Palestine Journal of Mathematics

Vol 13(4)(2024) , 47–55

LORENTZ-SASAKIAN SPACE FORMS ON W_0-CURVATURE TENSOR

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 53C15; Secondary 53C44.

Keywords and phrases: W_0-Curvature Tensors, Lorentz-Sasakian Space Forms, Lorentzian Manifold.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper..

Abstract In this article, Lorentz-Sasakian space forms on W_0-curvature tensor are investigated. Riemann, Ricci, concircular, and projective curvature tensors are discussed for $(2m+1)$-dimensional Lorentz-Sasakian space forms, and with the help of these curvature tensors, some special curvature conditions established on the W_0- curvature tensor are examined. Also, with the help of these curvature conditions, some properties of Lorentz-Sasakian space forms such as Einstein manifold, η-Einstein manifold, and real space form have been obtained.

1 Introduction

$\phi-$sectional curvature plays an important role in the Sasakian manifold. If the $\phi-$sectional curvature of a Sasakian manifold is constant, then the manifold is a Sasakian-space-form [1]. P. Alegre and D. Blair described generalized Sasakian space forms [2]. P. Alegre and D. Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. P. Alegre and A. Carriazo later discussed generalized indefinite Sasakian space forms [3]. Generalized indefinite Sasakian space forms are also called Lorentz-Sasakian space forms, and Lorentz manifolds are of great importance for Einstein's theory of Relativity. Sasakian space forms, generalized Sasakian space forms, and Lorentz-Sasakian space forms have been discussed by many scientists and important properties of these manifolds have been obtained ([4]-[8]). Similarly, Lorentz-Sasakian space forms can be considered in some important works on the subject ([9]-[10]).

In this article, Lorentz-Sasakian space forms on W_0-curvature tensor are investigated. Riemann, Ricci, concircular, and projective curvature tensors are discussed for $(2m+1)$-dimensional Lorentz-Sasakian space forms, and with the help of these curvature tensors, some special curvature conditions established on the W_0- curvature tensor are examined. Also, with the help of these curvature conditions, some properties of Lorentz-Sasakian space forms such as Einstein manifold, η-Einstein manifold, and real space form have been obtained.

2 Preliminary

Let $\tilde{M}$ be a $(2m+1)-$dimensional Lorentz manifold. If the $\tilde{M}$ Lorentz manifold with (ϕ, ξ, η, g) structure tensors satisfies the following conditions, this manifold is called a Lorentz-Sasakian manifold

$$\phi^2\varkappa_1 = -\varkappa_1 + \eta(\varkappa_1)\xi, \eta(\xi) = 1, \eta(\phi\varkappa_1) = 0,$$

$$g(\phi\varkappa_1, \phi\varkappa_2) = g(\varkappa_1, \varkappa_2) + \eta(\varkappa_1)\eta(\varkappa_2), \eta(\varkappa_1) = -g(\varkappa_1, \xi),$$

$$\left(\tilde{\nabla}_{\varkappa_1}\phi\right)\varkappa_2 = -g(\varkappa_1, \varkappa_2)\xi - \eta(\varkappa_2)\varkappa_1, \left(\tilde{\nabla}_{\varkappa_1}\xi\right) = -\phi\varkappa_1,$$

where, $\tilde{\nabla}$ is the Levi-Civita connection according to the Riemann metric g [5].

The plane section Π in $T_{\varkappa_1}\tilde{M}$. If the Π plane is spanned by $\varkappa_1$ and $\phi\varkappa_1$, this plane is called the ϕ-section. The curvature of the ϕ-section is called the ϕ-sectional curvature. If the Lorentz-Sasakian manifold has a constant ϕ-sectional curvature, this manifold is called the Lorentz-Sasakian space form and is denoted by $\tilde{M}(c)$. The curvature tensor of the Lorentz-Sasakian space form $\tilde{M}(c)$ is defined as

$$\begin{aligned}
&\tilde{R}(\varkappa_1, \varkappa_2)\varkappa_3 = \left(\tfrac{c-3}{4}\right)\{g(\varkappa_2, \varkappa_3)\varkappa_1 - g(\varkappa_1, \varkappa_3)\varkappa_2\} \\
&+ \left(\tfrac{c+1}{4}\right)\{g(\varkappa_1, \phi\varkappa_3)\phi\varkappa_2 - g(\varkappa_2, \phi\varkappa_3)\phi\varkappa_1 \\
&+2g(\varkappa_1, \phi\varkappa_2)\phi\varkappa_3 + \eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1 - \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 \\
&+g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\xi - g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\xi\},
\end{aligned} \tag{1}$$

for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \chi\left(\tilde{M}(c)\right)$. If we choose $\varkappa_1 = \xi$ in (1), we get

$$\tilde{R}(\xi, \varkappa_2)\varkappa_3 = -g(\varkappa_2, \varkappa_3)\xi - \eta(\varkappa_3)\varkappa_2. \tag{2}$$

