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ZARISKI TOPOLOGIES ON SKEW BRACES

A. Goswami and T. Dube

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Corresponding author: A. Goswami

Abstract By modifying the usual definition of a prime ideal in a skew brace A somewhat, we define what we call a skew-prime ideal of A, and denote by $\text{Spec}_s A$ the set of these ideals. We then endow this set with a Zariski topology in a manner akin to how the set of prime ideals of a commutative ring is endowed with the Zariski topology. We characterize the irreducible closed subsets of the resulting topological space, and prove that every irreducible closed subset of the space has a unique generic point. We give a sufficient condition for the space to be Noetherian. We study continuous maps between such spaces. Denoting the set of all ideals of A by $\text{Idl}\, A$, it turns out that $\text{Idl}\, A$, partially ordered by inclusion, is a multiplicative lattice with a certain multiplication introduced here for the first time. We end the paper by observing that $\text{Spec}(\text{Idl}\, A)$ is a spectral space.

1 Introduction

To study non-degenerate involutive set-theoretic solutions of the Yang–Baxter equation, RUMP introduced a new algebraic structure in [18] called a (left) brace, which was generalized into noncommutative setting by GUARNIERI & VENDRAMIN in [7] and called a (left) skew brace. Since the introduction of skew braces, several algebraic properties of them have been studied. JESPERS, KUBAT,VAN ANTWERPEN, & VENDRAMIN studied factorization of skew left braces through strong left ideals in [11] and proved analogs of Itô's theorem in the context of skew left braces, whereas in [12], they obtained a Wedderburn type decomposition for Artinian skew left braces and also proved analogues of a theorem of Wiegold, a theorem of Schur and its converse. In [3], CEDÓ, SMOKTUNOWICZ & VENDRAMIN studied series of left ideals of skew left braces that are analogs of upper central series of groups and using that they defined left and right nilpotent skew left braces. The concepts like ideals, series of ideals, prime, and semiprime ideals, Baer and Wedderburn radicals and solvability for skew braces have been explored in [14] by KONOVALOV, SMOKTUNOWICZ & VENDRAMIN, whereas some categorical aspects of skew braces have been studied in BOURN, FACCHINI, & POMPILI [2]. A study of braces and skew braces from multiplicative lattices point of view has been conducted in FACCHINI [5].

Skew braces are fairly new algebraic structures and much more study of their algebraic aspects yet to come. Compared to algebraic properties studied so far, little has been explored from the topological side. The aim of this paper is to initiate that aspect, and we do so by studying some of the (Zariski) topological properties of the spectrum of what we call skew-prime ideals of a skew brace. It is worth it to mention that in [14], it has been pointed out that "in the conference 'Groups, rings, and the Yang–Baxter equation,' Spa, 2017, Louis Rowen suggested that it could be interesting to study prime ideals of skew braces." To meet our requirements and to make analogy with the prime spectrum of a (commutative) ring, we propose a new definition of prime-type ideals of a skew brace called skew-prime ideals. We discuss separation axioms and irreducibility. Using a skew brace homomorphism, we obtain continuous maps between spaces of spectra and study properties of these maps. Finally, we prove a result on spectrality. We end the paper with some open questions.

2 Preliminaries

A *skew left brace* is a triple $(A, +, \circ)$ such that $(A, +)$ and $(A, \circ)$ are groups, and the operations satisfy the identity:

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

for all a, b, c in A, where $-a$ denotes the inverse of a with respect to the $+$ operation. By a skew brace, we will always mean a skew left brace. We say that $(A, +)$ is the additive group of A, whereas $(A, \circ)$ is the multiplicative group of A. The identity of the additive group of A coincides with that of the multiplicative group of A, and we denote the common identity element by e. The multiplicative inverse of an element a in A is denoted by a^{-1}. We write A as a skew brace to mean the triple $(A, +, \circ)$. For any set S, we denote by $\mathfrak{P}(S)$ the power set S.

Given a skew brace A, a new operation $*$ is defined as follows:

$$a * b = -a + a \circ b - b,$$

for all a, b in A. In particular, $a * b = \lambda_a(b) - b$, for all a, b in A, where λ_a is an automorphism of $(A, +)$ and is defined by:

$$\lambda_a(b) = -a + a \circ b.$$

If X and Y are subsets of a skew brace A, The notation $X * Y$ denotes the subset of A defined by $\{x * y \mid x \in X, y \in Y\}$.

An *ideal* I of a skew brace A is a normal subgroup of both the group structures $(A, +)$ and $(A, \circ)$ such that $\lambda_a(I) \subseteq I$ for all $a \in A$. We define the *trivial* or *zero* ideal of A as the singleton set $\{e\}$ of A and denote it by 0, whereas by a *proper* ideal I, we mean I is an ideal of A and $I \neq A$. If S is a subset of A, then $\langle S \rangle$ denotes the ideal generated by S. A *maximal ideal* M is a proper ideal of A and not properly contained in another proper ideal of A. We denote the set of all ideals of A by Idl A, whereas the set of all maximal ideals A is denoted by Max A. Involving the $*$ operation, a characterisation (see CEDÓ, SMOKTUNOWICZ, & VENDRAMIN [3, Lemma 1.8, Lemma 1.9]) of normal subgroups of any skew brace to be an ideal is given in the following proposition.

