Palestine Journal of Mathematics

Vol 13(4)(2024) , 483–503

On a Bi-nonlocal fourth-order difference problem involving the $p(k)$-Laplacian type operator

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: Bi-nonlocal problem, Kirchhoff type equation, Mixed boundary conditions, Mountain pass Theorem.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract In this paper, we study a class of bi-nonlocal fourth-order discrete problems involving $p(k)$-Laplacian operator in a finite dimensional Banach space. By using the variational method and the (S_+) mapping theory, we investigate the existence and multiplicity of nontrivial solutions, subject to the condition that the parameters are sufficiently large.

1 Introduction

The study of fourth-order difference equations with nonstandard $p(k)$-Laplacian operator is an attractive topic and has been the object of considerable attention in recent years.
Solving these problems requires several approaches such as fixed point theorems, lower and upper solutions, and Brower degree (see [1, 3, 5, 6, 7, 8, 9, 12, 19] and references therein). It is well known that critical point theory and variational methods are important tools to deal with the problems of differential equations.
The main goal of this work is to investigate under different assumptions on data, the existence and multiplicity of nontrivial weak solutions for the following nonlinear Mixed boundary value problem

$$\begin{cases} M_1(L_1(u))(-\triangle^2\varphi(k-2,\triangle^2 u(k-2))+q(k)|u(k)|^{p(k)-2}u(k)) \\ =\lambda M_2(L_2(u))f(k,u(k))+\mu g(k,u(k)), \quad k\in\mathbb{Z}[1,T], \\ u(-1)=u(T+2)=\triangle u(-1)=\triangle u(T+1)=0, \end{cases} \tag{1.1}$$

where $T \geq 2$ is a fixed positive integer, $\mathbb{Z}[a,b]$ denotes the discrete interval $\{a, a+1,\dots,b-1,b\}$ with a and b integers such that $a<b$, $\triangle u(k)=u(k+1)-u(k)$ is the forward difference operator, $\lambda,\mu>0$ are parameters, $p:\mathbb{Z}[-1,T+2]\to(2,\infty)$,

$q : \mathbb{Z}[1, T+2] \to (1, \infty)$ are given function,

$$L_1(u) = \sum_{k=1}^{T+2} \left(\frac{H(|\triangle^2 u(k-2)|^{p(k-2)})}{p(k-2)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right), \quad \varphi(k, v) = h(|v|^{p(k)})|v|^{p(k)-2} v$$

with increasing function h from $\mathbb{R}$ into $\mathbb{R}$, $H(t) = \int_0^t h(s)\, ds$, $L_2(u) = \sum_{k=1}^{T} F(k, u)$, where $F(k, u) = \int_0^u f(k, s)\, ds$ and $f, g : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Moreover, the functions $M_1, M_2 : [0, \infty) \to \mathbb{R}$ are continuous and nondecreasing functions.

For the rest of this article, we will use the notations

$$p^+ := \max_{k \in \mathbb{Z}[-1, T+2]} p(k), \quad p^- := \min_{k \in \mathbb{Z}[-1, T+2]} p(k),$$

$$q^+ := \max_{k \in \mathbb{Z}[1, T+2]} q(k), \quad \overline{q} := \sum_{k=1}^{T+2} q(k).$$

The study concerning the discrete fourth-order anisotropic problems has only been started, (see [24], [28]), and has been followed by other authors (see [25], [26], [29]).

To our knowledge, the first work which deals with the fourth-order discrete problems with exponent variables was done by Leszczynski (see [25]).

Recently, Nonlocal discrete problems involving the Kirchhoff operator type have been studied by many authors, and many important and interesting results have been established, we refer to [16, 23, 30, 34, 35].

If we take $h(t) = 1$ and $M_1(t) = M_2(t) = 1$ in problem (1.1), we obtain a fourth-order discrete problem with two parameters involving $p(k)$-Laplacian type operator

$$\begin{cases} -\triangle^2 \left(|\triangle^2 u(k-2)|^{p(k-2)-2} \triangle^2 u(k-2) \right) + q(k)|u(k)|^{p(k)-2} u(k) \\ = \lambda f(k, u(k)) + \mu g(k, u(k)), \quad k \in \mathbb{Z}[1, T], \\ u(-1) = u(T+2) = \triangle u(-1) = \triangle u(T+1) = 0. \end{cases} \tag{1.2}$$

If $\mu = 0$ in problem (1.2), we obtain the problem studied by Moghadam and al. in [29]. The authors investigate the existence of nontrivial solutions and their approach is based on a new critical point theorem due to Galewski and Galewska (see [13]).

In [11], a class of bi-nonlocal $p(x)$-Kirchhoff type problems with Dirichlet boundary conditions was initiated by Fan, considering the following Dirichlet boundary problem

$$\begin{cases} -a \left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\, dx \right) \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) = b \left(\int_\Omega F(x, u)\, dx \right) f(x, u), \quad x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{cases} \tag{1.3}$$

where a, and b are continuous functions. Under suitable conditions on a, b, and the Ambrosetti-Rabinowitz type condition on the nonlinear term f, the author proved the existence of at least a non-trivial solution or the existence of infinitely many problem solutions (1.3) by using variational methods.
In their paper [2], G. A. Afrouzi and al. investigate a class of bi-nonlocal problems with nonlinear Neumann boundary conditions and sublinear terms at infinity. Using (S_+) mapping theory and variational methods, the authors establish the existence of at least two non-trivial weak solutions for the problem provided that the parameters are large enough.
F. Jaafri and al. in [20], studied a bi-nonlocal elliptic problem involving $p(x)$-Biharmonic operator. By applying variational methods and under adequate conditions, they prove the existence of nontrivial weak solutions.
The authors in [4], are interested in the study of the multiplicity of weak solutions for a bi-nonlocal fractional $p(x,\cdot)-$Kirchhoff type problems. They have used the general three critical points theorem obtained by B. Ricceri (see [32]) as the basis for their approach.
In [31], A. Ouaziz and al., investigate a class of fractional Kirchhoff $p(\cdot;\cdot)-$Laplacian problems with variable exponents and indefinite weights on a smooth bounded domain. Using the Mountain pass theorem combined with variational techniques, the authors prove the existence of a nontrivial solution.
As the first equation in (1.1) contains the terms $M_i,\ i=1,2$, it is no longer a pointwise identity; therefore it is often called a nonlocal problem. This problem models several physical and biological systems, where u describes a process that depends on the average of itself, such as the population density. It should be noticed that problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2}=\left(T_0+\frac{Ea}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.4}$$

where $\rho>0$ is the mass per unit length, T_0 is the base tension, E is the Young modulus, a is the area of cross section and L is the initial length of the string.
Nonlocal equation (1.4) is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. This equation was first presented by Kirchhoff in 1876 (see [22]).
In [23], Koné and al. studied a class of discrete Mixed boundary value problems

$$\begin{cases}-M(A(k-1,\triangle u(k-1)))\triangle\left(a(k-1,\triangle u(k-1))\right)+|u(k)|^{p(k)-2}u(k)=f(k), & k\in\mathbb{Z}\left[1,T\right],\\ u(0)=\triangle u(T)=0.\end{cases} \tag{1.5}$$