If we choose $\varkappa_2 = \xi$ in (1), we have

$$\tilde{R}(\varkappa_1, \xi)\varkappa_3 = g(\varkappa_1, \varkappa_3)\xi + \eta(\varkappa_3)\varkappa_1. \tag{3}$$

Similarly, if we choose $\varkappa_3 = \xi$ in (1), we obtain

$$\tilde{R}(\varkappa_1, \varkappa_2)\xi = \eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2. \tag{4}$$

Also, if we take the inner product of both sides of (1) by the vector field ξ, we get

$$\eta\left(\tilde{R}(\varkappa_1, \varkappa_2)\varkappa_3\right) = \frac{c-1}{2}\left[g(\varkappa_2, \varkappa_3)\eta(\varkappa_1) - g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\right]. \tag{5}$$

Lemma 2.1. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. The following relations are provided for the Lorentz-Sasakian space forms [5].*

$$S(\varkappa_1, \varkappa_2) = \left[\frac{(m+2)c-(3m-2)}{2}\right]g(\varkappa_1, \varkappa_2) + \left[\frac{(c+1)(m+1)}{2}\right]\eta(\varkappa_1)\eta(\varkappa_2), \tag{6}$$

$$Q\varkappa_1 = \left[\frac{(m+2)c-(3m-2)}{2}\right]\varkappa_1 - \left[\frac{(c+1)(m+1)}{2}\right]\eta(\varkappa_1)\xi, \tag{7}$$

$$S(\varkappa_1, \xi) = -\left[\frac{(c+1)-4m}{2}\right]\eta(\varkappa_1), \tag{8}$$

$$Q\xi = -\left[\frac{(c+1)-4m}{2}\right]\xi. \tag{9}$$

W_0-curvature tensor on a semi-Riemann manifold was described by M. Tripathi and P. Gunam [5]. W_0-curvature tensor is the $(1,3)-$type tensor field defined as

$$W_0(\varkappa_1, \varkappa_2)\varkappa_3 = R(\varkappa_1, \varkappa_2)\varkappa_3 - \frac{1}{2m}\left[S(\varkappa_2, \varkappa_3)\varkappa_1 - g(\varkappa_1, \varkappa_3)Q\varkappa_2\right] \tag{10}$$

on a semi-Riemann manifold (M, g), where R, S, Q are the Riemann curvature tensor, Ricci curvature tensor, Ricci operator and of manifold, respectively.

Lemma 2.2. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. The following relations are provided for the W_0-curvature tensor in $\tilde{M}(c)$.*

$$W_0(\xi, \varkappa_2)\varkappa_3 = -\frac{(m+2)(c+1)}{4m}\left[g(\varkappa_2, \varkappa_3)\xi + \eta(\varkappa_3)\varkappa_2\right], \tag{11}$$

$$W_0(\varkappa_1, \xi)\varkappa_3 = \left(\frac{c+1}{4m}\right)\left[g(\varkappa_1, \varkappa_3)\xi + \eta(\varkappa_3)\varkappa_1\right], \tag{12}$$

$$\begin{aligned} W_0(\varkappa_1, \varkappa_2)\xi = {} & \tfrac{(m+2)(c+1)}{4m}\left[-\eta(\varkappa_1)\varkappa_2\right. \\ & \left.+\eta(\varkappa_1)\eta(\varkappa_2)\xi\right] + \left(\tfrac{c+1}{4m}\right)\eta(\varkappa_2)\varkappa_1, \end{aligned} \tag{13}$$

for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \chi(\tilde{M}(c))$.

3 Lorentz-Sasakian Space Forms On W_0-Curvature Tensor

In this section, the Lorentz-Sasakian space form $\tilde{M}(c)$ will be characterized according to some curvature conditions established between W_0-curvature tensor and Riemann, Ricci, concircular, and projective curvature tensor. Let us first examine the special curvature condition established between the W_0 and $\tilde{R}$ curvature tensors. Let us state and prove the following theorem.

