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FUZZY MULTI-HYPERRINGS

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Abstract *In this paper, we introduce the concept of fuzzy multi-hyperrings and obtain several related results. Also, we study different operations on fuzzy multi-hyperrings (such as intersection and union). Moreover, we define homomorphism and the direct product of fuzzy multi-hyperrings and investigate some of their main properties.*

1 Introduction

A fuzzy multiset is a generalization of a fuzzy set. Zadeh [21], in his theory of fuzzy sets, proposed using a membership function operating on the domain of all possible values. In recent years, several researchers studied various generalizations of fuzzy sets. Yager [20] discussed the fuzzy-bag structure and operations on fuzzy multisets, such as intersection and union. Girish [9] introduced the concepts of relation, function, and composition and Nazmul [15] defined a group on the multiset derived from the initial universal set. An element of a fuzzy multiset can occur more than once with possibly the same or different membership values. This new structure has many applications in mathematics and computer sciences.

Marty [12] introduced the concept of hyperstructure during the 8^{th} *Congress of Scandinavian Mathematicians in 1934. He defined the concept of a hypergroup as a generalization of a group. Krasner [11] introduced the notion of hyperrings. Moreover, many researchers studied this field and developed this theory and found related results. For example, Corsini and Leoreanu gave main works in hyperstructures in [5]. Also for some related study see [1], [3].*

Although many authors extended these concepts, study in the fuzzy multiset theory has not yet gained much ground, and it is still in its infant stage. Therefore the study of hyperring structure in a fuzzy-multiset context is helpful. In this paper, we introduce the notion of fuzzy multi-hyperrings and study some of their main properties. We show that the intersection of two fuzzy multi-hyperrings is again a fuzzy multi-hyperring, but their union may not be a fuzzy multi-hyperring in general. In addition, we define and study homomorphic properties and the direct product of fuzzy multi-hyperrings.

In the following, we study some basic definitions and results related to fuzzy multisets and hyperstructures.

A multiset is an unordered collection of objects (inverse of a standard Cantorian-set) elements are allowed to repeat. If X is a set, a multiset A drawn from X is characterized by a count function A or C_A *defined as* $C_A : X \longrightarrow \{0, 1, 2, \cdots\}$. *For each* $x \in X$, $C_A(x)$ *is the characteristic value of x in A, and it indicates the number of occurrences of the element x in A.*

Definition 1.1. [13, 17] Let *X* be a non-empty set. We represent a fuzzy multiset *A* drawn from the set *X* by a function CM_A such that $CM_A : X \longrightarrow Q$, where Q represents the set of all crisp multisets drawn from the unit interval $[0, 1]$. In particular, we represent a fuzzy multiset *A* by a higher order function $A : X \longrightarrow [0, 1] \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. For each

$x \in X$, CM_A is the characteristic value of *x* in *A* and indicates a crisp multiset drawn from $[0,1]$. Also, for each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$ and is denoted by $(\mu_A^1(x), \cdots, \mu_A^k(x))$; $\mu_A^1(x) \geq \cdots \geq \mu_A^k(x)$. If every $x \in X$ is mapped to a finite multiset of $\mathbb{N}$ under the count membership function CM_A, then *A* is said to be a finite fuzzy multiset of *X*. For all $x \in A$, we define $L(x; A) = \max\{j; \mu_A^j(x) \neq 0\}$. We consider an operation between two fuzzy multisets by an equal length. Thus if *A* and *B* are two fuzzy multisets at consideration, take $L(x; A, B) = \max\{L(x; A), L(x; B)\}$. If no ambiguity arises, then we denote the length of membership with $L(x)$.

Let A and B be two fuzzy multisets drawn from a set X. Then $A \subseteq B$ *if* $\mu_A^j(x) \leq \mu_B^j(x)$, *for all* $x \in X$, $j = 1, \cdots, L(x)$. *The intersection and union of A and B, denoted by* $A \cap B$ *and* $A \cup B$, *respectively, are defined by* $\mu_{A\cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x)$ *and* $\mu_{A\cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x)$, *respectively, where* $j = 1, \cdots, L(x)$ *and* $\wedge$, $\vee$ *are the minimum and the maximum operation, respectively. For all* $i = 1, 2, \cdots, \max\{L(x), L(y)\}$ *by* $CM_A(x) \geq CM_A(y)$, $CM_A(x) \wedge CM_A(y)$ *and* $CM_A(x) \vee CM_A(y)$ *we mean that* $\mu_A^i(x) \geq \mu_A^i(x)$, $\{\mu_A^i(x) \wedge \mu_A^i(y)\}$ *and* $\{\mu_A^i(x) \vee \mu_A^i(y)\}$, *respectively.*