By the direct method of the calculus of variation, they showed the existence of at least one weak solution.
Deng in [10], studied a class of fourth-order nonlinear difference equations. By using the critical point method, the author establishes various sets of sufficient conditions for the nonexistence and existence of solutions for mixed boundary value problems and gives some new results.
In [18], Hammouti and El Amrouss consider the boundary value problem for a fourth-order nonlinear p-Laplacian difference equation and prove the existence of at least two

nontrivial solutions. Their approach is mainly based on the variational method and critical point theory.
Inspired by the papers mentioned above, in this paper, we investigate the existence of weak solutions with Mixed boundary conditions for a nonlocal fourth-order problem. Under some conditions on data f and g, we obtain a multiplicity result by applying the minimum principle associated with the mountain pass theorem. The main result of this paper complements and improves some previous ones for the superlinear case when the Ambrosetti-Rabinowitz type conditions are imposed on the nonlinearities.
The remaining part of this article is organized as follows. Section 2 is devoted to mathematical preliminaries, several important inequalities, and the abstract critical point theorem. Section 3 is divided into two important parts, in the first one, we establish the necessary tools for the proofs of the existence of nontrivial solutions under some conditions on the functions f and g. In the second one, we prove the existence of at least two nontrivial weak solutions.

2 Preliminaries

In this section, we first establish the variational framework associated with problem (1.1). We consider the T-dimensional Banach space

$$E = \{u : \mathbb{Z}[-1, T+2] \longrightarrow \mathbb{R} \text{ such that } u(-1) = u(T+2) = \triangle u(-1) = \triangle u(T+1) = 0\}$$

equipped with the norm

$$\|u\| = \left(\sum_{k=1}^{T+2} \left(|\triangle^2 u(k-2)|^{p^-} + q(k)|u(k)|^{p^-} \right) \right)^{1/p^-}.$$

In the space E, we will also introduce the norm

$$\|u\|_{p^+} = \left(\sum_{k=1}^{T+2} \left(|\triangle^2 u(k-2)|^{p^+} + q(k)|u(k)|^{p^+} \right) \right)^{1/p^+}$$

and the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \mu > 0 : \sum_{k=1}^{T+2} \left(\left| \frac{\triangle u(k-2)}{\mu} \right|^{p(k-2)} + q(k) \left| \frac{u(k)}{\mu} \right|^{p(k)} \right) \leq 1 \right\}.$$

In the sequel, we will use the following inequality.

$$K\|u\|_{p^+} \leq \|u\| \leq 2^{\frac{p^+ - p^-}{p^+ p^-}} K\|u\|_{p^+}, \tag{2.1}$$

where $K := (\max\{T+2,\overline{q}\})^{\frac{p^+-p^-}{p^+p^-}}$.
Indeed, by weighted Hölder's inequality, we know that

$$\begin{aligned}\sum_{k=1}^{T+2}\left|\triangle^2 u(k-2)\right|^{p^-} &\leq \left(\sum_{k=1}^{T+2}(1)^{\frac{p^+}{p^+-p^-}}\right)^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T+2}\left(\left|\triangle^2 u(k-2)\right|^{p^-}\right)^{\frac{p^+}{p^-}}\right)^{\frac{p^-}{p^+}} \\ &\leq (T+2)^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T+2}|\triangle^2 u(k-2)|^{p^+}\right)^{\frac{p^-}{p^+}}.\end{aligned}$$

Using the same arguments, one has

$$\begin{aligned}\sum_{k=1}^{T+2} q(k)\left|u(k)\right|^{p^-} &\leq \left(\sum_{k=1}^{T+2} q(k)\right)^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T+2} q(k)\left(|u(k)|^{p^-}\right)^{\frac{p^+}{p^-}}\right)^{\frac{p^-}{p^+}} \\ &\leq \overline{q}^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T+2} q(k)|u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}.\end{aligned}$$

Then, we get from the above inequalities and the fact that $\dfrac{p^-}{p^+}\leq 1$,

$$\begin{aligned}\|u\|^{p^-} &\leq (\max\{T+2,\overline{q}\})^{\frac{p^+-p^-}{p^+}}\left(\left(\sum_{k=1}^{T+1}|\triangle^2 u(k-2)|^{p^+}\right)^{\frac{p^-}{p^+}}+\left(\sum_{k=1}^{T+2} q(k)|u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}\right) \\ &\leq 2^{1-\frac{p^-}{p^+}}(\max\{T+2,\overline{q}\})^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T+2}\left(|\triangle^2 u(k-2)|^{p^+}+|u(k)|^{p^+}\right)\right)^{p^-/p^+}.\end{aligned}$$

Therefore,

$$\|u\|\leq 2^{\frac{p^+-p^-}{p^+p^-}}(\max\{T+2,\overline{q}\})^{\frac{p^+-p^-}{p^+p^-}}\|u\|_{p^+}.$$

On the other hand, we get by weighted Hölder's inequality and the fact that $\dfrac{p^+}{p^-}\geq 1$,

$$\begin{aligned}\|u\|_{p^+}^{p^+} &\leq (\max\{T+2,\overline{q}\})^{\frac{p^--p^+}{p^-}}\left(\left(\sum_{k=1}^{T+2}|\triangle^2 u(k-2)|^{p^-}\right)^{\frac{p^+}{p^-}}+\left(\sum_{k=1}^{T+2} q(k)|u(k)|^{p^-}\right)^{\frac{p^+}{p^-}}\right) \\ &\leq (\max\{T+2,\overline{q}\})^{\frac{p^--p^+}{p^-}}\left(\sum_{k=1}^{T+2}\left(|\triangle^2 u(k-2)|^{p^-}+q(k)|u(k)|^{p^-}\right)\right)^{p^+/p^-}.\end{aligned}$$

Therefore,

$$(\max\{T+2,\overline{q}\})^{\frac{p^+-p^-}{p^+p^-}}\|u\|_{p^+}\leq\|u\|.$$

We conclude that

$$K\|u\|_{p^+} \le \|u\| \le 2^{\frac{p^+-p^-}{p^+p^-}} K\|u\|_{p^+}.$$

Moreover, we will also make use of the following norms,

$$\|u\|_\infty := \max\{|u(k)| : k \in \mathbb{Z}[-1, T+2]\}, \quad \text{for all} \quad u \in E.$$