Theorem 3.1. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. If $\tilde{M}(c)$ satisfies the curvature condition $W_0(\varkappa_1, \varkappa_2) \cdot \tilde{R} = 0$, $\tilde{M}(c)$ is the real space form.*

Proof. Let's assume that $\tilde{M}(c)$ satisfies the condition

$$\left(W_0(\varkappa_1, \varkappa_2) \cdot \tilde{R}\right)(\varkappa_4, \varkappa_5, \varkappa_3) = 0,$$

for all $\varkappa_1, \varkappa_2, \varkappa_4, \varkappa_5, \varkappa_3 \in \chi(\tilde{M}(c))$. So, we can easily write

$$\begin{aligned} & W_0(\varkappa_1, \varkappa_2)\tilde{R}(\varkappa_4, \varkappa_5)\varkappa_3 - \tilde{R}(W_0(\varkappa_1, \varkappa_2)\varkappa_4, \varkappa_5)\varkappa_3 \\ & -\tilde{R}(\varkappa_4, W_0(\varkappa_1, \varkappa_2)\varkappa_5)\varkappa_3 - \tilde{R}(\varkappa_4, \varkappa_5)W_0(\varkappa_1, \varkappa_2)\varkappa_3 = 0 \end{aligned} \tag{14}$$

If we choose $\varkappa_1 = \xi$ in (14) and use (11), we get

$$\begin{aligned} & -\tfrac{(m+2)(c+1)}{4m}\left\{g\left(\varkappa_2, \tilde{R}(\varkappa_4, \varkappa_5)\varkappa_3\right)\xi + \eta\left(\tilde{R}(\varkappa_4, \varkappa_5)\varkappa_3\right)\varkappa_2\right. \\ & -g(\varkappa_2, \varkappa_4)\tilde{R}(\xi, \varkappa_5)\varkappa_3 - \eta(\varkappa_4)\tilde{R}(\varkappa_2, \varkappa_5)\varkappa_3 \\ & -g(\varkappa_2, \varkappa_5)\tilde{R}(\varkappa_4, \xi)\varkappa_3 - \eta(\varkappa_5)\tilde{R}(\varkappa_4, \varkappa_2)\varkappa_3 \\ & \left.-g(\varkappa_2, \varkappa_3)\tilde{R}(\varkappa_4, \varkappa_5)\xi - \eta(\varkappa_3)\tilde{R}(\varkappa_4, \varkappa_5)\varkappa_2\right\} = 0. \end{aligned} \tag{15}$$

If we make use of (2), (3), (4) in (15), we have

$$\begin{aligned} & -\tfrac{(m+2)(c+1)}{4m}\left\{g\left(\varkappa_2, \tilde{R}(\varkappa_4, \varkappa_5)\varkappa_3\right)\xi + \eta\left(\tilde{R}(\varkappa_4, \varkappa_5)\varkappa_3\right)\varkappa_2\right. \\ & +g(\varkappa_2, \varkappa_4)g(\varkappa_5, \varkappa_3)\xi + g(\varkappa_2, \varkappa_4)\eta(\varkappa_3)\varkappa_5 \\ & -\eta(\varkappa_4)\tilde{R}(\varkappa_2, \varkappa_5)\varkappa_3 - g(\varkappa_2, \varkappa_5)g(\varkappa_4, \varkappa_3)\xi \\ & -g(\varkappa_2, \varkappa_5)\eta(\varkappa_3)\varkappa_4 - \eta(\varkappa_5)\tilde{R}(\varkappa_4, \varkappa_2)\varkappa_3 \\ & -g(\varkappa_2, \varkappa_3)\eta(\varkappa_5)\varkappa_4 + g(\varkappa_2, \varkappa_3)\eta(\varkappa_4)\varkappa_5 \\ & \left.-\eta(\varkappa_3)\tilde{R}(\varkappa_4, \varkappa_5)\varkappa_2\right\} = 0. \end{aligned} \tag{16}$$

If we choose $\varkappa_4 = \xi$ in (16) and use (2), we obtain

$$\frac{(m+2)(c+1)}{4m}\left\{\tilde{R}(\varkappa_2,\varkappa_5)\varkappa_3 - [g(\varkappa_2,\varkappa_3)\varkappa_5 - g(\varkappa_5,\varkappa_3)\varkappa_2]\right\} = 0.$$

It is clear from the last equation that $\tilde{M}(c)$ is the real space form. This completes the proof. □

Corollary 3.2. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. If $\tilde{M}(c)$ satisfies the curvature condition $W_0(\varkappa_1,\varkappa_2)\cdot\tilde{R} = 0$, $\tilde{M}(-1)$ is hyperbolic space.*

Let us examine the special curvature condition established between the W_0 and S Ricci curvature tensors. Let us state and prove the following theorem.