Let H be a non-empty set and $P^*(H)$ *be the family of all non-empty subsets of H. A mapping* $\circ : H \times H \longrightarrow P^*(H)$ *is called a binary hyperoperation on H and* $(H, \circ)$ *is called a hypergroupoid. A hypergroupoid* $(H, \circ)$ *is called a semihypergroup if* $x \circ (y \circ z) = (x \circ y) \circ z$, *for all* $x, y, z \in H$ *and is called a quasihypergroup if* $x \circ H = H = H \circ x$, *for all* $x \in H$. *The couple* $(H, \circ)$ *is called a hypergroup if it is a semihypergroup and a quasi-hypergroup. A hypergroup* $(H, \circ)$ *is called commutative if* $x \circ y = y \circ x$, *for all* $x, y \in H$.

Definition 1.2. [6, 11] A Krasner hyperring is an algebraic hyperstructure $(R, +, .)$ which satisfies the following axioms:
(1) $(R, +)$ is a canonical hypergroup, i.e.,
(i) for every $x, y, z \in R$, $x + (y + z) = (x + y) + z$;
(ii) for every $x, y \in R$, $x + y = y + x$;
(iii) there exist $0 \in R$ such that $0 + x = \{x\}$, for all $x \in R$;
(iv) for every $x \in R$ there exists a unique element $-x \in R$ such that $0 \in x - x$;
(v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.
(2) $(R, .)$ is a semigroup having zero as a bilateral absorbing element, i.e., $x.0 = 0.x = 0$;
(3) The multiplication "." is distributive to the hyperoperation "+".

In the above definition, for simplicity of notations, we write $0+x = x$ *instead of* $0+x = \{x\}$. *The following elementary facts follow easily from the axioms:* $-(-x) = x$ *and* $-(x+y) = -x-y$ *and* $-A = \{-x; x \in A\}$. *We say that a Krasner hyperring* $(R, +, .)$ *is commutative (with unit element) if* $(R, .)$ *is a commutative semigroup (with unit element). A non-empty subset A of R is a sub-hyperring of* $(R, +, .)$ *if* $a + b \subseteq A$, $-a \in A$ *and* $a.b \in A$. *Let R and S be two Krasner hyperrings. A map* $\varphi : R \longrightarrow S$ *is called a homomorphism if* $\varphi(x + y) \subseteq \varphi(x) + \varphi(y)$, $\varphi(x.y) = \varphi(x).\varphi(y)$ *and* $\varphi(0) = 0$, *for all* $x, y \in R$.

Example 1.3. [14, 18] Let $(H, \circ)$ be a finite group with *k* elements ($k > 3$) and consider "+" and "$\times$" on $G = H \cup \{0\}$ with:
$x + y = y + x = G \diagdown \{x, y\}$, for all $x, y \in H, x \neq y$; $x + x = \{x, 0\}$, for all $x \in H$; $x + 0 = 0 + x = \{x\}$, for all $x \in G$; $x \times y = x \circ y$, for all $x, y \in H$; $x \times 0 = 0$, for all $x \in G$. Then $(G, +, \times)$ is a Krasner hyperring.

Throughout this paper, let X be a Krasner hyperring and 0 *be the additive identity of X. We also omit all clear proofs throughout the article.*

2 Characterizations of Fuzzy multi-hyperrings

In this section, inspired by the definitions of the Krasner hyperring [11] and the fuzzy multiset [9], we introduce the concept of fuzzy multi-hyperring.

Definition 2.1. A fuzzy multiset *A* drawn from *X* is said to be a fuzzy multi (Krasner) hyperring over *X*, if for all $x, y \in X$ the count function of *A* i.e., CM_A satisfies the following conditions:

(i) $\bigwedge_{z \in x+y} CM_A(z) \geq CM_A(x) \wedge CM_A(y)$;

(ii) $CM_A(-x) \geq CM_A(x)$;

(iii) $CM_A(x.y) \geq CM_A(x) \wedge CM_A(y)$.

The set of all fuzzy multi-hyperrings over *X* is denoted by *FMHR(X)*.

