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CLASSIFICATION RESULTS ON NORMAL GENERIC LIGHTLIKE SUBMANIFOLDS

Nand Kishor Jha, Sangeet Kumar and Jatinder Kaur

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Corresponding Author: Sangeet Kumar

Abstract In this study, we investigate normal generic lightlike submanifolds of an indefinite Kaehler manifold. At first, we introduce two kinds of tensors and then propose the definition of a normal generic lightlike submanifold of an indefinite Kaehler manifold. Secondly, we find some necessary and sufficient conditions enabling a generic lightlike submanifold of an indefinite Kaehler manifold to be a normal generic lightlike submanifold. We also establish a classification theorem for the holomorphic bisectional curvature of a normal generic lightlike submanifold of an indefinite complex space form.

1 Introduction

The theory of submanifolds has emerged as one of the most fruitful area of research in differential geometry and has played a significant role in the development of the subject matter. With the development of the submanifold theory a large variety of submanifolds, namely, invariant submanifolds, anti-invariant submanifolds, Cauchy-Riemann submanifolds, semi-invariant submanifolds and generic submanifolds have been introduced and developed in complex and contact geometries by several authors (for details, see [1], [2], [3], [4], [5], [6]). Due to outstanding geometric features, these submanifolds help us to unfold the beauty of the subject matter. However, the class of generic submanifolds [5] has a unique geometric characteristic that in this case the normal bundle is mapped to the tangent bundle under the action of an almost complex structure $\bar{J}$. Due to this geometric feature, generic submanifolds form an exceptional category of semi-invariant submanifolds, which gives us more attractive and significant results.

On the other hand, from last few decades, geometers and physicists have shown substantial interest in the study of semi-Riemannian geometry due to their extensive applications in different fields. To generalize submanifold theory from Riemannian manifolds to semi-Riemannian manifolds, the class of lightlike submanifolds transpired eventually in the semi-Riemannian category. The geometry of lightlike submanifolds has a vast application area. For example, a lightlike submanifold produces models of different types of horizons such as Cauchy's horizons, event horizons, and Kruskal's horizons. The universe can correspond to a 4-dimensional submanifold enclosed in a $(4+n)$-dimensional space-time manifold. The general theory of lightlike submanifolds was introduced and developed by Kupeli [7] and Bejancu and Duggal [8]. The theory of lightlike submanifolds has been further developed by many others ([9], [10], [11], [12], [13], [14], [15], [16]). With due time, Duggal and Jin [17] introduced the general concept of generic lightlike submanifolds of indefinite Sasakian manifolds. Since then, various studies have been done by other researchers in the field of generic lightlike submanifolds ([18], [19], [20]). Furthermore, Dogan et al. [21] investigated screen generic lightlike submanifolds of indefinite Kaehler manifolds. In [22], the authors investigated the geometry of normal GCR-lightlike submanifolds of indefinite nearly Kaehler manifolds, and the concept of generic lightlike submanifolds is yet to be examined under normal conditions. Therefore, in view of the rich geometric features of generic lightlike submanifolds, it is interesting to investigate the geometry of normal generic lightlike submanifolds of a Kaehler manifold.

In the present paper, we investigate the study of normal generic lightlike submanifolds of an indefinite Kaehler manifold. We introduced two kinds of tensors and then proposed the definition of normal generic lightlike submanifolds. We derive some necessary and sufficient conditions for a generic lightlike submanifold of an indefinite Kaehler manifold to be a normal generic lightlike submanifold. Furthermore, we established a classification theorem for holomorphic bisectional curvature of a normal generic lightlike submanifold of an indefinite complex space form.

2 Preliminaries

2.1 Geometry of lightlike submanifolds

A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric g induced from $\bar{g}$, (for details, see [8]). For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T\bar{M}$. Thus, both TM and $TM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(TM) = TM \cap TM^{\perp}$, which is known as radical distribution of rank r, where $1 \leq r \leq m$. Then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is

$$TM = Rad(TM) \perp S(TM). \tag{2.1}$$

We consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. For any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local null frame $\{N_i\}$ of null sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM^{\perp})^{\perp}$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \text{ for any } i, j \in \{1, 2, .., r\}. \tag{2.2}$$

