\int_0^T |u' + \phi'|^p + \int_0^T J_p(u+\phi') - \frac{1}{p}\int_0^T |u'|^p + \int_0^T J_p(u) \geq z\int_0^T w\phi$$

Dividing the inequality by $z > 0$ then $z < 0$, then, the limit when z approaches 0 gives :

$$\frac{d}{dz}_{|z=0} J_1(u+z\phi) + \frac{d}{dz}_{|z=0} J_2(u+z\phi) = \int_0^T w\phi$$

Now, one has

$$\frac{1}{p}\int_0^T \frac{d}{dz}_{|z=0} |u' + z\phi'|^p + \int_0^T \frac{d}{dz}_{|z=0} J_p(u+z\phi) = \int_0^T w\phi$$

which yields to the following result

$$\begin{cases} \int_0^T |u'|^{p-2}u'\phi' + \int_0^T j(u)\phi = \int_0^T w\phi \\ u \in W^{1,p}_{per}(0,T) \end{cases}$$

Therefore,

$$\begin{cases} < -(|u'|^{p-2}u')' + j(u), \phi > = < w, \phi > \quad \forall \phi \in \mathcal{D}(0,T) \\ u \in W^{1,p}_{per}(0,T) \end{cases}$$

Thus,

$$\partial J(u) = -(|u'|^{p-2}u')' + j(.,u)$$

Using the properties of maximal monotone operators, we obtain $\forall \lambda \geq 0$ and $\forall f \in L^2(0,T)$ the existence and uniqueness of a solution to the problem

$$\begin{cases} \lambda u(x) - \partial J(u) = f(x) \\ u \in W^{1,p}_{per}(0,T) \end{cases} \tag{3.2}$$

where for $\lambda = 1$, the equation (3.2) is equivalent to our main problem and that concludes the proof.
Now, if $f \geq 0$ a.e. in (0,T), we introduce the function sign^- defined in $\mathbb{R}$ by the following sense

$$\text{sign}^- r = \begin{cases} -1 & \text{if } r < 0 \\ 0 & \text{if } r \geq 0 \end{cases}$$

as sign^- is an increasing function, one considers the convex function j_ε which is a twice differentiable function, such that

$$j'_\varepsilon(r) \to \text{sign}^- r \quad \text{when } \varepsilon \to 0$$

Let's take $j_\varepsilon'(u)$ as a test function, then we get

$$\int_0^T u j_\varepsilon'(u) + \int_0^T |u'|^{p-2}u' j_\varepsilon''(u) + \int_0^T j\,(t,u)\, j_\varepsilon'(u) = < f, j_\varepsilon'(u) > .$$

Using the convexity of j_ε, one can deduce that

$$\int_0^T |u'|^{p-2}u' j_\varepsilon''(u) \geq 0$$

for the other terms, we have

$$\lim_{\varepsilon\to 0}\int_0^T j(t,u) j_\varepsilon'(u) = \lim_{\varepsilon\to 0}\int_{[u\geq 0]} j(t,u) j_\varepsilon'(u) + \int_{[u<0]} j(t,u) j_\varepsilon'(u) = \int_{[u<0]} j(t,u) \geq 0,$$

which follows that

$$\lim_{\varepsilon\to 0}\int_0^T u j_\varepsilon'(u) \leq \lim_{\varepsilon\to 0}\int_0^T j_\varepsilon'(u) f,$$

this implies that

$$\int_0^T u^- \leq -\int_0^T f \leq 0$$

This leads to conclude that $u \geq 0$ a.e. in $[0, T]$.