Proposition 2.1. *Suppose A is a skew brace and I is a normal subgroup of $(A, +)$. Then I is an ideal of A if and only if $A * I \subseteq I$ and $I * A \subseteq I$.*

The following lemma is going to be used in the sequel. The proof of the first part of it is trivial, whereas for the second part, we refer the reader to KONOVALOV, SMOKTUNOWICZ, & VENDRAMIN [14, Lemma 3.7].

Lemma 2.2. *If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals of A, then $\bigcap_{\lambda \in \Lambda} I_\lambda$ is also an ideal of A. If I and J are ideals of A, so is $I + J$, where $I + J$ is defined as the additive subgroup of A generated by $\{i + j \mid i \in I,\ j \in J\}$.*

The binary sum of ideals of a skew brace can be extended to an "infinite sum", which we shall see in the next corollary. But before that, let us first define this infinite sum of ideals of a skew brace A. If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals of A, then the *sum* $\sum_{\lambda \in \Lambda} I_\lambda$ is a subset of A, which constitutes of elements of the form $\sum_{j=1}^{n} i_{\lambda_j}$, for some finite subset $\{\lambda_1, \ldots, \lambda_n\}$ of Λ, where $i_{\lambda_j} \in I_{\lambda_j}$. Using the fact that the sum of two ideals is an ideal, we can easily prove the following result.

Corollary 2.3. *If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals of A, then $\sum_{\lambda \in \Lambda} I_\lambda$ is also an ideal of A.*

Remark 2.4. (1). Recall that an associative ring $(R, +, \circ)$ is called a *radical ring* if $(R, *)$ is a group, where the operation $*$ is defined by $a * b = a + b - a \circ b$. RUMP [18] showed that $(R, +, *)$ is an example of a (left) *brace*. It is now natural to ask what should be a "ring-type" example of a skew brace, and here the problem starts. Since the addition operation of a ring is commutative, we can not expect an example even from a noncommutative ring. We rather need to consider a near-ring and apply the above method to have the expected example.

(2). Note that if a ring R is a radical ring, then every element of R is quasi-regular (see JACOBSON [10]) and such a ring can not have a multiplicative identity 1. Indeed, for $x \in R$,

$$x * 1 = x + 1 - x \circ 1 = x + 1 - x = 1 \neq 0,$$

a contradiction. Therefore, it follows that we can not have a "unital" brace or a "unital" skew brace (unless the braces or the skew braces are trivial). Furthermore, we observe that the identity element e of a skew brace A can not play the role of an unital element. Otherwise, A will not have any proper ideals (as by definition $e \in I$ for all ideals I of A).

3 Spectra of skew braces

The aim of this section is to introduce a Zariski topology on the set of "prime-like" ideals (see Definition 3.1) in a manner that seeks to mimic the Zariski topology on the set of prime ideas of a (commutative) ring. We formalize the definition of these ideals below. The reader will observe that our definition mimics that of prime ideals in commutative rings. We first introduce the following notation. If I and J are ideals of A, the ideal $I \circledast J$ is defined by

$$I \circledast J = \langle \{ i * j \mid i \in I,\, j \in J \} \rangle.$$

Definition 3.1. A proper ideal P of a skew brace A is called *skew-prime* if $I \circledast J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$, for all ideals I and J of A. We denote the set of all skew-prime ideals of A by $\operatorname{Spec_s} A$.

Note that $I * J$ used in the definition of prime ideals in KONOVALOV, SMOKTUNOWICZ & VENDRAMIN [14] is merely an additive subgroup of A, whereas $I \circledast J$ in Definition 3.1 is an ideal of A, which accords primeness in commutative rings.

Lemma 3.2. *If I and J are any two ideals of a skew brace A, then $I \circledast J \subseteq I \cap J$.*

Proof. If $i * j \in I \circledast J$, then $i * j = \lambda_i(j) - j \in J$ as J is an ideal. On the other hand,

$$i * j = -i + i \circ j - j = -i + j \circ \left(j^{-1} \circ i \circ j\right) - j = -i + j + \lambda_j \left(j^{-1} \circ i \circ j\right) - j \in I. \qquad \square$$

Proposition 3.3. *Suppose A is a skew brace. Then $(\operatorname{Idl} A, \subseteq, \circledast)$ is a multiplicative lattice.*

Proof. Notice that under subset inclusion $\subseteq$ relations, the zero ideal and A are respectively the bottom and top elements of $\operatorname{Idl} A$. That the pair $(\operatorname{Idl} A, \subseteq)$ is a complete lattice now follows from Lemma 2.2 and Corollary 2.3, whereas by Lemma 3.2 it follows that the complete lattice $(\operatorname{Idl} A, \circledast)$ is also multiplicative. $\square$

In a skew brace, maximal ideals need not be skew-prime. For example, in a finite two sided brace A with more than one element, each maximal ideal of A cannot be skew-prime. What we have instead is the following characterisation of maximal ideals that are skew-prime. The proof of this proposition is analogous to the proof of JESPERS, KUBAT, VAN ANTWERPEN, & L. VENDRAMIN [12, Proposition 3.6].