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M} \mid_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}) \tag{2.3}$$

and

$$T\bar{M} \mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.4}$$

Here, one should note that the screen distribution $S(TM)$ is not unique.
Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$, then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M and the second fundamental form h is a symmetric bilinear form on $\Gamma(TM)$ and A_U is a linear operator on M and is known as shape operator.
Moreover, we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.6}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.7}$$

where $X, Y \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.
Employing Eqs. (2.6) and (2.7), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.8}$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(A_W X, N), \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$, $W \in \Gamma(S(TM^\perp))$ and $N \in \Gamma(ltr(TM))$.
Denote the projection of TM on $S(TM)$ by P. From the decomposition Eq. (2.1) of tangent bundle of a lightlike submanifold, we can induce some new geometric objects on the screen distribution $S(TM)$ of M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X, PY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively.
Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇, respectively, then by straightforward calculations, we obtain

$$\begin{aligned}\bar{R}(X,Y)Z =& R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ &+ (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) + D^l(X, h^s(Y,Z)) \\ &- D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) \\ &+ D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z))\end{aligned} \tag{2.11}$$

and the equation of Codazzi is

$$\begin{aligned}(\bar{R}(X,Y)Z)^\perp =& (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) + D^l(X, h^s(Y,Z)) \\ &- D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) \\ &+ D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)),\end{aligned} \tag{2.12}$$

where

$$(\nabla_X h^s)(Y,Z) = \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z), \tag{2.13}$$

$$(\nabla_X h^l)(Y,Z) = \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z), \tag{2.14}$$

for any $X, Y, Z \in \Gamma(TM)$.

Definition 2.1. ([9]). A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$, if there exist a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M such that for $Y, Z \in \Gamma(TM)$

$$h(Y,Z) = H\bar{g}(Y,Z). \tag{2.15}$$

From Eq. (2.7), M is totally umbilical, if and only if, on each coordinate neighborhood u, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ satisfying

$$h^l(Y,Z) = H^l g(Y,Z), \quad h^s(Y,Z) = H^s g(Y,Z), \quad D^l(Y,V) = 0, \tag{2.16}$$

for any $Y, Z \in \Gamma(TM)$ and $V \in \Gamma(S(TM^\perp))$. A lightlike submanifold of an indefinite Kaehler manifold is said to be totally geodesic if $h(Y,Z) = 0$, for any $Y, Z \in \Gamma(TM)$.

2.2 Indefinite Kaehler manifolds

Let $\bar{M}$ be an indefinite almost Hermitian manifold with an almost complex structure $\bar{J}$ of type $(1,1)$ and indefinite Hermitian metric $\bar{g}$ such that for all $X, Y \in \Gamma(T\bar{M})$ (see [23]), we have

$$\bar{J}^2 = -I, \ \bar{g}(\bar{J}X, \bar{J}Y) = \bar{g}(X,Y).$$

An indefinite almost Hermitian manifold $\bar{M}$ is called an indefinite Kaehler manifold, if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is,

$$(\bar{\nabla}_X \bar{J})Y = 0, \ \forall \ X, Y \in \Gamma(TM). \tag{2.17}$$

2.3 Generic lightlike submanifolds

Definition 2.2. [18] Let M be a real r-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, M is said to be a generic lightlike submanifold if the screen distribution $S(TM)$ of M has the following decomposition:

$$\begin{aligned} S(TM) =&\bar{J}(S(TM)^{\perp}) \oplus_{orth} D_0 \\ =&\bar{J}(Rad(TM)) \oplus \bar{J}(ltr(TM)) \oplus_{orth} \bar{J}(S(TM^{\perp})) \oplus_{orth} D_0, \end{aligned} \tag{2.18}$$

where D_0 is a non-degenerate almost complex distribution on M with respect to $\bar{J}$, i.e., $\bar{J}(D_0) = D_0$ and D' is an r-lightlike distribution on $S(TM)$ such that $\bar{J}(D') \subset tr(TM)$, where $D' = \bar{J}(ltr(TM)) \oplus_{orth} \bar{J}(S(TM^{\perp}))$.

Therefore, by using Eq. (2.18) and the general decomposition of Eqs. (2.1) and (2.4) becomes

$$TM = D \oplus D', \quad T\bar{M} = D \oplus D' \oplus tr(TM),$$

where D is a $2r$-lightlike almost complex distribution on M such that $D = Rad(TM) \oplus_{orth} \bar{J}(Rad(TM)) \oplus_{orth} D_0$.
Consider Q, P_1 and P_2 be the projections from TM to D, $\bar{J}ltr(TM)$ and $\bar{J}S(TM^{\perp})$, respectively. Then for $Y \in \Gamma(TM)$, we have