□

4 Proof of the main result

Since $f \in \mathcal{M}_B^+(0,T)$, then there exists $f_n \in L^2(0,T)$ $f_n \geq 0$ such that $\|f_n\|_{L^1} \leq \|f\|_{\mathcal{M}_B}$, and which converges to f in $\mathcal{M}_B^+(0,T)$. According to the last theorem, there exists a weak nonnegative periodic solution u_n of

$$\begin{cases} u_n \in W^{1,p}_{per}(0,T),\ u_n \geq 0 \\ \displaystyle\int_0^T u_n\phi + \int_0^1 |u_n'|^{p-2}u_n'\phi' + \int_0^T j\,(t,u_n)\,\phi = \int_0^T f_n\phi \quad \forall\phi\in W^{1,p}_{per}(0,T) \end{cases} \tag{4.1}$$

In this section, the main ingredient for the proof of Theorem 2.4 will be developed. First, we derive some apriori estimates to finally prove that under appropriate additional assumptions, the solution of the approximated problem (4.1) converges to a weakly periodic solution of (1.1).
For the remainder of this paper, we denote by C any generic and nonnegative constant depending only on T.

Lemma 4.1. *Let u_n be the sequence defined as above, then, there exists a constant C such that:*

i) $\|u_n\|_p \leq C(T)\,\|u_n'\|_p$

ii) $\displaystyle\int_0^T |u_n|\,dt + \int_0^T |j(t,u_n)|\,dt \leq \|f\|_{\mathcal{M}_B}$

iii) $\left\||u_n'(t)|^{p-2}u_n'\right\|_{p'} \leq C(T)\,\|f\|_{\mathcal{M}_B}$

Proof. Let's take $\phi \equiv 1$ as a test function in (4.1) to get

$$\int_0^T u_n + \int_0^T j\,(t,u_n) = \int_0^T f_n \leq \|f\|_{\mathcal{M}_B}$$

since u_n and $j(t,u_n) \geq 0$, then one obtains $i)$ and $ii)$.
Finally, let θ_n be defined as $\theta_n = |u_n'|^{p-2}u_n'$. We deduce from (4.1) that

$$\theta_n(t)' = u_n + j(t,u_n) - f_n$$

Then θ_n is bounded in $VB(0,T)$. Therefore u_n is compact in $L^1(0,T)$. Then there exists a sub-sequence still denoted by u_n and $u \in L^1(0,T)$ such that, $u_n \to u$ strongly in L^1 and a.e in $(0,T)$. Moreover, θ_n is bounded in $L^{p'}(0,T)$, then, $\theta_n \rightharpoonup \theta$ weakly in $L^{p'}(0,T)$. □

According to i) and ii), one can deduce that u_n is bounded in $W^{1,p}_{per}(0,T)$. consequently, there exists a sub-sequence still denoted by u_n and $u \in W^{1,p}_{per}(0,T)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in} \quad W^{1,p}_{per}(0,T) \\ u_n \longrightarrow u & \text{strongly in} \quad \mathcal{C}([0,T]) \end{cases}$$

Therefore, according to (H_4) we get

$$j(t,u_n) \longrightarrow j(t,u) \text{ in } L^1(0,T).$$

Moreover, the almost everywhere convergence result of the gradient of [3] applies in this case. Hence, we deduce the existence a subsequence still denoted by u_n such that

$$u'_n \to u' \quad \text{a.e. in }]0,T[$$

According to (iii), we have that $|u'_n(t)|^{p-2}u'_n$ is bounded in $L^{p'}$. This implies that

$$\begin{cases} |u'_n(t)|^{p-2}u'_n \to |u'(t)|^{p-2}u' & \text{a.e. in }]0,T[\\ |u'_n(t)|^{p-2}u'_n \rightharpoonup |u'(t)|^{p-2}u' & \text{weakly in } L^{p'}(0,T), \end{cases}$$

which allows one to pass to the limit in (4.1) to obtain that u satisfies

$$\begin{cases} u \in W^{1,p}_{per}(0,T) \\ \displaystyle\int_0^T u\phi + \int_0^T (|u'|^{p-2}u')\phi' + \int_0^T j\,(t,u)\,\phi = < f, \phi > \quad \forall \phi \in W^{1,p}_{per}(0,T) \end{cases}$$

5 Numerical simulation

5.1 The Lattice Boltzmann method

The Lattice Boltzmann method is derived from the kinetic theory of gases as well as cellular automata, and models the fluid as a set of microscopic particles [8]. It is based on microscopic and mesoscopic kinetic equations, unlike traditional schemes based on macroscopic Navier-Stokes equations. Moreover, this method has the advantage of being fast to implement and allows simple boundary conditions to be expressed. The LBM method describes the evolution of the discretized particle distribution functions $f_i(x,t)$ in space x and time t, along the direction with a velocity c_i.