Note that for any $u \in E$ and any $k \in \mathbb{Z}[1, T]$, we have (see [29])

$$\|u\|_\infty \le \frac{(T+2)^{\dfrac{2p^- - 1}{p^-}}}{4}\|u\|.$$

Since E is of finite dimension, therefore there exists constants $0 < \theta_1 < \theta_2$ such that

$$\theta_1\|u\|_{p(\cdot)} \le \|u\| \le \theta_2\|u\|_{p(\cdot)}. \tag{2.2}$$

Now, let $\phi : E \longrightarrow \mathbb{R}$ be defined by

$$\phi(u) = \sum_{k=1}^{T+2}\left(|\triangle^2 u(k-2)|^{p(k-2)} + q(k)|u(k)|^{p(k)}\right). \tag{2.3}$$

If $u \in E$ then the following properties hold.

$$\|u\|_{p(\cdot)} < 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^+} \le \phi(u) \le \|u\|_{p(\cdot)}^{p^-}, \tag{2.4}$$

$$\|u\|_{p(\cdot)} > 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^-} \le \phi(u) \le \|u\|_{p(\cdot)}^{p^+}, \tag{2.5}$$

$$\|u_n - u\|_{p(\cdot)} \to 0 \quad \Rightarrow \quad \phi(u_n - u) \to 0 \;\; n \to +\infty. \tag{2.6}$$

To state the main result of this paper, we assume the following assumptions:

(M_1) There exist two positive constants m_0, m_1 such that

$$m_0 \le M_1(t) \le m_1, \quad \forall\, t \ge 0.$$

(M_2) There exist two positive constants $m_2, m_3 > 0$ such that

$$m_2 \le M_2(t) \le m_3, \quad \forall\, t \ge 0.$$

(H_1) $h : [0, +\infty) \to \mathbb{R}$ is an increasing continuous function and there exists $c_1, c_2 > 0$ such that

$$c_1 \le h(t) \le c_2, \quad \forall\, t \ge 0.$$

(H_2) There exists constant $c > 0$ such that

$$\left(h(|\xi|^{p(k)})|\xi|^{p(k)-2}\xi - h(|\eta|^{p(k)})|\eta|^{p(k)-2}\eta\right)(\xi - \eta) \ge c|\xi - \eta|^{p(k)}.$$

(F_1) $f : \mathbb{Z}[1, T+2] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(k,t)| \leq C_1(1 + |t|^{p(k)-1}) \text{ and for all } (k,t) \in \mathbb{Z}[1, T+2] \times \mathbb{R}$$

(G_1) $g : \mathbb{Z}[1, T+2] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|g(k,t)| \leq C_2(1 + |t|^{p(k)-1}) \text{ and for all } (k,t) \in \mathbb{Z}[1, T+2] \times \mathbb{R}$$

Let C be a positive constant. We say that a C^1-function $\gamma : \mathbb{R} \to [0, +\infty)$ verifies the property (Γ) if and only if

$$\gamma(t) \leq C|t|^p \quad \forall\, t \in \mathbb{R}, \tag{Γ}$$

where p is either p^- or p^+.
We consider four functions $K_i,\ i = 1, 2, 3, 4$, satisfying property (Γ) such that

$$K_i(t) \leq \begin{cases} C|t|^{p^+} & \text{for } \ i = 1, 3 \\ \\ C|t|^{p^-} & \text{for } \ i = 2, 4 \end{cases}$$

We introduce the following assumptions on the behavior of F and G at origin and at infinity:

(F_2) $\displaystyle\limsup_{t\to 0} \frac{F(k,t)}{K_1(t)} \leq 0$ uniformly in $k \in \mathbb{Z}[1, T+2]$;

(F_3) $\displaystyle\limsup_{t\to +\infty} \frac{F(k,t)}{K_2(t)} \leq 0$ uniformly in $k \in \mathbb{Z}[1, T+2]$;

(G_2) $\displaystyle\limsup_{t\to 0} \frac{G(k,t)}{K_3(t)} \leq 0$ uniformly in $k \in \mathbb{Z}[1, T+2]$;

(G_3) $\displaystyle\limsup_{t\to +\infty} \frac{G(k,t)}{K_4(t)} \leq 0$ uniformly in $k \in \mathbb{Z}[1, T+2]$;

where $F(k,t) = \displaystyle\int_0^t f(k,s)\, ds$ and $G(k,t) = \displaystyle\int_0^t g(k,s)\, ds$.
We can see that there are many functions K_i satisfying the property (Γ), for example

$$K_i(t) := \begin{cases} \dfrac{C}{3}|t|^{p^+} & \text{for } \ i = 1, 3 \\ \\ \dfrac{C}{3}|t|^{p^-} & \text{for } \ i = 2, 4. \end{cases}$$

Taking a C^1-function $w : \mathbb{R} \to \mathbb{R}$ such that $\lim\limits_{|t|\to 0} w(t) = \lim\limits_{|t|\to +\infty} w(t) = 0$, we deduce that $F(x,t) = G(x,t) = \dfrac{C}{3} w(t)|t|^p$ verify the conditions $(F_1) - (F_3)$ and $(G_1) - (G_3)$, where $p = p^+$ or $p = p^-$.

It is noticed that the function $h(t) = 1 + \dfrac{t}{\sqrt{1+t^2}}$ verify the hypothesis $(H_1) - (H_2)$. In this case, the problem (1.1) with $(H_1)-(H_2)$ can be considered as a generalized capillarity problem.
Next, we introduce the following quotients

$$S_{p^-,E} = \inf_{u \in E \setminus \{0\}} \frac{\sum_{k=1}^{T+2} \left(|\triangle^2 u(k-2)|^{p^-} + q(k)|u(k)|^{p^-} \right)}{\sum_{k=1}^{T+2} |u(k)|^{p^-}},$$

and

$$S_{p^+,E} = \inf_{u \in E \setminus \{0\}} \frac{\sum_{k=1}^{T+2} \left(|\triangle^2 u(k-2)|^{p^+} + q(k)|u(k)|^{p^+} \right)}{\sum_{k=1}^{T+2} |u(k)|^{p^+}}.$$

In the sequel, we need some definitions and lemmas which will be used later.

Definition 2.1. An element $u \in E$ is a critical point of the functional $J : E \to \mathbb{R}$ if

$$\langle J'(u), v \rangle = 0, \quad \text{for all} \quad v \in E.$$

Definition 2.2. Let E be a Banach space. Let $J \in C^1(E, \mathbb{R}$. For any sequence $\{u_n\} \subset E$, if $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to +\infty$ possesses a convergent subsequence, then we say that J satisfies the Palais–Smale condition ((PS) condition for short).