$$Y = QY + P_1Y + P_2Y, \tag{2.19}$$

applying $\bar{J}$ to Eq. (2.19), we obtain

$$\bar{J}Y = \phi Y + \omega P_1 Y + \omega P_2 Y, \tag{2.20}$$

and we can write the Eq. (2.20) as

$$\bar{J}Y = \phi Y + \omega Y, \tag{2.21}$$

where ϕY and ωY, respectively, denote the tangential and transversal components of $\bar{J}Y$. Similarly,

$$\bar{J}V = EV, \tag{2.22}$$

for $V \in \Gamma(tr(TM))$, where EV is the section of TM.
Then, differentiating Eq. (2.20) and using Eqs. (2.6), (2.7) and (2.22), we derive

$$(\nabla_Y \phi)Z = A_{\omega P_1 Z}Y + A_{\omega P_2 Z}Y + Eh(Y, Z), \tag{2.23}$$

$$D^s(Y, \omega P_1 Z) = -\nabla^s_Y \omega P_2 Z + \omega P_2 \nabla_Y Z - h^s(Y, \phi Z), \tag{2.24}$$

$$D^l(Y, \omega P_2 Z) = -\nabla^l_Y \omega P_1 Z + \omega P_1 \nabla_Y Z - h^l(Y, \phi Z), \tag{2.25}$$

for $Y, Z \in \Gamma(TM)$.

Example 2.3. Consider M be a submanifold of $(R^8_2, \bar{g})$ with signature $(+,+,-,+,+,-,+,+)$ given by the equations $u_3 = u_8$ and $u_5 = \sqrt{1 - u_6^2}$ with respect to basis $(\partial u_1, \partial u_2, \partial u_3, \partial u_4, \partial u_5, \partial u_6, \partial u_7, \partial u_8)$.
The tangent bundle of M is given by

$$U_1 = \partial u_1, \quad U_2 = \partial u_2, \quad U_3 = \partial u_3 + \partial u_8, \quad U_4 = \partial u_4,$$

$$U_5 = -u_6 \partial u_5 + u_5 \partial u_6, \quad U_6 = \partial u_7.$$

It is easy to see that M is a 1-lightlike submanifold with $Rad(TM) = Span\{U_3\}$ and $\bar{J}U_3 = U_4 - U_6 \in \Gamma(S(TM))$. Moreover, $\bar{J}U_1 = U_2$ and $\bar{J}U_2 = -U_1$ and therefore $D_0 = Span\{U_1, U_2\}$. By direct calculations, we get $S(TM^{\perp}) = Span\{V = x_5\partial x_5 + x_6\partial x_6\}$. Thus, $\bar{J}V = U_5$ and thus $\bar{J}S(TM^{\perp}) \subset S(TM)$. On the other hand, $ltr(TM)$ is spanned by $N = \frac{1}{2}(-\partial x_3 + \partial x_8)$. Then $\bar{J}N = -\frac{1}{2}(\partial x_4 + \partial x_7) = -\frac{1}{2}(U_4 + U_6)$ and $D' = \{\bar{J}N, \bar{J}V\}$. Thus, M is a proper 6-dimensional generic lightlike submanifold of $(R^8_2, \bar{g})$.

Firstly, we will prove a basic lemma for later use.

Lemma 2.4. *Suppose that M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then we have*

$$(\nabla_X \phi)Y = A_{\omega Y} X + Eh(X,Y) \tag{2.26}$$

and

$$(\nabla^t_X \omega)Y = -h(X, \phi Y), \tag{2.27}$$

for $X, Y \in \Gamma(TM)$ and

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y, \quad (\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla^t_X Y. \tag{2.28}$$

Proof. For any $X, Y \in \Gamma(TM)$, using Eqs. (2.5), (2.7), (2.17) and (2.21), we obtain

$$\begin{aligned}(\bar{\nabla}_X \bar{J})Y =& \nabla_X(\phi Y) + h(X, \phi Y) - A_{\omega Y} X + \nabla^t_X(\omega Y) \\ &- \phi(\nabla_X Y) - \omega(\nabla_X Y) - Eh(X,Y).\end{aligned} \tag{2.29}$$

On comparing the tangential and transversal components in Eq. (2.29), the result follows. □

Theorem 2.5. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D defines a totally geodesic foliation in M if and only if $Eh(X,Y) = 0$, for any $X, Y \in \Gamma(D)$.*