In this paper, the D1Q3 model is used, where D1 represents the one-dimensional domain and Q3 represents the three velocities. The Lattice Boltzmann Method (LBM) can be divided into two main steps: the collision step and the streaming step.

In the collision step, the LBM calculates the local equilibrium distribution function of the fluid particles at each node of the lattice, based on the current values of macroscopic variables such as density, velocity and temperature. In the streaming step, the LBM updates the distribution functions through moving the fluid particles along the lattice's discrete velocity vectors, based on the result of the particle's collision.

Most interestingly, the LBM method can solve both linear and non-linear partial differential equations.

The discrete LBM equation is:

$$f_i(x + v_i \triangle t, t + \triangle t) = f_i(x,t) + \frac{\triangle t}{\tau}[f_i^{eq}(x,t) - f_i(x,t)] \tag{5.1}$$

where $f_i(x,t)$ is the probability density of having a particle at position x and time t. τ denotes the equilibrium relaxation time, and f^{eq} designates the equilibrium distribution function.

5.2 Applications

In this section, we consider the equation (1.1) with $T = 1$. Let us first introduce the microscopic time s, $u = u(t, s)$ and set

$$k(t, s) = \left| \frac{\partial u(t, s)}{\partial t} \right|^{p-2}$$

and

$$F(t, s) = f - j(t, u(t, s)) - u(t, s)$$

The equation (1.1) will then be written

$$\frac{\partial}{\partial t} \left(k(t, s) \frac{\partial u}{\partial t} \right) = F(t, s) \tag{5.2}$$

In this study, the model $D1Q3$ is used [6], [17]. To do this, the interval $[0, 1]$ is descritized into m step points, and we take $dt = 0.01$, the microscopic time step $ds = 0.01$, and the speed $c = \dfrac{dt}{ds} = 1$.
We initialize the solution in $s = 0$ by $u \equiv 1$, and we define the distribution functions $f_0^{(eq)}$, $f_1^{(eq)}$, $f_2^{(eq)}$ as follows

$$f_i^{(eq)}(t, s) = \begin{cases} (w_0 - 1)\, u(t, s) & \text{for} \quad i = 0 \\ w_i u(t, s) & \text{for} \quad i = 1, 2 \end{cases} \tag{5.3}$$

where w_i are the weight coefficients which satisfy the following equations:

$$\sum_{i=0}^{2} \omega_i = 1 \qquad \sum_{i=0}^{2} \omega_i c_i = 0, \qquad \sum_{i=0}^{2} \omega_i c_i c_i = \alpha c^2 \tag{5.4}$$

For the $D1Q3$ model, these weights are given by:

$$w_0 = \frac{2}{3}, \quad w_1 = \frac{1}{6}, \quad w_2 = \frac{1}{6}$$

and the coefficients α, c_0, c_1 and c_{-1} are equal to

$$\alpha = \frac{1}{3}, c_0 = 0, c_1 = c, c_{-1} = -c.$$

Therefore, it can deduced that

$$\sum_{i=0}^{2} c_i f_i^{(eq)} = 0, \qquad \sum_{i=0}^{2} c_i c_i f_i^{(eq)} = \alpha c^2 u, \tag{5.5}$$

where the physical quantity u is given by (see [10])

$$u(t, s) = \frac{1}{1 - \omega_0} \sum_{i=1}^{2} f_i^{(\text{eq})}(t, s) = \frac{1}{1 - \omega_0} \sum_{i=1}^{2} f_i(t, s) \tag{5.6}$$