Definition 2.3. We say that a sequence $\{u_n\} \subset E$ is said to satisfy the $(PS)_c$ condition if

$$J(u_n) \to c \in \mathbb{R} \ \text{ and } \ J'(u_n) \to 0 \ \text{ as } \ n \to \infty.$$

Theorem 2.4. ([27]). *If the functional $J : E \to \mathbb{R}$ is weakly lower semicontinuous and coercive, i.e. $\lim_{\|x\| \to \infty} J(x) = \infty$, then there exists $x_0 \in E$ such that $\inf_{x \in E} J(x) = J(x_0)$. Moreover, if J has bounded linear Gâteaux derivative on E, then x_0 is also a critical point of J, i.e. $J'(x_0) = 0$.*

Lemma 2.5. *(Mountain pass lemma in [1]) Let $J \in C^1(E, \mathbb{R})$ satisfy the (PS) condition. Suppose that*

(i) $j(0) = 0$;

(ii) there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$;

(iii) there exists $u_1 \in E$ $\|u_1\| > \rho$ such that $J(u_1) < \alpha$.

Then J has a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$\inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u),$$

where $\Gamma = \{g \in C(E, \mathbb{R}) / g(0) = 0, g(1) = u_1\}$.

Let us define the functionals $\Phi, \Psi : E \to \mathbb{R}$ as follows.

$$\Phi(u) = \widehat{M_1}(L_1(u)), \tag{2.7}$$

$$\Psi(u) := \widehat{M_2}(L_2(u)), \tag{2.8}$$

$$I(u) := \sum_{k=1}^{T} G(k,u), \tag{2.9}$$

where

$$L_1(u) := \sum_{k=1}^{T+2} \left(\frac{H(|\triangle^2 u(k-2)|^{p(k-2)})}{p(k-2)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right), \tag{2.10}$$

$$L_2(u) := \sum_{k=1}^{T+2} F(k,u), \tag{2.11}$$

with

$$H(t) = \int_0^t h(s)\,ds, \quad F(k,\xi) = \int_0^\xi f(k,\tau)d\tau, \quad G(k,\xi) = \int_0^\xi f(k,\tau)d\tau.$$

The energy functional associated with the problem (1.1) is defined as $J_{\cdot}E \longrightarrow \mathbb{R}$,

$$J(u) = \Phi(u) - \lambda\Psi(u) - \mu I(u). \tag{2.12}$$

Thus, it is easy to verify that Φ, Ψ, and I are three functionals of class $C^1(E,\mathbb{R})$ whose Gâteaux derivatives at the point $u \in E$ are given by

$$\langle \Phi'(u), v\rangle = M_1(L_1(u))(\sum_{k=1}^{T+2}(\varphi(k-2, |\triangle^2 u(k-2)|^{p(k-1)-2}\triangle^2 u(k-2)\triangle^2 v(k-2) \tag{2.13}$$
$$q(k)|u(k)|^{p(k)-2}u(k)v(k))),$$

$$\langle \Psi'(u), v\rangle = M_2(L_2(u)) \sum_{k=1}^{T+2} f(k,u(k))v(k), \tag{2.14}$$

$$\langle I'(u), v\rangle = \sum_{k=1}^{T+2} g(k,u(k))v(k), \tag{2.15}$$

for all $u, v \in E$. We observe that

$$\langle J'(u), v\rangle = \langle \Phi'(u), v\rangle - \lambda \langle \Psi'(u), v\rangle - \mu \langle I'(u), v\rangle .$$

Definition 2.6. We say that $u \in E$ is a weak solution to the problem (1.1) if

$$M_1(L_1(u))(\sum_{k=1}^{T+2}(h(|\triangle^2 u(k-2)|^{p(k-2)})|\triangle^2 u(k-2)|^{p(k-2)-2}\triangle^2 u(k-2)\triangle^2 v(k-2)$$

$$+q(k)|u(k)|^{p(k)-2}u(k)v(k))) = \lambda M_2(L_2(u))\sum_{k=1}^{T+2} f(k,u(k))v(k) + \mu\sum_{k=1}^{T+2} g(k,u(k))v(k), \tag{2.16}$$

for any $v \in E$.

Note that $J \in C^1(E,\mathbb{R})$ with

$$\langle J'(u),v\rangle = M_1(L_1(u))(\sum_{k=1}^{T+1} h(|\triangle^2 u(k-2)|^{p(k-2)})|\triangle^2 u(k-2)|^{p(k-2)-2}\triangle^2 u(k-2)\triangle^2 v(k-2)$$

$$+\sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2}u(k)v(k)) - \lambda M_2(L_2(u))\sum_{k=1}^{T+2} f(k,u(k))v(k) - \mu\sum_{k=1}^{T+2} g(k,u(k))v(k)$$

for all $u, v \in E$.
Since $u(-1) = u(T+2) = \triangle u(-1) = \triangle u(T+1) = 0$, it is clear that

$$M_1(L_1(u))(\sum_{k=1}^{T+2}(h(|\triangle^2 u(k-2)|^{p(k-2)})|\triangle^2 u(k-2)|^{p(k-2)-2}\triangle^2 u(k-2)\triangle^2 v(k-2)$$

$$+q(k)|u(k)|^{p(k)-2}u(k)v(k)))$$

$$= \sum_{k=1}^{T+2} M_1(L_1(u))(-\triangle^2(h(|\triangle^2 u(k-2)|^{p(k-2)})|\triangle^2 u(k-2)|^{p(k-2)-2}\triangle^2 u(k-2))$$
$$+q(k)|u(k)|^{p(k)-2}u(k))v(k).$$

Then,

$$\langle J'(u),v\rangle = \sum_{k=1}^{T+2} M_1(L_1(u))(-\triangle^2(h(|\triangle^2 u(k-2)|^{p(k-2)})|\triangle^2 u(k-2)|^{p(k-2)-2}\triangle^2 u(k-2))$$

$$+q(k)|u(k)|^{p(k)-2}u(k))v(k) - \lambda\sum_{k=1}^{T+2} M_2(L_2(u))f(k,u(k))v(k) - \mu\sum_{k=1}^{T+2} g(k,u(k))v(k).$$

Thus, the critical points of J are exactly the solutions to the problem (1.1).

Now, we recall some auxiliary results to be used throughout the paper.