Proof: Using the definition of a generic lightlike submanifold, D defines a totally geodesic foliation in M if and only if $\nabla_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. In other words, D defines a totally geodesic foliation in M if and only if

$$\bar{g}(\nabla_X Y, \bar{J}\xi) = \bar{g}(\nabla_X Y, \bar{J}W) = 0,$$

for any $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Now from Eqs. (2.6) and (2.17), we drive

$$\bar{g}(\nabla_X Y, \bar{J}\xi)) \quad = \quad \bar{g}(\bar{\nabla}_X \bar{J}Y, \xi) = \bar{g}(h^l(X, \bar{J}Y), \xi)) \tag{2.30}$$

for $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TM)$. Similarly, using Eqs. (2.6) and (2.17), we get

$$\bar{g}(\nabla_X Y, \bar{J}W)) \quad = \quad \bar{g}(\bar{\nabla}_X \bar{J}Y, W) = \bar{g}(h^s(X, \bar{J}Y), W), \tag{2.31}$$

for $X, Y \in \Gamma(D)$ and $W \in \Gamma(S(TM^{\perp}))$. From Eqs. (2.30) and (2.31) it follows that D defines a totally geodesic foliation in M, if and only if, $h^s(X, \bar{J}Y)$ has no components in $S(TM^{\perp})$ and $h^l(X, \bar{J}Y)$ has no components in $ltr(TM)$. Thus, from Eq. (2.22), we acquire $\bar{J}h(X,Y) = Eh(X,Y) = 0$, which proves the assertion.

Theorem 2.6. *Consider M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribbution D' defines a totally geodesic foliation in M if and only if $A_{\omega Y} X \in \Gamma(D')$, for any $X, Y \in \Gamma(D')$.*

Proof. Firstly, let D' defines a totally geodesic foliation in M, then for any $X, Y \in \Gamma(D')$ from Eq. (2.23), we have $-A_{\omega Y} X - Eh(X,Y) = 0$, which further gives $-A_{\omega Y} X = Eh(X,Y)$, this yields that $A_{\omega Y} X \in \Gamma(D')$.
Conversely, suppose that $A_{\omega Y} X \in \Gamma(D')$ for $X, Y \in \Gamma(D')$ then from Eq. (2.23), we obtain $\phi \nabla_X Y = 0$, which implies that $\nabla_X Y \in \Gamma(D')$. Thus, the proof follows. □

Lemma 2.7. *Suppose that M be a totally umbilical generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then the distribution D' defines a totally geodesic foliation in M.*

Proof. For any $X, Y \in \Gamma(D')$, using Eq. (2.26), we obtain

$$\phi \nabla_X Y = -A_{\omega Y} X - Eh(X,Y).$$

On taking inner product of above equation with respect to $Z \in \Gamma(D_0)$, we have

$$\begin{aligned} g(\phi\nabla_X Y, Z) &= -g(A_{\omega Y}X, Z) - g(Eh(X,Y), Z) \\ &= \bar{g}(\bar{\nabla}_X \omega Y, Z) = \bar{g}(\bar{\nabla}_X \bar{J}Y, Z) \\ &= -\bar{g}(\bar{\nabla}_X Y, \bar{J}Z) = -\bar{g}(\bar{\nabla}_X Y, Z') \\ &= g(Y, \nabla_X Z'). \end{aligned} \tag{2.32}$$

where $Z' = \bar{J}Z \in \Gamma(D_0)$. Since $X \in \Gamma(D')$ and $Z \in \Gamma(D_0)$, then from Eqs. (2.24), (2.25) and (2.15), we derive

$$\omega P \nabla_X Z = h(X, \phi Z) = Hg(X, \phi Z) = 0.$$

Thus, $\omega P \nabla_X Z = 0$, which yields that $\nabla_X Z \in \Gamma(D_0)$. Then from Eq. (2.32) together with the non-degeneracy of D_0, we acquire $\phi\nabla_X Y = 0$. Hence, $\nabla_X Y \in \Gamma(D')$, which completes the proof. □

Theorem 2.8. *[18] Suppose that M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D' is integrable, if and only if,*

$$A_{\bar{J}Z}V = A_{\bar{J}V}Z,$$

for any $Z, V \in \Gamma(D')$.