Let's consider ε a rather small expansion parameter, $s_2 = \dfrac{s}{\varepsilon^2}$ and $t_1 = \dfrac{t}{\varepsilon}$ as before to derive the equation (5.2), we use the Chapman-Enskog analysis [14] to get

$$\begin{aligned} & f_i = f_i^{eq} + \varepsilon f_i^{(1)} + \varepsilon^2 f_i^{(2)} + O(\varepsilon^2), \\ & \partial_s = \varepsilon^2 \partial_{s_2} + O(\varepsilon^3), , \\ & \partial_t = \varepsilon \partial_{t_1} + O(\varepsilon^2), \\ & F = \varepsilon^2 F^{(2)} + O(\varepsilon^3) \end{aligned}$$

where ∂t_1 is the derivative with respect to t_1.
Now in order to solve numerically the Lattice Boltzmann equation, we propose to use the LB

algorithm which consists on the following two processes.
For the collision process, the three distribution functions are calculated at each node by the following formula

$$f_i(t+c_ids,s+ds)=(1-\frac{1}{\tau})f_i(t,s)+\frac{1}{\tau}f_i^{eq}(t,s)+ds\bar{w}_iF(t,s)$$

Where $\bar{\omega}_0=0,\bar{\omega}_i=\dfrac{1}{2}\quad(i=1,2)$.
For the propagation process, the only functions that propagates are f_1 and f_2. The function f_1 propagates along the direction $c_2=c$ while f_2 propagates along the direction $c_3=-c$.
By applying the Taylor expansion to the equation (5.2), one can derive the first and second-order equations in ε:

$$\begin{aligned}
&O(\varepsilon):D_if_i^{\mathrm{eq}}=-\tfrac{1}{\tau ds}f_i^{(1)}\\
&O\left(\varepsilon^2\right):\partial_{s_2}f_i^{\mathrm{eq}}+D_if_i^{(1)}+\frac{ds}{2}D_i^2f_i^{\mathrm{eq}}=-\frac{1}{\tau ds}f_i^{(2)}+\bar{w}_iF(t,s),
\end{aligned}$$

where $D_i=\mathbf{c}_i\cdot\partial_t$ and $D_i^2=\mathbf{c}_i^2\cdot\partial_{tt}$.
Substituting these two equations, the first one into the second to obtain

$$\partial_{s_2}f_i^{\mathrm{eq}}+D_i\left[\left(1-\frac{1}{2\tau}\right)f_i^{(1)}\right]=-\frac{1}{\tau ds}f_i^{(2)}+\bar{w}_iF(t,s)$$

By summing over i and using the fact that $\sum_{i=0}^{2}f_i^{(k)}=0$ for $k\geq 1$, $\sum_{i=0}^{2}f_i^{(eq)}=0$ and that $\sum_{i=0}^{2}\bar{w}_i=1$, we obtain

$$\sum_i c_if_i^{(1)}=-\tau ds\sum_i c_iD_if_i^{(eq)}$$

By using (5.3), one gets

$$\sum_{i=0}^{2}c_iD_{1i}f_i^{(eq)}=\alpha c^2\partial_{t_1}u.$$

Therefore

$$\sum_{i=0}^{2}c_if_i^{(1)}=-\tau ds\alpha c^2\partial_{t_1}u$$

Substitute this into the previous equality

$$-\partial_{t_1}\left(ds\alpha c^2\left(\tau-\frac{1}{2}\right)\partial_{t_1}u\right)=F^{(2)}$$

then we multiply by ε^2 to get

$$-\partial_t\left(ds\alpha c^2(\tau-\frac{1}{2})\partial_tu\right)=F$$

Hence, the function k and the relaxation parameter are related by:

$$k=ds\alpha c^2\left(\tau-\frac{1}{2}\right)\tag{5.7}$$

Calcul ∂_tu and $\tau(t,s)$:
Based on the Chapman-Enskog analysis, one can estimate $\varepsilon f_i^{(1)}$by$f_i-f_i^{\mathrm{eq}}$. Thus, we obtain:

$$\partial_tu=-\frac{\sum_{i=0}^{i=2}\mathbf{c}_i\left(f_i-f_i^{\mathrm{eq}}\right)}{\tau ds\alpha c^2}=-\frac{\sum_{i=0}^{i=2}\mathbf{c}_if_i}{\tau ds\alpha c^2}\tag{5.8}$$