Lemma 2.7. *Let $u \in E$ and $\|u\| > 1$. Then,*

$$\sum_{k=1}^{T+2}\left(|\triangle^2 u(k-2)|^{p(k-2)} + q(k)|u(k)|^{p(k)}\right) \geq \|u\|^{p^-} - (1+q^+)T - 1.$$

Proof. Let $u \in E$ be fixed. By a similar argument as in [16], we define

$$\beta_k := \begin{cases} p^+ & \text{if } |\triangle^2 u(k-2)| \leq 1 \\ p^- & \text{if } |\triangle^2 u(k-2)| > 1 \end{cases} \qquad \text{and} \qquad \delta_k := \begin{cases} p^+ & \text{if } |u(k)| \leq 1 \\ p^- & \text{if } |u(k)| > 1, \end{cases}$$

for each $k \in \mathbb{Z}[1, T+2]$.
For $u \in E$ with $\|u\| > 1$, one has

$$\begin{aligned}
&\sum_{k=1}^{T+2} \left(|\triangle^2 u(k-1)|^{p(k-1)} + q(k)|u(k)|^{p(k)} \right) \\
&\geq \sum_{k=1,\beta_k=p^+}^{T+2} \left|\triangle^2 u(k-2)\right|^{p^+} + \sum_{k=1,\beta_k=p^-}^{T+2} \left|\triangle^2 u(k-2)\right|^{p^-} \\
&+ \sum_{k=1,\delta_k=p^+}^{T+2} q(k)\,|u(k)|^{p^+} + \sum_{k=1,\delta_k=p^-}^{T+2} q(k)\,|u(k)|^{p^-} \\
&= \sum_{k=}^{T+2} |\triangle^2 u(k-1)|^{p^-} - \sum_{k=1,\beta_k=p^+}^{T+2} \left(|\triangle^2 u(k-2)|^{p^-} - |\triangle^2 u(k-2)|^{p^+} \right) \\
&+ \sum_{k=1}^{T+2} q(k)|u(k)|^{p^-} - q^+ \sum_{k=1,\delta_k=p^+}^{T+2} \left(|u(k)|^{p^-} - |u(k)|^{p^+} \right) \\
&\geq \sum_{k=1}^{T+2} |\triangle^2 u(k-2)|^{p^-} - (T+2) + \sum_{k=1}^{T} q(k)|u(k)|^{p^-} - q^+(T+2) \\
&= \|u\|^{p^-} - (1+q^+)(T+2).
\end{aligned}$$

□

3 Existence of nontrivial weak solutions

In this section, we study the existence of weak solutions of (1.1).

Lemma 3.1. *The functional L_1 given in* (2.10) *is weakly lower semicontinuous.*

Proof. Since H is a convex and nondecreasing function. Moreover $t \to |t|^{p(k)}$ is convex, therefore L_1 is convex.
Thus, it is enough to show that L_1 is lower semicontinuous. For this, we fix $u \in E$ and $\epsilon > 0$. Since L_1 is convex, we deduce that for any $v \in E$

$$
\begin{aligned}
L_1(v) \quad & \geq L_1(u) + \langle L_1'(u), v-u\rangle \\
& \geq L_1(u) - \sum_{k=1}^{T+2}(h(|\triangle^2 u(k-2)|^{p(k-2)})|\triangle^2 u(k-2)|^{p(k-2)-1}|\triangle^2 v(k-2) - \triangle^2 u(k-2)| \\
& -q(k)|u(k)|^{p(k)-1}|v(k)-u(k)|) \\
& \geq L_1(u) - c_2\sum_{k=1}^{T+2}|\triangle^2 u(k-2)|^{p(k-2)-1}|\triangle^2 v(k-2) - \triangle^2 u(k-2)| \\
& -\sum_{k=1}^{T+2} q(k)|u(k)|^{p(k)-1}|v(k)-u(k)| \\
& \geq L_1(u) - c_2\left(\sum_{k=1}^{T+2}|\triangle^2 u(k-2)|^{\frac{p^-(p(k-2)-1)}{p^--1}}\right)^{\frac{p^--1}{p^-}}\left(\sum_{k=1}^{T+2}\left|\Delta^2 v(k-1) - \Delta^2 u(k-2)\right|^{p^-}\right)^{\frac{1}{p^-}} \\
& -\left(\sum_{k=1}^{T+2} q(k)|u(k)|^{\frac{p^-(p(k)-1)}{p^--1}}\right)^{\frac{p^--1}{p^-}}\left(\sum_{k=1}^{T+2} q(k)\,|v(k)-u(k)|^{p^-}\right)^{\frac{1}{p^-}} \\
& \geq L_1(u) - \overline{K}\left\{\left(\sum_{k=1}^{T+2}\left|\triangle^2 v(k-2) - \triangle^2 u(k-2)\right|^{p^-}\right)^{\frac{1}{p^-}} + \left(\sum_{k=1}^{T+2} q(k)\,|v(k)-u(k)|^{p^-}\right)^{\frac{1}{p^-}}\right.
\end{aligned}
$$

where $\overline{K} = \max(K_1, K_2)$ with

$$
K_1 = c_2\left(\sum_{k=1}^{T+2}|\triangle^2 u(k-2)|^{\frac{p^-(p(k-2)-1)}{p^--1}}\right)^{\frac{p^--1}{p^-}} \quad \text{and} \quad K_2 = \left(\sum_{k=1}^{T+2} q(k)|u(k)|^{\frac{p^-(p(k)-1)}{p^--1}}\right)^{\frac{p^--1}{p^-}}.
$$

Taking the above inequalities and $\frac{1}{p^-} \leq 1$ into account, we get

$$
\begin{aligned}
L_1(v) \quad & \geq L_1(u) - 2^{\frac{p^--1}{p^-}}\max(K_1, K_2)\|v-u\| \\
& \geq I(u) - \epsilon
\end{aligned}
$$

for all $v \in E$ with $\|v-u\| < \delta = \dfrac{\epsilon}{2^{\frac{p^--1}{p^-}}\max(K_1, K_2)}$. □

Lemma 3.2. *The functional* Φ *given in* (2.7) *is weakly lower semicontinuous.*

Proof. Let $\{u_n\}$ be a sequence that converges weakly in E. By using Lemma (3.2), we have

$$
\liminf_{n\to+\infty} L_1(u_n) \geq L_1(u),
$$

and from the continuity and monotonicity of the function $t \to \widehat{M_1}(t)$, we get

$$\liminf_{n\to+\infty} \widehat{M_1}(L_1(u_n)) \geq \widehat{M_1}(\liminf_{n\to+\infty} L_1(u_n)) \geq \widehat{M_1}(L_1(u)) = \Phi(u).$$

So, the functional Φ is weakly lower semicontinuous in E. □

Lemma 3.3. *The functionals L_2 and I given by (2.9) and (2.11) are weakly continuous.*

Proof. Let $\{u_n\}$ be a sequence converging weakly to u in E. We want to show that

$$\lim_{n\to+\infty} \sum_{k=1}^{T+2} F(k, u_n(k)) = \sum_{k=1}^{T+2} F(k, u(k)).$$