Theorem 2.9. *For a totally umbilical generic lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$, the distribution D' is always integrable.*

Proof. Suppose that for any $X, Y \in \Gamma(D')$ then using Eqs. (2.26), (2.28) with Lemma (2.7), we have $A_{\omega Y}X = -Eh(X,Y)$, which implies that $A_{\omega Y}X \in \Gamma(D')$. Further using the symmetric property of the second fundamental form h, we get $A_{\omega Y}X = A_{\omega X}Y$. Thus, in view of the Theorem (2.8), the result follows. □

3 Normal generic lightlike submanifolds of indefinite Kaehler manifolds

We define two tensor fields S and S^* as

$$S(X,Y) = [\phi,\phi](X,Y) - 2Ed\omega(X,Y) \tag{3.1}$$

and

$$S^*(Y,X) = (L_Y\phi)X, \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$, where

$$[\phi,\phi](X,Y) = [\phi X, \phi Y] + \phi^2[X,Y] - \phi([\phi X, Y] + [X, \phi Y]), \tag{3.3}$$

$$d\omega(X,Y) = \frac{1}{2}\{\nabla^t_X(\omega Y) - \nabla^t_Y(\omega X) - \omega[X,Y]\} \tag{3.4}$$

and the Lie derivative of ϕ with respect to $Y \in \Gamma(TM)$ is given by

$$(L_Y\phi)X = [Y, \phi X] - \phi[Y, X], \tag{3.5}$$

for any $X \in \Gamma(TM)$.
Since ∇ and ∇^t are torsion free, therefore we can rewrite Eqs. (3.3) and (3.4), respectively, as:

$$[\phi,\phi](X,Y) = (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X - \phi((\nabla_X\phi)Y - (\nabla_Y\phi)X) \tag{3.6}$$

and

$$d\omega(X,Y) = \frac{1}{2}\{(\nabla^t_X\omega)Y - (\nabla^t_Y\omega)X\}. \tag{3.7}$$

Then using Eqs. (3.6) and (3.7) in Eq. (3.1), we get

$$\begin{aligned} S(X,Y) =& (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X - \phi((\nabla_X\phi)Y - (\nabla_Y\phi)X) \\ &- B\{(\nabla^t_X\omega)Y - (\nabla^t_Y\omega)X\}. \end{aligned} \tag{3.8}$$

Further using Eqs. (2.26) and (2.27) in Eq. (3.8), we derive

$$S(X,Y) = (A_{\omega Y}\phi X - \phi A_{\omega Y}X) - (A_{\omega X}\phi Y - \phi A_{\omega X}Y) \tag{3.9}$$

Now, we define a normal generic lightlike submanifold M of an indefinite Kaehler manifold as follows:

Definition 3.1. A generic lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is said to be normal, if the tensor field S vanishes identically on M, that is, if

$$S(X,Y) = 0, \ \forall\ X, Y \in \Gamma(TM). \tag{3.10}$$

Theorem 3.2. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that the distribution D' is integrable. Then M is normal, if and only if,*

$$A_{\omega Y}\phi X = \phi A_{\omega Y}X, \tag{3.11}$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Proof. Assume that M is normal, then Eq. (3.11) follows directly from Eqs. (3.9) and (3.10). Conversely, let M be a generic lightlike submanifold of an indefinite Kaehler manifold satisfying Eq. (3.11). Now for $X, Y \in \Gamma(D)$, the result follows directly from Eq. (3.9). Next, let $X \in \Gamma(D)$ and $Y \in \Gamma(D')$, then from Eq. (3.9), we obtain $S(X,Y) = A_{\omega Y}\phi X - \phi A_{\omega Y}X$, which on using Eq. (3.11) gives that $S(X,Y) = 0$. Similarly, for $X \in \Gamma(D')$ and $Y \in \Gamma(D)$, from Eq. (3.9), we have $S(X,Y) = 0$. Finally for $X, Y \in \Gamma(D')$, we have $\phi X = 0, \phi Y = 0$. Further, by hypothesis D' is integrable, therefore the result follows from Eq. (3.9). □

Corollary 3.3. *Using the Theorem (3.2), a totally umbilical generic lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is normal, if and only if, $A_{\omega Y}\phi X = \phi A_{\omega Y}X$, for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.*

Suppose that $\{F_1, F_2, F_3, ..., F_q\}$ is a local field of orthogonal frames for $S(TM^{\perp})$. Denote A_i, the fundamental tensor of Weingarten with respect to $V_i = \bar{J}F_i$, then in view of the above theorem, we conclude

Corollary 3.4. *A generic lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ with D' integrable is normal, if and only if, the fundamental tensors of Weingarten A_i commute with ϕ on invariant distribution, that is, if and only if*

$$A_i \circ \phi = \phi \circ A_i. \tag{3.12}$$

Next using (2.5), we derive

$$\nabla_X F_i = \phi A_{\bar{J}F_i}X - E\nabla^t_X \bar{J}F_i \tag{3.13}$$

and

$$\nabla^t_X \bar{J}F_i = \omega \nabla_X Fi. \tag{3.14}$$

Definition 3.5. A vector field X is said to be a D-Killing vector field, if

$$g(\nabla_Z X, Y) + g(Z, \nabla_Y X) = 0,$$

for any $Y, Z \in \Gamma(D)$.