Proposition 3.4. *A maximal ideal M of A is skew-prime if and only if $A^2 \nsubseteq M$, where $A^2 = A \circledast A$.*

Since the operation $*$ is neither associative nor commutative, an expression like $a^n = a * a^{n-1}$ does not make sense for us. Therefore, we cannot adapt the "elementwise" definition of the radical of an ideal as we have for commutative rings. Following the definition of the radical of an ideal of a noncommutative ring (see LAM [15, Theorem 10.7]), we propose the following.

Definition 3.5. The *radical* of an ideal I in a skew brace A, denoted by $\operatorname{Rad} I$, is defined as

$$\operatorname{Rad} I = \bigcap_{I \subseteq P \in \operatorname{Spec_s} A} P.$$

The *nil radical* $\operatorname{Nil} A$ of A is the radical of the zero ideal (see Section 2) of A. So, $\operatorname{Nil} A$ is the intersection of all skew-prime ideals of A.

4 Zariski topology

Let S be a subset of a skew brace A. If $H(S) = \{P \in \operatorname{Spec_s} A \mid S \subseteq P\}$, then it is easy to see that $H(S) = H(\langle S\rangle)$. Therefore, it is sufficient to consider the set of all ideals of a skew brace brace A to define closed sets of Zariski topology on $\operatorname{Spec_s} A$.

Definition 4.1. The *Zariski* topology on $\operatorname{Spec_s} A$ is imposed by considering the collection of sets $\{H(I)\}_{I\in \mathrm{Idl}\, A}$ as closed sets, where

$$H(I) = \{P \in \operatorname{Spec_s} A \mid I \subseteq P\} \qquad (I \in \mathrm{Idl}\, A).$$

The following proposition shows that Definition 4.1 indeed makes sense for skew braces.

Proposition 4.2. *The collection $\{H(I)\}_{I\in \mathrm{Idl}\, A}$ of sets satisfies the following properties:*

(i) $H(A) = \emptyset$ *and* $H(0) = \operatorname{Spec_s} A$.

(ii) $H(I) \cup H(J) = H(I \cap J) = H(I \circledast J)$ *for all* $I, J \in \mathrm{Idl}\, A$.

(iii) $\bigcap_{\lambda\in\Lambda} H(I_\lambda) = H\left(\sum_{\lambda\in\Lambda} I_\lambda\right)$ *for all* $I_\lambda \in \mathrm{Idl}\, A$ *and* $\lambda \in \Lambda$.

Proof. (i) Since by Definition 3.1, $A \notin \operatorname{Spec_s} A$, we have $H(A) = \emptyset$. Since $\{e\} \subseteq P$ for all P in $\operatorname{Spec_s} A$, we have $H(0) = \operatorname{Spec_s} A$.

(ii) By Definition 4.1, it follows that $H(I) \cup H(J) \subseteq H(I \cap J)$ for any two ideals I and J of A. On the other hand, the containment $H(I) \cup H(J) \supseteq H(I \cap J)$ follows by Lemma 3.2. If $P \in H(I \circledast J)$, then $I \circledast J \subseteq P$, and hence $I \subseteq P$ or $J \subseteq P$, and that gives $P \in H(I) \cup H(J)$. The inclusion $H(I \circledast J) \supseteq H(I) \cup H(J)$ follows from Lemma 3.2.

(iii) Note that if $\{I_\lambda\}_{\lambda\in\Lambda}$ is a family of ideals of A, then by Corollary 2.3, $\sum_{\lambda\in\Lambda} I_\lambda$ is an ideal of A. By Definition 4.1 we also have $\bigcap_{\lambda\in\Lambda} H(I_\lambda) \supseteq H\left(\sum_{\lambda\in\Lambda} I_\lambda\right)$. Conversely, if P is in $\bigcap_{\lambda\in\Lambda} H(I_\lambda)$, then $I_\lambda \subseteq P$ for all $\lambda \in \Lambda$, and hence $\sum_{\lambda\in\Lambda} I_\lambda \subseteq P$. This proves that $\bigcap_{\lambda\in\Lambda} H(I_\lambda) \subseteq H\left(\sum_{\lambda\in\Lambda} I_\lambda\right)$. □

Lemma 4.3. *If $I \in \mathrm{Idl}\, A$, then $H(I) = H(\mathrm{Rad}\, I)$.*

Proof. The proof follows from Definition 3.5. □

Theorem 4.4. *For a subset S of* $\operatorname{Spec_s} A$, *let $K(S) = \bigcap_{I\in S} I$. The operator K has the following properties.*

(i) $K(\emptyset) = R$ *and* $K\left(\bigcup_{\lambda\in\Lambda} T_\lambda\right) = \bigcap_{\lambda\in\Lambda} K\left(T_\lambda\right)$.

(ii) *If T is a subset of X and I is an ideal of R then $KH(I) = \mathrm{Rad}\, I$, and $HK(S)$ is the closure of S in* $\operatorname{Spec_s} A$.