By using (F_1), we have

$$\begin{aligned}\left|\sum_{k=1}^{T+2} [F(k, u_n(k)) - F(k, u(k))]\right| &\leq \sum_{k=1}^{T+2} |f(k, u + \theta_n(u_n - u)|\, |u_n - u| \\ &\leq C_1 \sum_{k=1}^{T+2} \left(1 + |u + \theta_n(u_n - u)|^{p(k)-1}\right) |u_n - u| \\ &\leq C_1 \sum_{k=1}^{T+2} \left(1 + |u + \theta_n(u_n - u)|^{p^*-1}\right) |u_n - u| \\ &\leq C_1 \left[(T+2)^{1-\frac{1}{p^-}} + \|u + \theta_n(u_n - u)\|^{p^*-1}\right] \|u_n - u\|\end{aligned}$$

where $0 \leq \theta_n(k) \leq 1$ and

$$\beta^{p^*} := \begin{cases} \beta^{p^+} & \text{if } \ \beta > 1 \\ \beta^{p^-} & \text{if } \ \beta \leq 1. \end{cases}$$

Moreover, the sequence $\{\|u + \theta_n(u_n - u)\|\}$ is bounded in E. Hence, passing to the limit we obtain the result for F.
In the same way, one shows that

$$\lim_{n\to+\infty} \sum_{k=1}^{T+2} G(k, u_n(k)) = \sum_{k=1}^{T+2} G(k, u(k)).$$

□

Lemma 3.4. *The functional J is weakly lower semicontinuous.*

Proof. For all $u \in E$, we have $J(u) = \Phi(u) - \lambda\Psi(u) - \mu I(u)$.
By Lemma 3.2, the functional Φ is weakly lower semicontinuous.
It remains to prove that Ψ and I are weakly lower semicontinuous. Let $\{u_n\}$ be a sequence that converges weakly in E. By the weak continuity of the functional L_2, we have

$$\liminf_{n\to+\infty} L_2(u_n) = L_2(u).$$

Combining this with the continuity of the function $t \to \widehat{M_2}(t)$, we get

$$\liminf_{n\to+\infty} \Psi(u_n) = \liminf_{n\to+\infty} \widehat{M_2}(L_2(u_n)) = \widehat{M_2}(\liminf_{n\to+\infty} L_2(u_n)) = \widehat{M_2}(L_2(u)) = \Psi(u).$$

In the same manner, we prove that I is weakly lower semicontinuous.
Thus, we deduce that J is weakly lower semicontinuous in E.

□

Lemma 3.5. *The functional J is coercive and bounded from below.*

Proof. To prove the coercivity of J, we may assume that $\|u\| > 1$ and we get from (M_1) and Lemma 2.7, one has

$$\begin{aligned}
\Phi(u) = \quad & \widehat{M_1}(L_1(u)) \\
\geq \quad & m_0 L_1(u) \\
\geq \quad & m_0 \sum_{k=1}^{T+2} \left(\frac{H(|\triangle^2 u(k-2)|^{p(k-2)})}{p(k-2)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right) \\
\geq \quad & m_0 \sum_{k=1}^{T+2} \left(c_1 \frac{|\triangle^2 u(k-2)|^{p(k-2)}}{p^+} + \frac{q(k)}{p^+} |u(k)|^{p(k)} \right) \\
\geq \quad & \frac{m_0}{p^+} \min(c_1, 1) \sum_{k=1}^{T+2} \left(|\triangle^2 u(k-2)|^{p(k-2)} + q(k)|u(k)|^{p(k)} \right) \\
\geq \quad & \frac{m_0}{p^+} \min(c_1, 1) \left[\|u\|^{p^-} - (1+q^+)T - 1 \right] \\
= \quad & \frac{m_0}{p^+} \min(c_1, 1) \|u\|^{p^-} - \frac{m_0(1+q^+)(T+2)}{p^+} \min(c_1, 1)
\end{aligned}$$

Moreover, let C as (Γ). By (F_1), (F_3) and (G_1), $(G3)$, there exists some constants $C_\lambda, C'_\mu > 0$ such that

$$\lambda F(k,u) \leq \frac{m_0 \min(c_1, 1) S_{p^-,E}}{4Cm_3 p^+} K_2(u) + C_\lambda \quad \text{for all} \quad k \in \mathbb{Z}[1.T],$$

and

$$\mu G(k,u) \leq \frac{m_0 \min(c_1, 1) S_{p^-,E}}{4Cp^+} K_4(u) + C'_\mu \quad \text{for all} \quad k \in \mathbb{Z}[1.T].$$

Then, by the above inequalities, we have

$$
\begin{aligned}
J(u) = \quad & \Phi(u) - \lambda\Psi(u) - \mu I(u) \\
\geq \quad & m_0 L_1(u) - \lambda m_3 L_2(u) - \mu I(u) \\
\geq \quad & \frac{m_0}{p^+} \min(c_1,1)\|u\|^{p^-} - \frac{m_0(1+q^+)(T+2)}{p^+} \min(c_1,1) - \frac{m_0 \min(c_1,1) S_{p^-,E}}{4Cp^+} \sum_{k=1}^{T+2} K_2(u) - \bar{C}_\lambda \\
& - \frac{m_0 \min(c_1,1) S_{p^-,E}}{4Cp^+} \sum_{k=1}^{T+2} K_2(u) - \bar{C'}_\mu \\
\geq \quad & \frac{m_0}{p^+} \min(c_1,1)\|u\|^{p^-} - \frac{m_0(1+q^+)(T+2)}{p^+} \min(c_1,1) - \frac{m_0 \min(c_1,1) S_{p^-,E}}{4p^+} \sum_{k=1}^{T+2} |u|^{p^-} - \bar{C}_\lambda \\
& - \frac{m_0 \min(c_1,1) S_{p^-,E}}{4p^+} \sum_{k=1}^{T+2} |u|^{p^-} - \bar{C'}_\mu \\
\geq \quad & \frac{m_0}{p^+} \min(c_1,1)\|u\|^{p^-} - \frac{m_0(1+q^+)(T+2)}{p^+} \min(c_1,1) - \frac{m_0 \min(c_1,1)}{4p^+} \|u\|^{p^-} - \bar{C}_\lambda \\
& - \frac{m_0 \min(c_1,1)}{4p^+} \|u\|^{p^-} - \bar{C'}_\mu \\
\geq \quad & \frac{m_0}{2p^+} \min(c_1,1)\|u\|^{p^-} - \frac{m_0(1+q^+)(T+2)}{p^+} \min(c_1,1) - \bar{C}_\lambda - \bar{C'}_\mu
\end{aligned}
$$

Hence, since $p^- > 1$, as $\|u\| \to +\infty$, we can conclude $J(u) \to +\infty$ and bounded from below on E. □

From Lemmas 3.1–3.5 and by applying the minimum principle in [33], the functional J admits a global minimizer $\tilde{u}$ which is also the solution of problem (1.1). The following lemma shows that $\tilde{u}$ is a nontrivial solution provided that λ and μ are large enough.