Next, we give another necessary and sufficient condition for a generic lightlike submanifold to be normal. Thus, we have

Theorem 3.6. *A necessary and sufficient condition for a generic lightlike submanifold of an indefinite Kaehler manifold with D' being integrable to be normal is that $F_i, (i = 1, 2, 3, ..., q)$ be D-Killing vector fields.*

Proof. For $Y, Z \in \Gamma(D)$, from Eq. (3.13), we have

$$g(\nabla_Z F_i, Y) + g(Z, \nabla_Y Fi) = g(\phi A_{\bar{J}F_i} Z, Y) + g(Z, \phi A_{\bar{J}F_i} Y), \tag{3.15}$$

Then employing Eq. (2.8), we derive

$$\begin{aligned} g(Z, \phi A_{\bar{J}F_i} Y) &= -g(\phi Z, A_{\bar{J}F_i} Y) = -\bar{g}(h^s(Y, \phi Z), \bar{J}F_i) \\ &= -\bar{g}(\bar{\nabla}_{\phi Z} Y, \bar{J}F_i) = \bar{g}(Y, \bar{\nabla}_{\phi Z} \bar{J}F_i) \\ &= -g(Y, A_{\bar{J}F_i} \phi Z). \end{aligned} \tag{3.16}$$

Further from Eqs. (3.15) and (3.16), we acquire

$$g(\nabla_Z F_i, Y) + g(Z, \nabla_Y F_i) = g(\phi A_{\bar{J}F_i} Z - A_{\bar{J}F_i} \phi Z, Y). \tag{3.17}$$

Hence, the result follows from Eq. (3.17) and the Corollary (3.4). □

The Lie derivative of ϕ with respect to $Y \in \Gamma(TM)$ is given by

$$(L_Y \phi)X = [Y, \phi X] - \phi[Y, X], \tag{3.18}$$

for any $X \in \Gamma(TM)$. Then the normal generic lightlike submanifold can be characterized by another tensor field

$$S^*(Y, X) = (L_Y \phi)X, \tag{3.19}$$

for any $X, Y \in \Gamma(TM)$.

Theorem 3.7. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with D' being integrable and satisfying*

$$P(\nabla_X Y) = 0, \tag{3.20}$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$. Then M is a normal generic lightlike submanifold, if and only if

$$S^*(Y, X) = 0, \tag{3.21}$$

for all $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Proof. In view of Theorem (3.2), it follows that M is a normal generic lightlike submanifold if and only if $S(X, Y) = 0$ for all $X \in \Gamma(D)$ and $Y \in \Gamma(D')$. Now for $X \in \Gamma(D)$ and $Y \in \Gamma(D')$, using Eqs. (3.1) and (3.3), we derive

$$S(X, Y) = \phi([Y, \phi X] - \phi[Y, X]) - 2E d\omega(X, Y). \tag{3.22}$$

Since ∇^t is a torsion free connection and further using Eq. (2.27), Eq. (3.7) becomes

$$d\omega(X, Y) = \frac{1}{2} h(\phi X, Y) \tag{3.23}$$

which further gives

$$2E d\omega(X, Y) = E h(\phi X, Y). \tag{3.24}$$

Next using Eqs. (3.2), (3.5) and (3.24) in Eq. (3.22), we have

$$S(X, Y) = \phi(S^*(Y, X)) - E h(\phi X, Y). \tag{3.25}$$

Also for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$, from Eq. (2.27), we obtain

$$h(Y, \phi X) = \omega \nabla_Y X. \tag{3.26}$$

Applying $\bar{J}$ on both sides of the above Eq. (3.26), we get

$$E h(Y, \phi X) = -P \nabla_Y X.$$

Thus, Eq. (3.25) gives that

$$S(X,Y)=\phi(S^*(Y,X))+P(\nabla_Y X). \tag{3.27}$$

Now, assume first that M is a normal generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then from Eq. (3.25), we achieve