Theorem 3.3. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. If $\tilde{M}(c)$ satisfies the curvature condition $W_0(\varkappa_1,\varkappa_2)\cdot S = 0$, $\tilde{M}(c)$ is either a real space form or an Einstein manifold.*

Proof. Let's assume that $\tilde{M}(c)$ satisfies the condition

$$(W_0(\varkappa_1,\varkappa_2)\cdot S)(\varkappa_4,\varkappa_5) = 0,$$

for all $\varkappa_1,\varkappa_2,\varkappa_4,\varkappa_5 \in \chi\left(\tilde{M}(c)\right)$. So, we can easily write

$$S(W_0(\varkappa_1,\varkappa_2)\varkappa_4,\varkappa_5) + S(\varkappa_4,W_0(\varkappa_1,\varkappa_2)\varkappa_5) = 0. \tag{17}$$

If we choose $\varkappa_1 = \xi$ in (17) and use (11), we get

$$\begin{aligned}&-\tfrac{(m+2)(c+1)}{4m}\{g(\varkappa_2,\varkappa_4)S(\xi,\varkappa_5) + \eta(\varkappa_4)S(\varkappa_2,\varkappa_5)\\ &+g(\varkappa_2,\varkappa_5)S(\xi,\varkappa_4) + \eta(\varkappa_5)S(\varkappa_4,\varkappa_2)\} = 0.\end{aligned} \tag{18}$$

If we choose $\varkappa_4 = \xi$ in (18) and use (8), we have

$$-\frac{(m+2)(c+1)}{4m}\left\{S(\varkappa_2,\varkappa_5) - \frac{(c+1)-4m}{2}g(\varkappa_2,\varkappa_5)\right\} = 0.$$

It is clear from the last equation that either $c=-1$, that is, $\tilde{M}(c)$ is a real space form, or $\tilde{M}(c)$ is an Einstein manifold. □

Let us now Lorentz-Sasakian space form characterize with the curvature condition established on the W_0-curvature tensor itself. Let us state and prove the following theorem.

Theorem 3.4. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. If $\tilde{M}(c)$ satisfies the curvature condition $W_0(\varkappa_1,\varkappa_2)\cdot W_0 = 0$, $\tilde{M}(c)$ is either a real space form or an $\eta-$Einstein manifold provided $c \neq 3, c \neq -1$.*

Proof. Let's assume that $\tilde{M}(c)$ satisfies the condition

$$(W_0(\varkappa_1,\varkappa_2)\cdot W_0)(\varkappa_4,\varkappa_5,\varkappa_3) = 0,$$

for all $\varkappa_1,\varkappa_2,\varkappa_4,\varkappa_5,\varkappa_3 \in \chi\left(\tilde{M}(c)\right)$. So, we can easily write

$$\begin{aligned}&W_0(\varkappa_1,\varkappa_2)W_0(\varkappa_4,\varkappa_5)\varkappa_3 - W_0(W_0(\varkappa_1,\varkappa_2)\varkappa_4,\varkappa_5)\varkappa_3\\ &-W_0(\varkappa_4,W_0(\varkappa_1,\varkappa_2)\varkappa_5)\varkappa_3 - W_0(\varkappa_4,\varkappa_5)W_0(\varkappa_1,\varkappa_2)\varkappa_3 = 0.\end{aligned} \tag{19}$$

If we choose $\varkappa_1 = \xi$ in (19) and use (11), we get

$$\begin{aligned}&-\tfrac{(m+2)(c+1)}{4m}\{g(\varkappa_2,W_0(\varkappa_4,\varkappa_5)\varkappa_3)\xi + \eta(W_0(\varkappa_4,\varkappa_5)\varkappa_3)\varkappa_2\\ &-g(\varkappa_2,\varkappa_4)W_0(\xi,\varkappa_5)\varkappa_3 - \eta(\varkappa_4)W_0(\varkappa_2,\varkappa_5)\varkappa_3\\ &-g(\varkappa_2,\varkappa_5)W_0(\varkappa_4,\xi)\varkappa_3 - \eta(\varkappa_5)W_0(\varkappa_4,\varkappa_2)\varkappa_3\\ &-g(\varkappa_2,\varkappa_3)W_0(\varkappa_4,\varkappa_5)\xi - \eta(\varkappa_3)W_0(\varkappa_4,\varkappa_5)\varkappa_2\} = 0.\end{aligned} \tag{20}$$