Example 2.2. In the following, we present different examples of fuzzy multi-hyperrings:

(i) Let $X = \frac{\mathbb{Z}_{12}}{H} = \{xH \mid x \in \mathbb{Z}_{12}\}$, where $\mathbb{Z}_{12}$ is the set of all congruence classes of integers modulo 12 and *H* is its multiplicative subgroup of units. So $H = \{\bar{1}, \bar{5}, \bar{7}, \bar{11}\}$. Clearly, $\bar{2}H = \bar{10}H$, $\bar{3}H = \bar{9}H$, $\bar{4}H = \bar{8}H$ and so $X = \{\bar{0}H, \bar{1}H, \bar{2}H, \bar{3}H, \bar{4}H, \bar{6}H\}$. Consider the hyperoperation "$\dot{+}$" by $\bar{x}H \dot{+} \bar{y}H = \{\bar{z}H; (\bar{x}H \dot{+} \bar{y}H) \cap \bar{z}H \neq \emptyset\}$ and the binary operation "." by $\bar{x}H.\bar{y}H = \bar{x.y}H$. Then $(X, \dot{+}, .)$ is a Krasner hyperring [4]. It is clear that $A = \{\langle\frac{(1,0.8,0.5)}{\bar{0}H}\rangle, \langle\frac{(0.6,0,2)}{\bar{1}H}\rangle, \langle\frac{(0.8,0.5)}{\bar{2}H}\rangle, \langle\frac{(0.6,0,2)}{\bar{3}H}\rangle, \langle\frac{(0.9,0.6)}{\bar{4}H}\rangle, \langle\frac{(0.8,0.5)}{\bar{6}H}\rangle\}$ is a fuzzy multi-hyperring over *X*.

(ii) Let $X = \{0, 1, 2\}$ be a set with the hyperoperation "+" and the binary operation "." defined as follows:

+	0	1	2
0	0	1	2
1	1	1	$\{0, 1, 2\}$
2	2	$\{0, 1, 2\}$	2

.	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Then $(X, +, .)$ is a Krasner hyperring [6]. It is clear that

$$A = \{\langle\frac{(1,0.6,0.4)}{0}\rangle, \langle\frac{(0.3,0.2,0.1)}{1}\rangle, \langle\frac{(0.3,0.2,0.1)}{2}\rangle\}$$

is a fuzzy multi-hyperring over *X*.

(iii) Let $X = \{0, 1, 2, 3\}$ be a set with the hyperoperation "+" defined as follows:

+	0	1	2	3
0	0	1	2	3
1	1	{0, 1}	3	{2,3}
2	2	3	0	1
3	3	{2,3}	1	{0, 1}

and the binary operation "." defined as $x.y = 2$, if $x, y \in \{2, 3\}$ and $x.y = 0$, otherwise. Then $(X, +, .)$ is a Krasner hyperring [19]. It is clear that

$$A = \{\langle\frac{(1,0.8,0.3)}{0}\rangle, \langle\frac{(0.5,0.4)}{1}\rangle, \langle\frac{0.5,0.5,0.3}{2}\rangle, \langle\frac{(0.5,0.4)}{3}\rangle\}$$

is a fuzzy multi-hyperring over *X*.

(iv) Let $X = \{0, 1\}$ be a set with the hyperoperation "+" defined as follows:

$$0 + 1 = 1 + 0 = \{1\}, 1 + 1 = \{0, 1\} \text{ and } 0 + 0 = \{0\},$$

and the binary operation "." defined as the usual multiplication. Then *(X, +, .)* is a Krasner hyperring [16]. It is clear that $A = \{\langle\frac{(1,0.5,0.3)}{0}\rangle, \langle\frac{(0.9,0.4,0.2)}{1}\rangle\}$ is a fuzzy multi-hyperring over *X*

(v) Let *a* be a fixed element in *X*. Consider fuzzy multiset *A* drown from *X* by count function CM_A as $CM_A(x) = CM_A(a)$, for all $x \in X$. Then *A* is a fuzzy multi-hyperring over *X* and is called the constant fuzzy multi-hyperring over *X*.

(vi) Let $X = \{0,1\}$. Consider hyperoperation "+" as $x+y = \{z; z \leq x\}$, if $x = y$, and $x+y = x \vee y$, if $x \neq y$, where $\vee$ is the maximum and "." is the ordinary multiplication. Then X is a Krasner hyperring. It is clear that $A = \{< \frac{(0.4,0.2)}{0} >, < \frac{(0.3,0.1)}{1} >\}$ is a fuzzy multi-hyperring over X.

(vii) Let $X = \{0,1,2,3\}$ be a set with the hyperoperation "+" and the binary operation "." as follows:

+	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,2\}$	$\{1,3\}$	$\{2\}$
2	$\{2\}$	$\{1,3\}$	$\{0,2\}$	$\{1\}$
3	$\{3\}$	$\{2\}$	$\{1\}$	$\{0\}$

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3

Then $(X,+,.)$ is a Krasner hyperring [2]. It is clear that $A = \{\langle\frac{(1,0.7,0.6)}{0}\rangle, \langle\frac{(0.5,0.4)}{2}\rangle\}$ is a fuzzy multi-hyperring over X.