Using (5.7) and the fact that $k(t,s)=|\partial_tu(t,s)|^{p-2}$ to deduce the expression of τ

$$\tau(t,s)=\frac{1}{2}+\frac{1}{ds\alpha c^2}\left|\partial_tu(t,s)\right|^{p-2}\tag{5.9}$$

5.3 Numerical simulation

Numerical simulations were also performed using Matlab software. The LBM method was used to solve the equation (1.1). Then, an LBM algorithm was created to solve the proposed quasi-linear periodic equation. The proposed algorithm reads as follows:

Algorithm 1 The LBM algorithm for the proposed quasilinear periodic equation

Initialization: tolerance ε_0, and u_0, $f_i^{(0)}$, $f_i^{(eq)}$, $\alpha = \frac{1}{3}$, $w = [\frac{2}{3}, \frac{1}{6}, \frac{1}{6}]$, $\bar{w} = [0, \frac{1}{2}, \frac{1}{2}]$, ds, dt, $c = \frac{dt}{ds}$, $F^{(0)} = -u_0 - j(., u_0) + f$, $u^{(1)} = u_0$

Repeat until $\left\|u^{(k+1)} - u^{(k)}\right\|_p < \varepsilon_0$

Calculate $\partial_t u^{(k)}$ using the formulate (5.8)

Calculate τ_k using the formulate (5.9)

Calculate $F^{(k)} = -u^{(k)} - j(t, u^{(k)}) + f$

Calculate f_i using the formulate (5.5)

Collision: $f_i^{(k)} = f_i^{(k)} - \frac{1}{\tau_k}(f_i^{(k)} - f_i^{(eq)^{(k)}}) + ds\bar{w}_i F^{(k)}$

Streaming: $f_i^{(k+1)} = f_i^{(k)}$

Calculate: $u^{(k+1)} = 3(f_1^{(k+1)} + f_2^{(k+1)})$

end until: when the convergent rule is met.

In order to strengthen this method, the numerical solution obtained in the following two examples was calculated:

- **Example 1:**

$$\begin{cases} u(t) - (|u'(t)|^{p-2}\, u'(t))' + u(t)^4 = 1 + sin(\pi t) \quad \text{for} \quad 0 < t < 1 \\ u(0) = u(1) \\ u'(0) = u'(1) \end{cases} \tag{5.10}$$

The initial condition is taken $u_0 \equiv 1$, and $ds = dt = 0.01$.
The solutions corresponding to $p = 2$, $p = 4$ and $p = 7$ are shown respectively in Figure 1, Figure 2 and Figure 3.