If we replace $(11),(12),(13)$ in (20) and make the necessary arrangements, we obtain

$$\begin{aligned} &-\tfrac{(m+2)(c+1)}{4m}\{g(\varkappa_2, W_0(\varkappa_4,\varkappa_5)\varkappa_3)\xi+\eta(W_0(\varkappa_4,\varkappa_5)\varkappa_3)\varkappa_2 \\ &+\tfrac{(m+2)(c+1)}{4m}[g(\varkappa_2,\varkappa_4)g(\varkappa_5,\varkappa_3)\xi+g(\varkappa_2,\varkappa_4)\eta(\varkappa_3)\varkappa_5 \\ &+g(\varkappa_2,\varkappa_3)\eta(\varkappa_4)\varkappa_5-g(\varkappa_2,\varkappa_3)\eta(\varkappa_4)\eta(\varkappa_5)\xi] \\ &-\eta(\varkappa_4)W_0(\varkappa_2,\varkappa_5)\varkappa_3-\tfrac{c+1}{4m}[g(\varkappa_2,\varkappa_5)g(\varkappa_4,\varkappa_3)\xi \\ &+g(\varkappa_2,\varkappa_5)\eta(\varkappa_3)\varkappa_4+g(\varkappa_2,\varkappa_3)\eta(\varkappa_5)\varkappa_4] \\ &-\eta(\varkappa_5)W_0(\varkappa_4,\varkappa_2)\varkappa_3-\eta(\varkappa_3)W_0(\varkappa_4,\varkappa_5)\varkappa_2\}=0. \end{aligned} \tag{21}$$

If we choose $\varkappa_4=\xi$ in (21) and use $(10),(11)$, we get

$$\begin{aligned} &-\tfrac{(m+2)(c+1)}{4m}\Big\{\tfrac{(m+2)(c+1)}{4m}[g(\varkappa_2,\varkappa_3)\varkappa_5-g(\varkappa_5,\varkappa_3)\varkappa_2] \\ &-\tilde{R}(\varkappa_2,\varkappa_5)\varkappa_3+\tfrac{1}{2m}S(\varkappa_5,\varkappa_3)\varkappa_2-\tfrac{1}{2m}g(\varkappa_2,\varkappa_3)Q\varkappa_5 \\ &-\tfrac{c+1}{4m}g(\varkappa_2,\varkappa_3)\eta(\varkappa_5)\xi\Big\}=0. \end{aligned} \tag{22}$$

If we choose $\varkappa_3=\xi$ in (22), we get

$$\begin{aligned} &-\tfrac{(m+2)(c+1)}{4m}\Big\{\tfrac{(m+1)(c+1)}{4m}\eta(\varkappa_5)\varkappa_2+\tfrac{c+1}{4m}\eta(\varkappa_2)\eta(\varkappa_5)\xi \\ &+\tfrac{m(3-c)-2(c+1)}{4m}\eta(\varkappa_2)\varkappa_5+\tfrac{1}{2m}\eta(\varkappa_2)Q\varkappa_5\Big\}=0. \end{aligned} \tag{23}$$

If we choose $\varkappa_2=\xi$ in (23) and later, we take inner product of both side of the equation by $\varkappa_3\in\chi(\tilde{M}(c))$, we obtain

$$-\frac{(m+2)(c+1)}{4m}\left\{\frac{1}{2m}S(\varkappa_5,\varkappa_3)+\lambda_1 g(\varkappa_5,\varkappa_3)+\lambda_2\eta(\varkappa_5)\eta(\varkappa_3)\right\}=0, \tag{24}$$

where

$$\lambda_1=\frac{m(3-c)-2(c+1)}{4m},$$

and

$$\lambda_2=-\frac{(m+2)(c+1)}{4m}.$$

Thus it is clear from (24) that $\tilde{M}(c)$ is either a real space form or an $\eta-$Einstein manifold provided $c\neq-1, c\neq 3$. This completes the proof. □

Corollary 3.5. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. If $\tilde{M}(c)$ satisfies the curvature condition $W_0(\varkappa_1,\varkappa_2)\cdot W_0=0$, $\tilde{M}(c)$ is Einstein manifold if and only if $c=-1$ and $c\neq 3$.*