Proof. (i) The first assertion follows from the empty intersection property. For the second, the fact $S_\lambda \subseteq \bigcup_{\lambda\in\Lambda} S_\lambda$ implies $K(S_\lambda) \supseteq K(\bigcup_{\lambda\in\Lambda} S_\lambda)$, and hence $\bigcap_{\lambda\in\Lambda} K(S_\lambda) \supseteq K(\bigcup_{\lambda\in\Lambda} S_\lambda)$. For the other half of the inclusion, let $S_\lambda = \{I_{\alpha,\lambda}\}_{\alpha\in L}$ and let $x \in \bigcap_{\lambda\in\Lambda} K(S_\lambda)$. Then $x \in \bigcap_{\lambda\in\Lambda}(\bigcap_{\alpha\in L} I_{\alpha,\lambda})$; whence $x \in K(\bigcup_{\lambda\in\Lambda} S_\lambda)$.

(ii) For the first assertion we observe that

$$a \in KH(I) \Leftrightarrow a \in P \text{ for every } P \text{ with } P \in H(I) \Leftrightarrow a \in P \text{ for every } P \supseteq I.$$

Therefore, $KH(I) = \bigcap_{P\supseteq I} P = \mathrm{Rad}\, I$. For the second claim, if a closed set $H(X)$ (for some subset X of A) contains S, then $X \subseteq I$ for all $I \in S$; which subsequently implies $X \subseteq K(S)$ and hence $H(X) \supseteq HK(S)$. Since $T \subseteq HK(S)$, and $HK(S)$ is the smallest closed set of $\operatorname{Spec_s} A$ containing S, we have the desired claim. □

Following Kock [13], we can represent the relation between H and K categorically as follows: We observe that the poset map K is a right adjoint of the map H. The unit of the adjunction $H \dashv K$ is $\eta\colon \langle S\rangle \mapsto KH(S) = \langle S\rangle$. Hence the full subcategory $\mathrm{Fix}\,\eta = \{S \in \mathfrak{P}(A) \mid \eta_S$ is an isomorphism$\}$ is the set of ideals of R. The counit of the adjunction $H \dashv K$ is $\epsilon\colon T \mapsto HK(T) = \overline{T}$. Therefore $\mathrm{Fix}\,\epsilon = \{T \in \mathfrak{P}(\operatorname{Spec_s} A)^{\mathrm{op}} \mid \epsilon_T$ is an isomorphism$\}$ is the set of closed subsets. Therefore the adjunction $H \dashv K$ restricts to an adjoint equivalence between the categories Idl A and Closed(Spec$_s$ A). By considering open sets of the topology on Spec$_s$ A, the above isomorphism of categories become Idl $A \approx$ Open(Spec$_s$ A). The following diagram summarises the above inter-relations.

$$\begin{array}{ccccc} & \mathfrak{P}(A) & \underset{K}{\overset{H}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} & \mathfrak{P}(\mathrm{Spec_s}\, A)^{\mathrm{op}} & \\ & \uparrow & & \uparrow & \\ \mathrm{Fix}\,\eta = & \mathrm{Idl}\, A & \underset{\longleftarrow}{\overset{\longrightarrow}{\approx}} & \mathrm{Closed}(\mathrm{Spec}\, A)^{\mathrm{op}} & = \mathrm{Fix}\,\epsilon. \end{array}$$

Lemma 4.5. *Suppose A is a skew brace and $P, Q \in$ Spec$_s$ A. Then*

(i) cl$\{P\} = H(P)$*, where* cl$\{P\}$ *is the closure of* $\{P\}$.

(ii) $P \in$ cl$\{Q\}$ *if and only if* $P \in H(Q)$.

Proof. (i) Since $H(P)$ is a closed set containing P and cl$\{P\}$ is the smallest closed set containing P, we immediately have cl$\{P\} \subseteq H(P)$. For the converse, suppose cl$\{P\} = H(I)$ for some ideal I of A. Clearly, $P \in$ cl$\{P\}$, hence $I \subseteq P$. Thus any skew-prime ideal that contains P will also contain I. Hence, $H(P) \subseteq H(I) =$ cl$\{P\}$.

(ii) Observe that by (i), $P \in$ cl$\{Q\}$ if and only if $P \in H(Q)$, and this occurs if and only if $Q \subseteq P$. □

Proposition 4.6. *Every* Spec$_s$ A *is a T_0-space.*

Proof. Let P and Q be two distinct points of Spec$_s$ A such that $P \nsubseteq Q$. By Lemma 4.5 we have $Q \notin$ cl$\{P\} = H(P) \ni P$, which implies there exists a closed set $H(P)$ containing P that does not contain Q. □

With a further restriction on a skew brace, we obtain the following

Proposition 4.7. *If A is a skew brace with $A^2 \nsubseteq M$ for all $M \in$* Max A, *then* Spec$_s$ A *is a T_1-space if and only if* Spec$_s$ $A =$ Max A.