Lemma 3.6. *Assume that there exist $d > 0$ and $k_0 \in \mathbb{Z}([1, T+2])$ such that $F(k_0, d) > 0$ and $G(k_0, d) > 0$. Moreover, there exist two constants $\lambda^*, \mu* > 0$ such that for all $\lambda > \lambda^*$ and $\mu > \mu*$, we have $\inf_{u \in E} J(u) < 0$, then $\tilde{u}$ is a nontrivial solution.*

Proof. Let $w \in E$ such that

$$
w(t) := \begin{cases} d \ \text{ for } \ k = k_0 \\ \\ 0 \text{ for every } \ k \in \mathbb{Z}([1, T+2]) \setminus \{k_0\} \end{cases}
$$

Then, we have

$$\begin{aligned}
J(w) = \quad & \Phi(w) - \lambda\Psi(w) - \mu I(w) \\
\leq \quad & m_1 L_1(w) - \lambda m_2 L_2(w) - \mu L_3(w) \\
\leq \quad & \frac{m_1}{p^-}\max(c_2, 1)\left(\sum_{k=1}^{T+2}\left(|\triangle^2 w(k-1)|^{p(k-1)} + q(k)|w(k)|^{p(k)}\right)\right) - \lambda m_2 \sum_{k=1}^{T} F(k, w) \\
- \quad & \mu \sum_{k=1}^{T} G(k, w) \\
\leq \quad & \frac{m_1}{p^-}\max(c_2, 1)\left(d^{p(k_0)} + 2d^{p(k_0-1)} + d^{p(k_0-2)} + \overline{q}d^{p(k_0)}\right) - \lambda m_2 F(k_0, d) - \mu G(k_0, d) \\
\leq \quad & \frac{m_1}{p^-}\max(c_2, 1) d^{p^*}(4 + \overline{q}) - \lambda F(k_0, d) - \mu G(k_0, d) \\
< \quad & 0
\end{aligned}$$

for all $\lambda > \lambda^*$ and $\mu > \mu*$ large enough, where

$$\beta^{p^*} := \begin{cases} \beta^{p^+} & \text{if } \beta > 1 \\ \beta^{p^-} & \text{if } \beta \leq 1. \end{cases}$$

□

We aim to obtain the second weak solution $\overline{u} \in E$ by applying the Mountain pass theorem in [1]
It suffices for this purpose to show that for all $\lambda > \lambda^*$ and $\mu > \mu*$, J has the geometry of the mountain pass theorem.

Lemma 3.7. *There exist some constants $\rho, r > 0$ such that $\rho \in (0, \|\tilde{u}\|)$ and $J(u) \geq r$ for all $u \in E$ with $\|u\| = \rho$.*

Proof. Let $\|u\|$ be small enough such that $\|u\| = \rho \in (0, 1)$.
Let C as (Γ). By (F_1), (F_2) and (G_1), (G_2), there exists some constants $C_\lambda, C'_\mu > 0$ such that

$$\lambda F(k, u) \leq \frac{m_0 \min(c_1, 1) S_{p^+, E}}{4Cm_3 p^+} K_2(u) + C_\lambda |u|^\gamma \quad \text{for all} \quad k \in \mathbb{Z}[1.T],$$

and

$$\mu G(k, u) \leq \frac{m_0 \min(c_1, 1) S_{p^+, E}}{4Cp^+} K_4(u) + C'_\mu |u|^\gamma \quad \text{for all} \quad k \in \mathbb{Z}[1.T],$$

where $\gamma > p^+$.

Thus, by using inequality (2.1), we have

$$\begin{aligned}
J(u) = \quad & \Phi(u) - \lambda\Psi(u) - \mu I(u) \\
\geq \quad & m_0 L_1(u) - \lambda m_3 L_2(u) - \mu I(u) \\
\geq \quad & \frac{m_0}{p^+}\min(c_1,1)\left(\sum_{k=1}^{T+2}\left(|\triangle^2 u(k-2)|^{p(k-1)} + q(k)|u(k)|^{p(k)}\right)\right) - \lambda m_3 \sum_{k=1}^{T+2} F(k,u) \\
& -\mu \sum_{k=1}^{T+2} G(k,u) \\
\geq \quad & \frac{m_0}{p^+}\min(c_1,1)\left(\sum_{k=1}^{T+2}\left(|\triangle^2 u(k-2)|^{p^+} + q(k)|u(k)|^{p^+}\right)\right) - \frac{m_0 \min(c_1,1) S_{p^+,E}}{4Cp^+}\sum_{k=1}^{T+2} K_1(u) \\
& -C_\lambda \sum_{k=1}^{T+2}|u|^\gamma - \frac{m_0\min(c_1,1)S_{p^+,E}}{4Cp^+}\sum_{k=1}^{T+2} K_3(u) - C'_\mu \sum_{k=1}^{T+2}|u|^\gamma \\
\geq \quad & \frac{m_0}{p^+}\min(c_1,1)K^{-1}\|u\|^{p^+} - \frac{m_0\min(c_1,1)}{4p^+}K^{-1}\|u\|^{p^+} - [\max(T+2,\bar{q})]^{1-\frac{\gamma}{p^-}} C_\lambda \|u\|^\gamma \\
& -\frac{m_0\min(c_1,1)}{4p^+}K^{-1}\|u\|^{p^+} - [\max(T+2,\bar{q})]^{1-\frac{\gamma}{p^-}} C'_\mu\|u\|^\gamma \\
\geq \quad & \frac{m_0}{2p^+}\min(c_1,1)K^{-1}\|u\|^{p^+} - [\max(T+2,\bar{q})]^{1-\frac{\gamma}{p^-}} C_\lambda\|u\|^\gamma \\
& -[\max(T+2,\bar{q})]^{1-\frac{\gamma}{p^-}} C'_\mu\|u\|^\gamma \\
= \quad & \left(\frac{m_0}{2p^+K}\min(c_1,1) - \tilde{C}_\lambda\|u\|^{\gamma-p^+} - \tilde{C'}_\mu\|u\|^{\gamma-p^+}\right)\|u\|^{p^+}
\end{aligned}$$

□

where $\tilde{C}_\lambda = [\max(T+2,\bar{q})]^{1-\frac{\gamma}{p^-}} C_\lambda \geq 0$ and $\tilde{C'}_\mu = [\max(T+2,\bar{q})]^{1-\frac{\gamma}{p^-}} C'_\mu \geq 0$. Since $\gamma > p^+$, we find positive constants r, ρ such that $\|\tilde{u}\| > \rho$ and $J(u) \geq r$ for any $u \in E$ with $\|u\| = \rho$.