Now let's take the projective curvature tensor defined as

$$P(\varkappa_1,\varkappa_2)\varkappa_3=R(\varkappa_1,\varkappa_2)\varkappa_3-\frac{1}{2m}[S(\varkappa_2,\varkappa_3)\varkappa_1-S(\varkappa_1,\varkappa_3)\varkappa_2] \tag{25}$$

and characterize the manifold for the curvature condition written with the help of the projective curvature tensor. If we choose $\varkappa_1=\xi,\varkappa_2=\xi,$ and $\varkappa_3=\xi$ respectively in (25), we have

$$\begin{aligned} P(\xi,\varkappa_2)\varkappa_3=&-\tfrac{(m+2)(c+1)}{4m}g(\varkappa_2,\varkappa_3)\xi \\ &-\tfrac{c+1}{4m}\eta(\varkappa_3)\varkappa_2-\tfrac{(c+1)(m+1)}{4m}\eta(\varkappa_2)\eta(\varkappa_3)\xi, \end{aligned} \tag{26}$$

$$P(\varkappa_1,\xi)\varkappa_3 = \tfrac{(m+2)(c+1)}{4m} g(\varkappa_1,\varkappa_3)\xi$$
$$+\tfrac{c+1}{4m}\eta(\varkappa_3)\varkappa_1 + \tfrac{(c+1)(m+1)}{4m}\eta(\varkappa_1)\eta(\varkappa_3)\xi, \tag{27}$$

$$P(\varkappa_1,\varkappa_2)\xi = \frac{c+1}{4m}\left[\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2\right]. \tag{28}$$

Let us state and prove the following theorem.

Theorem 3.6. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. If $\tilde{M}(c)$ satisfies the curvature condition $W_0(\varkappa_1,\varkappa_2)\cdot P = 0$, $\tilde{M}(c)$ is a real space form.*

Proof. Let's assume that $\tilde{M}(c)$ satisfies the condition

$$(W_0(\varkappa_1,\varkappa_2)\cdot P)(\varkappa_4,\varkappa_5,\varkappa_3) = 0,$$

for all $\varkappa_1,\varkappa_2,\varkappa_4,\varkappa_5,\varkappa_3 \in \chi\left(\tilde{M}(c)\right)$. So, we can easily write

$$W_0(\varkappa_1,\varkappa_2)P(\varkappa_4,\varkappa_5)\varkappa_3 - P(W_0(\varkappa_1,\varkappa_2)\varkappa_4,\varkappa_5)\varkappa_3$$
$$-P(\varkappa_4,W_0(\varkappa_1,\varkappa_2)\varkappa_5)\varkappa_3 - P(\varkappa_4,\varkappa_5)W_0(\varkappa_1,\varkappa_2)\varkappa_3 = 0. \tag{29}$$

If we choose $\varkappa_1 = \xi$ in (29) and use (11), we get

$$-\tfrac{(m+2)(c+1)}{4m}\{g(\varkappa_2,P(\varkappa_4,\varkappa_5)\varkappa_3)\xi + \eta(P(\varkappa_4,\varkappa_5)\varkappa_3)\varkappa_2$$
$$-g(\varkappa_2,\varkappa_4)P(\xi,\varkappa_5)\varkappa_3 - \eta(\varkappa_4)P(\varkappa_2,\varkappa_5)\varkappa_3$$
$$-g(\varkappa_2,\varkappa_5)P(\varkappa_4,\xi)\varkappa_3 - \eta(\varkappa_5)P(\varkappa_4,\varkappa_2)\varkappa_3$$
$$-g(\varkappa_2,\varkappa_3)P(\varkappa_4,\varkappa_5)\xi - \eta(\varkappa_3)P(\varkappa_4,\varkappa_5)\varkappa_2\} = 0. \tag{30}$$

If it is written instead of (26), (27), (28) in (30) and necessary arrangements are made, we have

$$-\tfrac{(m+2)(c+1)}{4m}\{g(\varkappa_2,P(\varkappa_4,\varkappa_5)\varkappa_3)\xi + \eta(P(\varkappa_4,\varkappa_5)\varkappa_3)\varkappa_2$$
$$+\tfrac{(m+2)(c+1)}{4m}\left[g(\varkappa_2,\varkappa_4)g(\varkappa_5,\varkappa_3)\xi - g(\varkappa_2,\varkappa_5)g(\varkappa_4,\varkappa_3)\xi\right]$$
$$+\tfrac{(m+1)(c+1)}{4m}\left[g(\varkappa_2,\varkappa_4)\eta(\varkappa_5)\eta(\varkappa_3)\xi - g(\varkappa_2,\varkappa_5)\eta(\varkappa_4)\eta(\varkappa_3)\xi\right]$$
$$+\tfrac{c+1}{4m}[g(\varkappa_2,\varkappa_4)\eta(\varkappa_3)\varkappa_5 - g(\varkappa_2,\varkappa_5)\eta(\varkappa_3)\varkappa_4 - g(\varkappa_2,\varkappa_3)\eta(\varkappa_5)\varkappa_4$$
$$+g(\varkappa_2,\varkappa_3)\eta(\varkappa_4)\varkappa_5] - \eta(\varkappa_4)P(\varkappa_2,\varkappa_5)\varkappa_3 - \eta(\varkappa_5)P(\varkappa_4,\varkappa_2)\varkappa_3$$
$$-\eta(\varkappa_3)P(\varkappa_4,\varkappa_5)\varkappa_2\} = 0. \tag{31}$$