Proof. By Theorem 3.4 it follows that Spec$_s$ $A \supseteq$ Max A. Let Spec$_s$ $A \neq$ Max A. Then there exists $P \in$ Spec$_s$ A such that $P \notin$ Max A, and that implies there exists $M \in$ Max A such that $P \subsetneq M$. Then $M \in H(P) =$ cl$\{P\}$, where the equality follows from Lemma 4.5(i). Hence $\{P\}$ is not closed and hence Spec$_s$ A is not a T_1-space. Conversely, suppose that Spec$_s$ $A =$ Max A. Then $H(P) = \{P\}$ for all $P \in$ Spec$_s$ A and hence every singleton set is closed, and that implies Spec$_s$ A is a T_1-space. □

Let us recall the notion of irreducible topological spaces and some of their properties. Readers may consult GÖRTZ & WEDHORN [8] for further details on irreducibility.

Definition 4.8. If X is a topological space, a closed subset S is *irreducible* if S is not the union of two properly smaller closed subsets $S_1, S_2 \subsetneq S$. A maximal irreducible subset of a topological space X is called an *irreducible component* of X. A point x in a closed subset S is called a *generic point* of S if $S =$ cl$(\{x\})$.

Recall the following basic properties on irreducibility (see ALTMAN & KLEIMAN [1, Lemma 16.50]).

Proposition 4.9. *Let X be a topological space.*

(i) *A subspace Y of X is irreducible if and only if* cl(Y) *is irreducible.*

(ii) *The irreducible components of X are closed and cover X.*

Lemma 4.10. $\{H(P)\}_{P \in \mathrm{Spec_s}\, A}$ *are precisely the irreducible closed subsets of* Spec$_s$ A.

Proof. Let $P \in \operatorname{Spec_s} A$. Since $\{P\}$ is irreducible, by Proposition 4.9(i) and Lemma 4.5(i), so is $H(P)$. Let $H(I)$ be an irreducible closed subset of $\operatorname{Spec_s} A$ and if possible, $I \notin \operatorname{Spec_s} A$. This implies there exist ideals I_1 and I_2 such that $I_1 \nsubseteq I$ and $I_2 \nsubseteq I$, but $I_1 \circledast I_2 \subseteq I$. Then we have:

$$H(\langle I, I_1\rangle) \cup H(\langle I, I_2\rangle) = H(\langle I, I_1\rangle \cap \langle I, I_2\rangle) = H(\langle I, I_1 * I_2\rangle) = H(I),$$

where the first two equalities follow from Proposition 4.2(ii). Since $H(\langle I, I_1\rangle) \neq H(I)$ and $H(\langle I, I_2\rangle) \neq H(I)$, the closed set $H(I)$ is not irreducible, a contradiction. □

Theorem 4.11. *Every irreducible closed subset of* $\operatorname{Spec_s} A$ *has a unique generic point.*

Proof. The existence of generic point follows from Lemma 4.5 and Lemma 4.10, whereas the uniqueness part follows from Proposition 4.6. □

Note that if we have a decreasing chain of skew-prime ideals, then the intersection of the ideals in the chain gives the minimal element, which is either a skew-prime ideal or the zero ideal, and in the second case, with a vacuous argument, it is also a skew-prime ideal. This confirms the existence of the minimal skew-prime ideals of A and hence allows us to obtain the following lemma.

Lemma 4.12. *The irreducible components of* $\operatorname{Spec_s} A$ *are the closed sets $H(P)$, where P is a minimal skew-prime ideal of A.*

Proof. By Proposition 4.9(ii), irreducible components of $\operatorname{Spec_s} A$ are closed. If P is a minimal skew-prime ideal, then by Lemma 4.10, $H(P)$ is irreducible. If $H(P)$ is not a maximal irreducible subset of $\operatorname{Spec_s} A$, then there exists a maximal irreducible subset $H(Q)$ with $Q \in \operatorname{Spec_s} A$ such that $H(P) \subsetneq H(Q)$. This implies that $P \in H(Q)$ and hence $Q \subsetneq P$, contradicting the minimality property of P. □

Proposition 4.13. $\operatorname{Spec_s} A$ *is irreducible if and only if* $\operatorname{Nil} A$ *is skew-prime.*

Proof. If $\operatorname{Nil} A$ is skew-prime, then it is the minimum skew-prime ideal. Therefore, $\operatorname{Spec_s} A$ has only one maximal irreducible component, namely $H(\operatorname{Nil} A)$, and since by Proposition 4.9(ii) the maximal irreducible components cover $\operatorname{Spec_s} A$, the irreducible component $H(\operatorname{Nil} A)$ must be equal to $\operatorname{Spec_s} A$. Hence $\operatorname{Spec_s} A$ is irreducible by Lemma 4.12. Conversely, suppose $\operatorname{Spec_s} A$ is irreducible. Then by Proposition 4.9(ii), there exists $P \in \operatorname{Spec_s} A$ such that $H(P) = \operatorname{Spec_s} A$. Hence $P \subseteq Q$ for all $Q \in \operatorname{Spec_s} A$, and so is $\operatorname{Nil} A$, *i.e.*, $P \subseteq \operatorname{Nil} A$. By Definition 3.5, $\operatorname{Nil} A \subseteq P$, and hence $\operatorname{Nil} A$ is skew-prime. □

Before we discuss continuous maps, let us first recall some of the basic facts about skew brace homomorphisms and quotient skew braces.