$$\phi(S^*(Y,X))=0, \quad P(\nabla_Y X)=0,$$

which implies that

$$QS^*(X,Y)=0, \quad P(\nabla_Y X)=0. \tag{3.28}$$

Again from Eqs. (3.2) and (3.5), we get

$$P(S^*(Y,X))=P(\nabla_Y \phi X-\nabla_{\phi X}Y), \tag{3.29}$$

which on using hypothesis along with second part of Eq. (3.28), yields that

$$PS^*(Y,X)=0.$$

Conversely, suppose that M is a generic lightlike submanifold of indefinite Kaehler manifold $\bar{M}$ satisfying (3.21). Then using hypothesis and Eq. (3.21) in Eq. (3.29), we get

$$P(\nabla_Y \phi X)=0. \tag{3.30}$$

Thus, using Eqs. (3.21) and (3.30) in Eq. (3.27), we obtain $S(X,Y)=0$, which shows that M is a normal generic lightlike submanifold. □

Finally, we present a characterization theorem for holomorphic bisectional curvature of a normal generic lightlike submanifold of an indefinite Kaehler manifold. Before proceeding to the main theorem, we define holomorphic bisectional curvature as follows:

Definition 3.8. The holomorphic bisectional curvature for the pair of unit vector fields $\{X,Y\}$ on $\bar{M}$ is given by

$$\bar{H}(X,Y)=\bar{g}(\bar{R}(X,\bar{J}X)Y,\bar{J}Y),$$

where $X\in\Gamma(D_0)$ and $Y\in\Gamma(S(TM^\perp))$.

Definition 3.9. Suppose that M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then the distribution D_0 is said to be parallel with respect to the induced connection ∇, if $\nabla_X Y\in\Gamma(D_0)$, for any $Y\in\Gamma(D_0)$ and $X\in\Gamma(TM)$.

Lemma 3.10. *Suppose that M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ and D_0 is parallel distribution with respect to ∇, then $\nabla_X W\in\Gamma(D')$ for any $X\in\Gamma(D_0)$ and $W\in\Gamma(S(TM^\perp))$.*

Proof. By Definition (3.8), for $X,Y\in\Gamma(D_0)$ and $W\in\Gamma(D')$,

$$\begin{aligned} g(\phi\nabla_X W,Y)&=-g(\nabla_X W,\phi Y)=-\bar{g}(\bar{\nabla}_X W,\phi Y)\\ &=\bar{g}(W,\bar{\nabla}_X\phi Y)=g(W,\nabla_X\phi Y)=0. \end{aligned}\tag{3.31}$$

Then the non-degeneracy of D_0 implies that $\phi\nabla_X W=0$, which proves the result. □

Theorem 3.11. *Suppose that M be a normal generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with the distribution D' being integrable. If D_0 is parallel with respect to the induced connection ∇, then*

$$\begin{aligned} \bar{H}(X,W)&=||h^s(\bar{J}X,W)||^2+||h^s(X,W)||^2+\bar{g}(h^s(\bar{J}X,W),\bar{J}[X,W])\\ &-\bar{g}(h^s(X,W),\bar{J}[\bar{J}X,W])+4\bar{g}(h^s(X,W),\bar{J}h^s(\bar{J}X,W))\\ &+\bar{g}(h^l(\bar{J}X,W),A_{\bar{J}W}X)-\bar{g}(h^l(X,W),A_{\bar{J}W}\bar{J}X), \end{aligned}\tag{3.32}$$

for any vector fields $X\in\Gamma(D_0)$ and $W\in\Gamma(S(TM^\perp))$.

Proof. For $X \in \Gamma(D_0)$ and $W \in \Gamma(S(TM^{\perp}))$, then the equation of Codazzi (2.12) becomes

$$\bar{g}(\bar{R}(X,\bar{J}X)W,\bar{J}W) = \bar{g}((\nabla_X h^s)(\bar{J}X,W),\bar{J}W) - \bar{g}((\nabla_{\bar{J}X}h^s)(X,W),\bar{J}W) + \bar{g}(D^s(X,h^l(\bar{J}X,W)),\bar{J}W) - \bar{g}(D^s(\bar{J}X,h^l(X,W)),\bar{J}W). \quad (3.33)$$