If we choose $\varkappa_4 = \xi$ in (31) and use (25), (26), we get

$$-\tfrac{(m+2)(c+1)}{4m}\left\{\tfrac{(m+1)(c+1)}{4m}[g(\varkappa_2,\varkappa_5)\eta(\varkappa_3)\xi - \eta(\varkappa_3)\eta(\varkappa_5')\varkappa_2\right.$$
$$+g(\varkappa_2,\varkappa_3)\eta(\varkappa_5)\xi + 2\eta(\varkappa_2)\eta(\varkappa_5)\eta(\varkappa_3)\xi] - \tilde{R}(\varkappa_2,\varkappa_5)\varkappa_3$$
$$-\tfrac{(m+2)(c+1)}{4m}g(\varkappa_5,\varkappa_3)\varkappa_2 + \tfrac{1}{2m}S(\varkappa_5,\varkappa_3)\varkappa_2 - \tfrac{1}{2m}S(\varkappa_2,\varkappa_3)\varkappa_5$$
$$\left.+\tfrac{c+1}{4m}g(\varkappa_2,\varkappa_3)\varkappa_5\right\} = 0. \tag{32}$$

If we choose $\varkappa_3 = \xi$ in (23) and later, we take inner product of both side of the equation by $\xi \in \chi\left(\tilde{M}(c)\right)$, we obtain

$$-\frac{(m+2)(m+1)(c+1)^2}{16m^2}\left\{g(\varkappa_2,\varkappa_5) - \eta(\varkappa_2)\eta(\varkappa_5)\right\} = 0.$$

It is clear from the last equation that $c = -1$ ie $\tilde{M}(c)$ is the real space form. This completes the proof. □

Finally, let's take the concircular curvature tensor defined as

$$\tilde{Z}(\varkappa_1,\varkappa_2)\varkappa_3 = R(\varkappa_1,\varkappa_2)\varkappa_3 - \frac{r}{2m(2m+1)}\left[g(\varkappa_2,\varkappa_3)\varkappa_1 - g(\varkappa_1,\varkappa_3)\varkappa_2\right] \tag{33}$$

and characterize the manifold for the curvature condition written with the help of the concircular curvature tensor. If we choose $\varkappa_1 = \xi, \varkappa_2 = \xi$ and $\varkappa_3 = \xi$ respectively in (33), we have

$$\tilde{Z}(\xi,\varkappa_2)\varkappa_3 = -\lambda\left[g(\varkappa_2,\varkappa_3)\xi + \eta(\varkappa_3)\varkappa_2\right], \tag{34}$$

$$\tilde{Z}(\varkappa_1,\xi)\varkappa_3 = \lambda\left[g(\varkappa_1,\varkappa_3)\xi + \eta(\varkappa_3)\varkappa_1\right], \tag{35}$$

$$\tilde{Z}(\varkappa_1,\varkappa_2)\xi = -\left[\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2\right], \tag{36}$$

where

$$\lambda = \left[1 + \frac{r}{2m(2m+1)}\right]$$

and r is the scalar curvature of $\tilde{M}(c)$. Let us state and prove the following theorem.

Theorem 3.7. *Let $\tilde{M}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. If $\tilde{M}(c)$ satisfies the curvature condition $W_0(\varkappa_1,\varkappa_2)\cdot\tilde{Z} = 0$, $\tilde{M}(c)$ is a real space form.*

Proof. Let's assume that $\tilde{M}(c)$ satisfies the condition

$$\left(W_0(\varkappa_1,\varkappa_2)\cdot\tilde{Z}\right)(\varkappa_4,\varkappa_5,\varkappa_3) = 0,$$

for all $\varkappa_1,\varkappa_2,\varkappa_4,\varkappa_5,\varkappa_3 \in \chi\left(\tilde{M}(c)\right)$. So, we can easily write

$$\begin{aligned} &W_0(\varkappa_1,\varkappa_2)\tilde{Z}(\varkappa_4,\varkappa_5)\varkappa_3 - \tilde{Z}(W_0(\varkappa_1,\varkappa_2)\varkappa_4,\varkappa_5)\varkappa_3 \\ &-\tilde{Z}(\varkappa_4,W_0(\varkappa_1,\varkappa_2)\varkappa_5)\varkappa_3 - \tilde{Z}(\varkappa_4,\varkappa_5)W_0(\varkappa_1,\varkappa_2)\varkappa_3 = 0. \end{aligned} \tag{37}$$