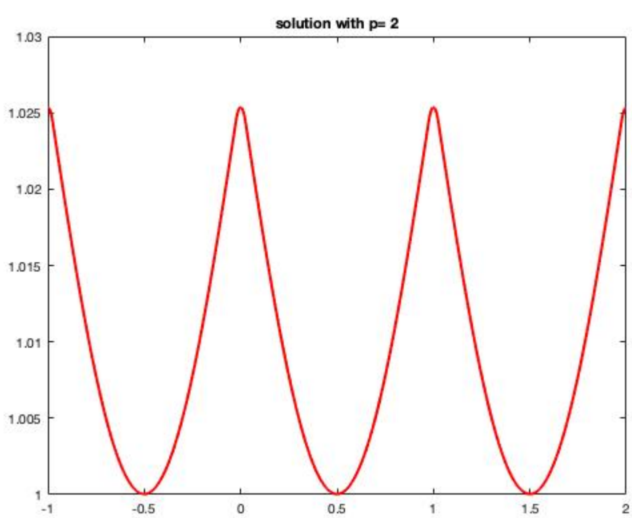

Figure 1. Output solution for p=2

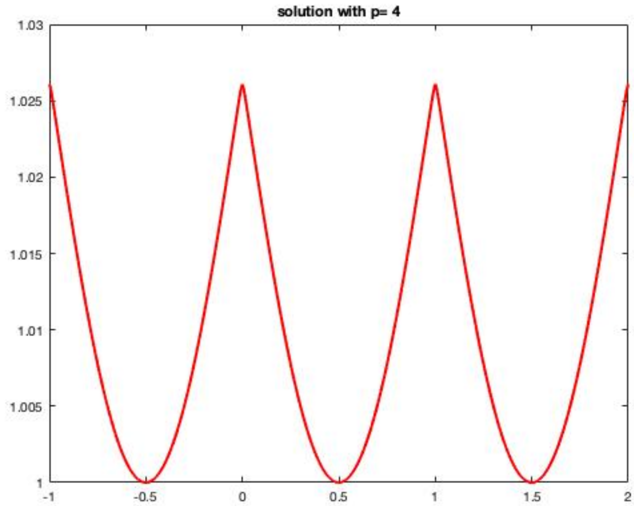

Figure 2. Output solution for p=4

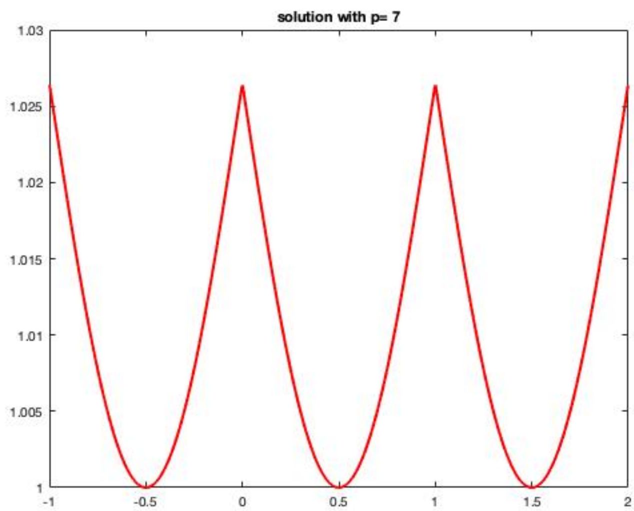

Figure 3. Output solution for p=7

- **Example 2:**

$$\begin{cases} u - (|u'|^{p-2}\,u')' + u^4 = \delta_{\frac{1}{2}} & \text{dans} \quad]0,1[\\ u(0) = u(1) \\ u'(0) = u'(1) \end{cases} \tag{5.11}$$

The initial condition is $u_0 \equiv 1$, and $ds = dt = 0.01$.
Using the sequence presented in equation (2.1) to approach $\delta_{\frac{1}{2}}$, we obtain the solutions corresponding to $p = 2$, $p = 4$ and $p = 7$, shown respectively in Figure 4, Figure 5 and Figure 6.