Definition 4.14. A *skew subbrace* S of a skew brace A is a subgroup of $(A, +)$ and $(A, \circ)$. If A and B are skew braces, a *skew brace homomorphism* is a map $f\colon A \to B$ of A into B such that $f(a + a') = f(a) + f(a')$, $f(a \circ a') = f(a) \circ f(a')$ for all $a, a' \in A$. The *kernel* of f is the subset $\ker f = \{a \in A \mid f(a) = e\}$ of A. The *image* of a skew brace homomorphism f is the subset $\operatorname{im} f = \{b \in B \mid b = f(a)$ for some $a \in A\}$ of B. Let $f\colon A \to B$ be a skew brace homomorphism and let I, J be ideals of B and A respectively. The *contraction* of I and the *extension* of J are respectively denoted by I^c and J^e, and they are, respectively, $f^{-1}(I)$ and $\langle f(J)\rangle$.

Lemma 4.15. *Let I and J respectively be ideals of skew left braces A and B. Let $f\colon A \to B$ be a skew brace homomorphism. Then*

(i) $\ker f$, $f^{-1}(J)$ *are ideals of A and* $\operatorname{im} f$ *is a skew subbrace of B*;

(ii) $a \circ I = a + I$ *for every $a \in A$ and $A/I = \{a + I \mid a \in A\}$ is a skew left brace under the operation:* $(a + I) + (b + I) = (a + b) + I$ *and* $(a + I) \circ (b + I) = (a \circ b) + I$;

(iii) *there is a bijection between ideals of A/I and ideals of A containing I*;

(iv) $f(I \circledast J) = f(I) \circledast f(J)$ *for all* $I, J \in \operatorname{Idl} A$.

Some parts of the following proposition (and indeed the proofs we give) mirror their commutative ring analogues.

Proposition 4.16. *Suppose* $f\colon A \to B$ *is a skew brace homomorphism and define the map* $f^!\colon \operatorname{Spec} B \to \operatorname{Spec_s} A$ *by* $f^!(P) = f^{-1}(P)$*, where* $P \in \operatorname{Spec} B$*. Then*

(i) *every skew-prime ideal of* A *is a contracted ideal if and only if* $f^!$ *is surjective;*

(ii) *if every skew-prime ideal of* B *is an extended ideal then* $f^!$ *is injective;*

(iii) $f^!$ *is continuous;*

(iv) *if* f *is surjective, then* $\operatorname{im}(f^!)$ *is homeomorphic to* $H(\ker f)$*;*

(v) *the image of* $f^!$ *is dense in* $\operatorname{Spec_s} A$ *if and only if* $\ker f \subseteq \operatorname{Nil} A$*.*

Proof. (i) Let $P \in \operatorname{Spec_s} A$ and $P = Q^c$ for some $Q \in \operatorname{Spec} B$. Then, $f^!(Q) = f^{-1}(Q) = P$. Hence $f^!$ is surjective. Conversely, if $f^!$ is surjective, then for any $P \in \operatorname{Spec_s} A$, we have $P = f^!(Q) = Q^c$, as required.

(ii) Let $Q \in \operatorname{Spec} B$ and $Q = P^e$. Then, $f^!(Q) = P^{ec} \supseteq P$. Suppose $f^!(Q) = f^!(Q')$. Then, $P^{ec} = (P')^{ec}$, where $Q' = (P')^e$, this implies $P^{ece} = (P')^{ece}$, which implies $P^e = (P')^e$ and that $Q = Q'$.

(iii) To show $f^!$ is continuous, we first show that $f^{-1}(P) \in \operatorname{Spec_s} A$, whenever $P \in \operatorname{Spec} B$. Let $I * J \subseteq f^{-1}(P)$, where $I, J \in \operatorname{Idl} A$. Then $f(I * J) \subseteq P$, and by Lemma 4.15(iv) we have $f(I) * f(J) \subseteq P$. Since P is skew-prime, by Definition 3.1 either $f(I) \subseteq P$ or $f(J) \subseteq P$, and that implies either $I \subseteq f^{-1}(P)$ or $J \subseteq f^{-1}(P)$. If $H(I)$ is a closed subset of $\operatorname{Spec_s} A$, then for any $Q \in \operatorname{Spec} B$, we have:

$$Q \in (f^!)^{-1}(H(I)) \Leftrightarrow f^!(Q) \in H(I) \Leftrightarrow f^{-1}(Q) \in H(I) \Leftrightarrow I \subseteq f^{-1}(Q) \Leftrightarrow Q \in H(\langle f(I) \rangle).$$

(iv) By Lemma 4.15(i), $\operatorname{im}(f^!)$ and $\ker f$ are ideals of A. We first show that $\operatorname{cl}(f^!(H(I))) = H(f^{-1}(I))$ for any $I \in \operatorname{Idl} B$ and for that it is sufficient to show: $f^!(H(I)) = H(f^{-1}(I))$. We observe:

$$P \in H(f^{-1}(I)) \Leftrightarrow f(f^{-1}(I)) \subseteq f(P) \Leftrightarrow P = f^{-1}(f(P)) = f^!(f(P)) \in f^!(H(I)),$$

and by taking $I = 0$, we obtain $\operatorname{im}(f^!) = H(\ker f)$. This implies $f^!$ induces a continuous bijection between $\operatorname{im}(f^!)$ and $H(\ker f)$. The continuity of $(f^!)^{-1}$ also now follows from the above. Hence we have the desired homeomorphism.