If we choose $\varkappa_1 = \xi$ in (37) and use (11), we get

$$\begin{aligned} &-\tfrac{(m+2)(c+1)}{4m}\left\{g\left(\varkappa_2,\tilde{Z}(\varkappa_4,\varkappa_5)\varkappa_3\right)\xi + \eta\left(\tilde{Z}(\varkappa_4,\varkappa_5)\varkappa_3\right)\varkappa_2\right. \\ &-g(\varkappa_2,\varkappa_4)\tilde{Z}(\xi,\varkappa_5)\varkappa_3 - \eta(\varkappa_4)\tilde{Z}(\varkappa_2,\varkappa_5)\varkappa_3 \\ &-g(\varkappa_2,\varkappa_5)\tilde{Z}(\varkappa_4,\xi)\varkappa_3 - \eta(\varkappa_5)\tilde{Z}(\varkappa_4,\varkappa_2)\varkappa_3 \\ &\left.-g(\varkappa_2,\varkappa_3)\tilde{Z}(\varkappa_4,\varkappa_5)\xi - \eta(\varkappa_3)\tilde{Z}(\varkappa_4,\varkappa_5)\varkappa_2\right\} = 0. \end{aligned} \tag{38}$$

If it is written instead of $(34)\,,(35)\,,(36)$ in (38) and necessary arrangements are made, we have

$$\begin{aligned}
&-\tfrac{(m+2)(c+1)}{4m}\left\{g\left(\varkappa_2,\tilde{Z}\left(\varkappa_4,\varkappa_5\right)\varkappa_3\right)\xi+\eta\left(\tilde{Z}\left(\varkappa_4,\varkappa_5\right)\varkappa_3\right)\varkappa_2\right.\\
&+\lambda\left[g\left(\varkappa_2,\varkappa_4\right)g\left(\varkappa_5,\varkappa_3\right)\xi+g\left(\varkappa_2,\varkappa_4\right)\eta\left(\varkappa_3\right)\varkappa_5\right.\\
&-g\left(\varkappa_2,\varkappa_5\right)g\left(\varkappa_4,\varkappa_3\right)\xi-g\left(\varkappa_2,\varkappa_5\right)\eta\left(\varkappa_3\right)\varkappa_4\\
&\left.-g\left(\varkappa_2,\varkappa_3\right)\eta\left(\varkappa_5\right)\varkappa_4+g\left(\varkappa_2,\varkappa_3\right)\eta\left(\varkappa_4\right)\varkappa_5\right]\\
&-\eta\left(\varkappa_4\right)\tilde{Z}\left(\varkappa_2,\varkappa_5\right)\varkappa_3-\eta\left(\varkappa_5\right)\tilde{Z}\left(\varkappa_4,\varkappa_2\right)\varkappa_3\\
&\left.-\eta\left(\varkappa_3\right)\tilde{Z}\left(\varkappa_4,\varkappa_5\right)\varkappa_2\right\}=0.
\end{aligned} \tag{39}$$

If we choose $\varkappa_4=\xi$ in (39) and use $(33)\,,(34)$, we obtain

$$-\frac{(m+2)\,(c+1)}{4m}\left\{-\tilde{R}\left(\varkappa_2,\varkappa_5\right)\varkappa_3-\left[g\left(\varkappa_5,\varkappa_3\right)\varkappa_2-g\left(\varkappa_2,\varkappa_3\right)\varkappa_5\right]\right\}=0.$$

This completes the proof. □

4 Conclusion

In this article, Lorentz-Sasakian space forms on W_0-curvature tensor are investigated. Riemann, Ricci, concircular, and projective curvature tensors are discussed for $(2m+1)$-dimensional Lorentz-Sasakian space forms, and with the help of these curvature tensors, some special curvature conditions established on the W_0- curvature tensor are examined. Also, with the help of these curvature conditions, some properties of Lorentz-Sasakian space forms such as Einstein manifold, η-Einstein manifold, and real space form have been obtained.

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Received: 2022-06-10
Accepted: 2023-09-30