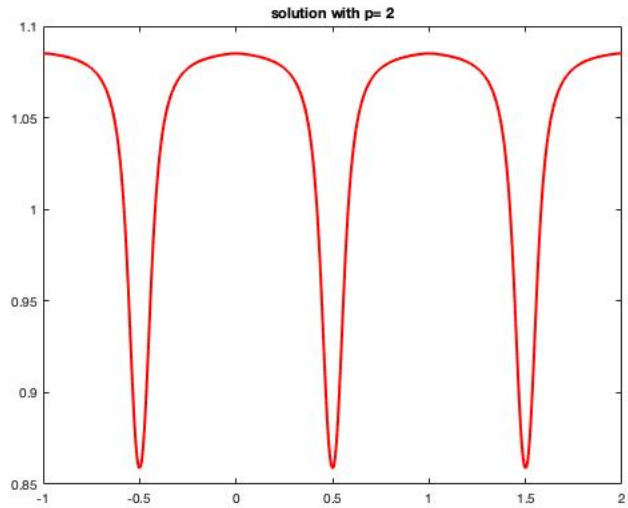

Figure 4. Output solution for p=2

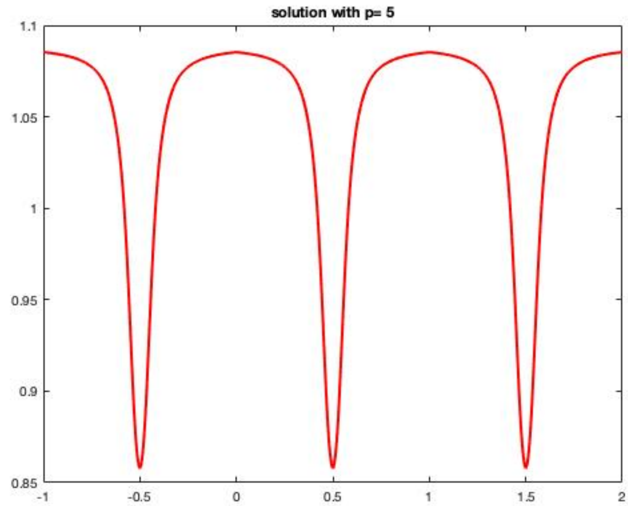

Figure 5. Output solution for p=4

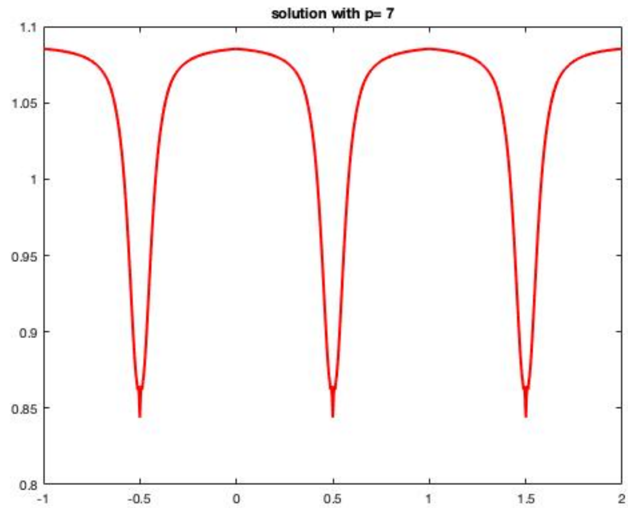

Figure 6. Output solution for p=7.

6 Conclusion

In this paper, we are interested in the mathematical analysis and numerical simulation of a class of periodic nonlinear equations with non-regular data. When the data are regular, the existence and uniqueness of the periodic solution using the optimization method are proved. If the data are not regular (for example Radon measures), then a sequence of periodic solutions based on the regular case have been constructed.
Once the a priori estimates have been obtained, we have shown that we can extract a subsequence that converges to the solution of the problem. Subsequently, a numerical algorithm based on the Lattice Boltzmann Method (LBM) was proposed to simulate the periodic solutions. Numerical simulations affirm the efficiency and robustness of the proposed algorithm.
Furthermore, in the near future, the main focus will be on the analysis of other numerical methods for the simulation of periodic solutions such as FEM (Finite Element Method), ANN (Artificial Neural Networks), etc..... , as well as the analysis of the differences in performance between these methods, comparing their accuracy, time consumption, etc...
Additionally, the research to be carried out as an extension of this work consists in applying the proposed approach to the following general periodic system:

$$\begin{cases} u_i(t) - (\mathcal{A}_i(t, u_i, u_i'))' + j_i\,(t, u, u') = f_i \quad \text{in } (0, T) \\ u_i(0) = u_i(T), \quad u_i'(0) = u_i'(T) \end{cases} \tag{6.1}$$

where $\mathcal{A}_i$ is an operator of type Leray-Lions.
Since the data are only nonnegative Radon measure, and the main operator of is strongly nonlinear, then the classical numerical methods such as finite element, finite difference etc... cannot be used here. For these reasons, the Lattice Boltzmann Method (LBM) is used. This is a relatively new method compared with the classical approaches usually used in numerical simulation (finite element method, etc.).

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