By using Eq. (2.13), we get

$$\begin{aligned}\bar{H}(X,W) &= \bar{g}(\nabla^s_X h^s(\bar{J}X,W),\bar{J}W) - \bar{g}(h^s(\nabla_X\bar{J}X,W),\bar{J}W)\\ &- \bar{g}(h^s(\bar{J}X,\nabla_X W),\bar{J}W) - \bar{g}(\nabla^s_{\bar{J}X}h^s(X,W),\bar{J}W)\\ &+ \bar{g}(h^s(\nabla_{\bar{J}X}X,W),\bar{J}W) + \bar{g}(h^s(X,\nabla_{\bar{J}X}W),\bar{J}W)\\ &+ \bar{g}(D^s(X,h^l(\bar{J}X,W)),\bar{J}W) - \bar{g}(D^s(\bar{J}X,h^l(X,W)),\bar{J}W). \quad (3.34)\end{aligned}$$

Now using Eqs. (2.7) and (2.17), we have

$$\bar{g}(\nabla^s_X h^s(\bar{J}X,W),\bar{J}W) = ||h^s(\bar{J}X,W)||^2 - 2\bar{g}(h^s(\bar{J}X,W),\bar{J}h^s(X,W)) - 2\bar{g}(h^s(\bar{J}X,W),\bar{J}(\nabla_X W)) + \bar{g}(h^s(\bar{J}X,W),\bar{J}[X,W]) \quad (3.35)$$

and similarly

$$\bar{g}(\nabla^s_{\bar{J}X}h^s(X,W),\bar{J}W) = -||h^s(X,W)||^2 - 2\bar{g}(h^s(X,W),\bar{J}h^s(\bar{J}X,W)) - 2\bar{g}(h^s(X,W),\bar{J}(\nabla_{\bar{J}X}W)) + \bar{g}(h^s(X,W),\bar{J}[\bar{J}X,W]). \quad (3.36)$$

By using Eq. (2.9), we have

$$\bar{g}(D^s(X,h^l(\bar{J}X,W)),\bar{J}W) = \bar{g}(A_{\bar{J}W}X,h^l(\bar{J}X,W)) \quad (3.37)$$

and

$$\bar{g}(D^s(\bar{J}X,h^l(X,W)),\bar{J}W) = \bar{g}(A_{\bar{J}W}\bar{J}X,h^l(X,W)). \quad (3.38)$$

Then using Eqs. (3.35)-(3.38) in Eq. (3.34), we obtain

$$\begin{aligned}\bar{H}(X,W) &= ||h^s(\bar{J}X,W)||^2 - 2\bar{g}(h^s(\bar{J}X,W),\bar{J}h^s(X,W))\\ &- 2\bar{g}(h^s(\bar{J}X,W),\bar{J}(\nabla_X W)) + \bar{g}(h^s(\bar{J}X,W),\bar{J}[X,W])\\ &- \bar{g}(h^s(\nabla_X\bar{J}X,W),\bar{J}W) - \bar{g}(h^s(\bar{J}X,\nabla_X W),\bar{J}W) + ||h^s(X,W)||^2\\ &+ 2\bar{g}(h^s(X,W),\bar{J}h^s(\bar{J}X,W)) + 2\bar{g}(h^s(X,W),\bar{J}(\nabla_{\bar{J}X}W))\\ &- \bar{g}(h^s(X,W),\bar{J}[\bar{J}X,W]) + \bar{g}(h^s(\nabla_{\bar{J}X}X,W),\bar{J}W)\\ &+ \bar{g}(h^s(X,\nabla_{\bar{J}X}W),\bar{J}W) + \bar{g}(A_{\bar{J}W}X,h^l(\bar{J}X,W))\\ &- \bar{g}(A_{\bar{J}W}\bar{J}X,h^l(X,W)). \quad (3.39)\end{aligned}$$

Further using Corollary (3.3) and Lemma (3.10) in Eq. (3.39), the proof follows. □

As an immediate consequence of above theorem, we have

Corollary 3.12. *Let M be a mixed geodesic normal generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with the distribution D' being integrable. If D_0 is parallel with respect to the induced connection ∇, then $\bar{H}(X,W) = 0$ for $X \in \Gamma(D_0)$ and $W \in \Gamma(S(TM^{\perp}))$.*

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Author information

Nand Kishor Jha, Department of Mathematics, Chandigarh University, Gharuan, SAS Nagar, Punjab, India.
E-mail: nandkishorjha1982@gmail.com

Sangeet Kumar, Department of Mathematics, Sri Guru Teg Bahadur Khalsa College, Sri Anandpur Sahib, punjab, India.
E-mail: sp7maths@gmail.com

Jatinder Kaur, Department of Mathematics, Chandigarh University, Gharuan, SAS Nagar, Punjab, India.
E-mail: jksaini83@gmail.com

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