(v) Note that $\operatorname{im}(f^!)$ is dense in $\operatorname{Spec_s} A$ if and only if $\operatorname{cl}(\operatorname{im} f^!) = H(\ker f) = \operatorname{Spec_s} A$, and this occurs if and only if $\ker f \subseteq P$ for all $P \in \operatorname{Spec_s} A$, and that holds if and only if $\ker f \subseteq \operatorname{Nil} A$. □

Corollary 4.17. *For a skew brace* A*, the spaces* $\operatorname{Spec_s} A$ *and* $\operatorname{Spec}(A/\operatorname{Nil} A)$ *are homeomorphic.*

Proof. Note that by Lemma 4.15(ii), $A/\operatorname{Nil} A$ is a skew brace. On applying Proposition 4.16(iv) on the homomorphism $A \to A/\operatorname{Nil} A$ we obtain the desired result. □

Proposition 4.18. *Let* I *be an ideal of a skew brace* A *and* $J = I^c$ *be an ideal of a skew brace* A'*. Let* $\bar{\phi}\colon A/I \to A'/J$ *be the skew brace homomorphism induced by the skew brace homomorphism* $f\colon A \to A'$*. Then the restriction of the map* $\phi_*\colon \operatorname{Spec_s} A' \to \operatorname{Spec_s} A$ *to* $H(J)$ *is the map* $\bar{\phi}_*\colon \operatorname{Spec}(A'/J) \to \operatorname{Spec}(A/I)$*.*

Proof. Note that below the left commutative diagram induces the right commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ {\scriptstyle \pi^I}\downarrow & & \downarrow{\scriptstyle \pi^J} \\ A/I & \xrightarrow{\bar{\phi}} & A'/J \end{array} \qquad\qquad \begin{array}{ccc} \operatorname{Spec_s} A & \xleftarrow{\phi_*} & \operatorname{Spec_s} A' \\ {\scriptstyle \pi^I_*}\uparrow & & \uparrow{\scriptstyle \pi^J_*} \\ \operatorname{Spec}(A/I) & \xrightarrow{\bar{\phi}_*} & \operatorname{Spec}(A'/J). \end{array}$$

Moreover, $K \in H(J)$ implies $\phi_*(K) = K^c \supseteq J^c = I^{ec} \supseteq I$, that is, $\phi_*(H(J)) \subseteq H(I)$. Now the desired result follows from Proposition 4.16(iv). □

Definition 4.19. According to JESPERS, KUBAT, VAN ANTWERPEN, & VENDRAMIN [12, Definition 4.1] the weight $\omega(A)$ of non-zero skew brace A is defined as the minimal number of elements of A needed to generate A (as an ideal). By convention, we put $\omega(A) = 1$ if $A = 0$.

Recall that a skew left brace A is said to be *Noetherian* if every ascending chain of ideals of A is eventually stationary. It follows immediately that a skew left brace is Noetherian if and only if all its ideals have finite weight. Also recall that a topological space X is said to be *Noetherian* if the closed subsets of X satisfy the descending chain condition.

Proposition 4.20. *If all ideals of A have finite weight, then* $\mathrm{Spec_s}\, A$ *is a Noetherian space.*

Proof. It suffices to show that a descending chain of closed sets $H(I_1) \supseteq H(I_2) \supseteq \cdots$ in $\mathrm{Spec_s}\, A$ satisfy descending chain condition. By Lemma 4.3, $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of ideals in A. Since A is Noetherian, it stabilizes at some $n \in \mathbb{N}$. Hence, $H(I_n) = H(I_{n+k})$ for any k. Thus $\mathrm{Spec_s}\, A$ is Noetherian. □

Corollary 4.21. *The set of minimal skew-prime ideals in a Noetherian skew brace is finite.*

Proof. By Proposition 4.20, $\mathrm{Spec_s}\, A$ is Noetherian, thus $\mathrm{Spec_s}\, A$ has finitely many irreducible components. By Lemma 4.12, every irreducible closed subset of $\mathrm{Spec_s}\, A$ is of the form $H(P)$, where P is a minimal skew-prime ideal. Thus $H(P)$ is irreducible components if and only if P is minimal skew-prime. Hence, A has only finitely many minimal skew-prime ideals. □

The following well-known result (see ALTMAN & KLEIMAN [1, Exercise 16.70]) holds for any topological space and hence for $\mathrm{Spec_s}\, A$.

Proposition 4.22. *If A is a skew brace, then the following are equivalent:*

(i) $\mathrm{Spec_s}\, A$ *is Noetherian.*

(ii) *Every open subspace of* $\mathrm{Spec_s}\, A$ *is compact.*

(iii) *Every subspace of* $\mathrm{Spec_s}\, A$ *is compact.*

Recall that in the sense of HOCHSTER [9], a topological space is called *spectral* if it is quasi-compact, sober, admits a basis of quasi-compact open subspaces that is closed under finite intersections.