Lemma 3.8. *The mapping Ψ' and I' weakly-strongly continuous, namely*

$$u_n \rightharpoonup u \text{ in } E,\ n \to \infty,\ \text{imply } \Psi'(u_n) \to \Psi'(u) \text{ and } I'(u_n) \to I'(u) \text{ in } E^*$$

Proof. Let $\{u_n\}$ be a sequence that converges weakly to u in E. From (F_1) and (G_1), the mappings $N_f, N_g : E \to E^*$ defined by $\langle N_f(u), v\rangle = \sum_{k=1}^{T} f(k,u(k))v(k)$ and $\langle N_g(u), v\rangle = \sum_{k=1}^{T} g(k,u(k))v(k)$ are weakly-strongly continuous.
By the weak continuity of the functional L_2 combined with the continuity of the function M_2, we obtain that the mappings Ψ', I' are weakly-strongly continuous. □

Lemma 3.9. *The function L_1' is of type (S_+), which means that*

$$u_n \rightharpoonup u \text{ in } E\ , n \to \infty \text{ and } \limsup_{n\to+\infty} \langle L_1'(u_n), u_n - u\rangle \le 0,$$

implies $u_n \to u$ in E.

Proof. We define the mappings $A, B : E \to E^*$ respectively by

$$\langle A(u), v\rangle = \sum_{k=1}^{T+2} q(k)|u(k)|^{p(k)-2}u(k)v(k) \quad \forall\, u, v \in E$$

$$\langle B(u), v\rangle = \sum_{k=1}^{T+2} h(|\triangle^2 u(k-2)|^{p(k-2)})|\triangle^2 u(k-2)|^{p(k-2)-2}\triangle u(k-2)\triangle v(k-2) \quad \forall\, u, v \in E$$

Let $\{u_n\}$ be a sequence such that converges weakly to u in E and

$$\limsup_{n\to+\infty} \langle A(u_n) + B(u_n), u_n - u\rangle \le 0.$$

The fact that $\{u_n\}$ converges weakly to u, we have

$$\lim_{n\to+\infty} \langle A(u) + B(u), u_n - u\rangle = 0.$$

By using (H_2) and inequality (2.6), we have

$$\begin{aligned}
0 &\ge \limsup_{n\to+\infty} [\langle A(u_n) + B(u_n), u_n - u\rangle - \langle A(u) + B(u), u_n - u\rangle] \\
&= \limsup_{n\to+\infty} [\langle B(u_n) - B(u), u_n - u\rangle - \langle A(u_n) - A(u), u_n - u\rangle] \\
&\ge \bar{c} \limsup_{n\to+\infty} \phi(u_n - u) \\
&\ge \bar{c} \limsup_{n\to+\infty} \left[\min\left(\|u_n - u\|_{p(.)}^{p^+}, \|u_n - u\|_{p(.)}^{p^-}\right)\right]
\end{aligned}$$

Thus, the sequence $\{u_n\}$ converges to u in E. This proves that the functional L_1' is of type (S_+).

□

By Lemma 3.9, it is clear that the mapping Φ' is of type (S_+). Indeed Assume that $\{u_n\}$ is a sequence that converges weakly to u in E.

$$\limsup_{n\to+\infty} \langle \Phi'(u), , u_n - u\rangle = \limsup_{n\to+\infty} M_1(L_1(u_n)) \langle L_1'(u_n), u_n - u\rangle \le 0.$$

From (M_1), we have

$$\limsup_{n\to+\infty} \langle L_1'(u_n), u_n - u\rangle \le 0.$$

By Lemma 3.9, we obtain that $\{u_n\}$ converges to u in E.

Lemma 3.10. *The functional J' is of type (S_+).*

Proof. We have that $J(u) = \Phi(u) - \lambda\Psi(u) - \mu I(u)$. It is clear that J' is of type (S_+), since Φ' is of type (S_+) and the mappings Ψ', I' are weakly-strongly continuous. □

Lemma 3.11. *The functional J satisfies the Palais-Smale condition in E.*

Proof. From Lemma 3.5, the functional J is coercive. Let $\{u_n\}$ be a Palais-Smale sequence for J in E.
We have

$$J(u_n) \to c, \quad J'(u_n) \to 0, \quad \text{as } n \to +\infty. \tag{3.1}$$

Since J is coercive, relation (3.1) given that the sequence $\{u_n\}$ is bounded in E. Moreover E is reflexive space, we can extract a subsequence of $\{u_n\}$ denoted still by $\{u_n\}$, such that it converges weakly to u in E. The convergence $J'(u_n) \to 0$ implies that $\langle J'(u_n), u_n - u\rangle \to 0$. As J' is of type (S_+), we obtain that $u_n \to u$ in E, and the proof of the lemma is completed. □

We state our main result as follows.

Theorem 3.12. *Assume that $(F_1) - (F_3)$, $(G_1) - (G_3)$, $(H_1) - (H_2)$ and $(M_1) - (M_3)$ holds. Moreover, we suppose that there exists $d > 0$ and $k_0 \in \mathbb{Z}[1, T+2]$ such that $F(k_0, d) >$ and $G(k_0, d) > 0$. Then for all λ and μ enough large, problem (1.1) admits at least two distinct, nontrivial weak solutions.*

Proof. By Lemmas 3.1–3.6, problem (1.1) has a nontrivial weak solution $\tilde{u}$ as the global minimizer of J.
Put

$$\bar{c} = \inf_{\varphi\in\Gamma} \max_{u\in\varphi([0,1])} J(u)$$

where $\Gamma = \{\varphi \in C(E, [0,1])/\varphi(0) = 0,\ \varphi(1) = \tilde{u}\}$.
From Lemmas 3.7–3.11, all assumptions of the mountain pass theorem in [1] are satisfied, $J(\tilde{u}) < 0$ and $\|\tilde{u}\| > \rho$.
Then $\bar{c}$ is a critical value of J, i.e, there exists $\bar{u} \in E$ such that $J(\bar{u}) = \bar{c}$ and $\langle J'(\bar{u}), \varphi\rangle = 0$ for all $\varphi \in E$. Moreover, $\bar{u}$ is not trivial and $\bar{u} \neq \tilde{u}$ since $J(\bar{u}) = \bar{c} > 0 > J(\tilde{u})$. Theorem 3.12 is proved. □

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Received: 2023-06-04
Accepted: 2024-01-12