Theorem 4.23. $\mathrm{Spec}(\mathrm{Idl}\, A)$ *is a spectral space.*

Proof. Note that by Proposition 3.3, $\mathrm{Idl}\, A$ is a multiplicative lattice. Since every skew brace A is finitely generated (by e) as an ideal of itself, by FACCHINI, FINOCCHIARO, & JANELIDZE [6, Theorem 11.5] we have the desired claim. □

Concluding remarks 1. The following are some natural topological questions that arise when we study Zariski topology on a skew-prime spectrum of an algebraic structure and we do not have answers to these questions for skew braces.

• It would be interesting to study S-Zariski topology and clopen topology (in the sense of JARBOUI [16] and [17] respectively) on the prime spectrum of a skew brace.

• Although the partition of unity property is a sufficient condition for compactness of skew-prime spectrum of a ring with identity (see DUBE & GOSWAMI [4]), we can not apply the same argument for skew braces. In the context of a skew brace A, when we try to apply the finite intersection property, although we obtain a finite sum representation of the identity e of A, but from that we cannot conclude the same for an arbitrary element of A.

• If a is an element of a skew brace A such that $a^2 = a$, then we immediately see that $a = e$. This means a skew brace does not have nontrivial idempotent elements. For a commutative ring R (with identity), without having any nontrivial idempotent elements implies $\mathrm{Spec}\, R$ is connected. Therefore, one may expect the same for a skew brace. Once again not having zero divisors and "multiplicative identity" in a skew brace lead to the problem of checking the connectivity.

• HOCHSTER [9] proved that the prime spectrum of a commutative ring (with 1) endowed with Zariski topology is spectral. We proved $\mathrm{Spec}(\mathrm{Idl}\, A)$ is spectral (see Theorem 4.23), but it could be interesting to know whether $\mathrm{Spec_s}\, A$ itself is spectral.

• Proposition 4.20 gives a sufficiant condition for $\mathrm{Spec_s}\, A$ to be Noetherian. But it would be nice to characterize Notherian spaces as well as obtain a similar result for Artinian spaces.

References

[1] A. B. Altman and S. L. Kleiman, *A term of commutative algebra,* Worldwide Center of Mathematics, (2018).

[2] D. Bourn, A. Facchini, and M. Pompili, *Aspects of the category SKB of skew braces*, Comm. Algebra, **51**(5), 2129–2143, (2023).

[3] F. Cedó, A. Smoktunowicz, L. Vendramin, *Skew left braces of nilpotent type*, Proc. Lond. Math. Soc., (3) **118**(6), 1367–1392, (2019).

[4] T. Dube and A. Goswami, *Ideal spaces: An extension of structure spaces of rings*, J. Algebra Appl., 22(11), 2350245 (18 pages) (2023).

[5] A. Facchini, *Algebraic structures from the point of view of complete multiplicative lattices* in Algebra and coding theory, 113–131, Contemp. Math., 785, Amer. Math. Soc., (2023).

[6] A. Facchini, C. A. Finocchiaro, and G. Janelidze, *Abstractly constructed prime spectra*, Algebra Univers., **83**(8), 1–38, (2022).

[7] L. Guarnieri and L. Vendramin. *Skew braces and the Yang-Baxter equation*, Math. Comp, **86**, 2519–2534, (2017).

[8] U. Görtz and T. Wedhorn, *Algebraic geometry I. Schemes – with examples and exercises*, 2nd ed., Springer, (2020).

[9] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Am. Math. Soc., **142**, 43–60, (1969).

[10] N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquium Publications, vol. **37**, Providence, (1956).

[11] E. Jespers, Ł. Kubat, A. Van Antwerpen, and L. Vendramin, *Factorizations of skew braces*, Math. Ann., **375** (3-4), 1649–1663, (2019).

[12] E. Jespers, Ł. Kubat, A. Van Antwerpen, and L. Vendramin, *Radical and weight of skew braces and their applications to structure groups of solutions of the Yang–Baxter equation*, Adv. Math., **385**, 107767, (2021).

[13] J. Kock, Remarks on spectra, supports, and Hochster duality, (a letter to Giordano), (2007).

[14] A. Konovalov, A. Smoktunowicz, and L. Vendramin, *On skew braces and their ideals*, Exp. Math., **30**(1), 95–104, (2021).

[15] T. Y. Lam, *A first course in noncommutative rings*, 2nd ed., Springer, (2001).

[16] N. Jarboui, A note on *S-Zariski topology*, Palest. J. Math., **11**(3), 348–350, (2022).

[17] N. Jarboui, *A note on clopen topologies*, Palest. J. Math., **11**(4), 26–27, (2022).

[18] W. Rump, *Braces, radical rings, and the quantum Yang–Baxter equation*, J. Algebra, **307**(1), 153–170, (2007).

Author information

A. Goswami, (1) Department of Mathematics and Applied Mathematics, University of Johannesburg, P.O. Box 524, Auckland Park 2006. (2) National Institute for Theoretical and Computational Sciences (NITheCS), South Africa.
E-mail: agoswami@uj.ac.za

T. Dube, (1) Department of Mathematical Sciences, University of South Africa, P.O. Box 392, 0003 Pretoria. (2) National Institute for Theoretical and Computational Sciences (NITheCS), South Africa.
E-mail: dubeta@unisa.ac.za

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