Proposition 2.3. *Let A be a fuzzy multi-hyperring over X with*

$$CM_A(x) = (\mu^1(x), \mu^2(x), \cdots, \mu^k(x)), \text{ for all } x \in X.$$

Consider the complement of A by

$$CM_{A'}(x) = (1-\mu^k(x), \cdots, 1-\mu^2(x), 1-\mu^1(x)), \text{ for all } x \in X.$$

Then $A = (A')'$.

Proof. Let $x \in X$. Then $CM_{(A')'}(x) = (1-(1-\mu^1(x)), 1-(1-\mu^2(x)), \cdots, 1-(1-\mu^k(x))) = (\mu^1(x), \mu^2(x), \cdots, \mu^k(x)) = CM_A(x)$. □

Example 2.4. Let $(X,+,.)$ be the Krasner hyperring defined in Example 2.2(vi). Consider

$$A = \{\langle\frac{(0.5,0.4)}{0}\rangle, \langle\frac{(0.3,0.2)}{1}\rangle\}.$$

Then $A' = \{\langle\frac{(0.6,0.5)}{0}\rangle, \langle\frac{(0.8,0.7)}{1}\rangle\}$. This example shows that A' is not a fuzzy multi-hyperring of X in general because $\bigwedge_{z\in 1+1} CM_{A'}(z) \ngeq CM_A(1)$.

Theorem 2.5. *Let A be a fuzzy multi-hyperring over X. Then for all $x \in X$, we have*

(i) $CM_A(0) \geq CM_A(x)$;

(ii) $CM_A(-x) = CM_A(x)$.

Proof. Let $x \in X$.
(i) Since A is a fuzzy multi-hyperring over X, we have

$$CM_A(0) \geq \bigwedge_{z\in -x+x} CM_A(z) \geq CM_A(-x) \wedge CM_A(x) \geq CM_A(x).$$

(ii) It is sufficient to prove that $CM_A(x) \geq CM_A(-x)$. For this aim, we have

$$CM_A(x) = CM_A(-(-x)) \geq CM_A(-x).$$

□

Theorem 2.6. *A fuzzy multiset A is a fuzzy multi-hyperring over X if and only if for all $x, y \in X$,*

(i) $\bigwedge_{z\in x-y} CM_A(z) \geq CM_A(x) \wedge CM_A(y)$;

(ii) $CM_A(x.y) \geq CM_A(x) \wedge CM_A(y)$.

Proof. Let $x, y \in X$. If A is a fuzzy multi-hyperring over X, then

$$CM_A(x.y) \geq CM_A(x) \wedge CM_A(y).$$

Moreover,

$$\begin{aligned}\bigwedge_{z\in x-y} CM_A(z) &= \bigwedge_{z\in x+(-y)} CM_A(z) \geq CM_A(x)\wedge CM_A(-y)\\ &\geq CM_A(x)\wedge CM_A(y),\end{aligned}$$

Conversely, if the conditions hold, then

$$CM_A(0) \geq \bigwedge_{z\in x-x} CM_A(z) \geq CM_A(x)\wedge CM_A(x) = CM_A(x).$$

Thus

$$CM_A(-x) \geq \bigwedge_{z\in 0-x} CM_A(z) \geq CM_A(0)\wedge CM_A(x) = CM_A(x).$$

Hence

$$\begin{aligned}\bigwedge_{z\in x+y} CM_A(z) &= \bigwedge_{z\in x-(-y)} CM_A(z) \geq CM_A(x)\wedge CM_A(-y)\\ &\geq CM_A(x)\wedge CM_A(y).\end{aligned}$$

Now, the condition *(ii)* completes the proof. □

Proposition 2.7. *Let A and B be two fuzzy multi-hyperrings over X. Consider $-A$ by $CM_{-A}(x) = CM_A(-x)$, for all $x \in X$. Then*

(i) *$-A$ is a fuzzy multi-hyperring over X;*

(ii) *$-(-A) = A$;*

(iii) *if $A \subseteq B$, then $-A \subseteq -B$.*

Proof. *(i)* Let $x, y \in X$. Since *A* is a fuzzy multi-hyperring over *X*, then

$$\begin{aligned}\bigwedge_{z\in x+y} CM_{-A}(z) &= \bigwedge_{z\in x+y} CM_A(-z) = \bigwedge_{t\in(-(x+y))} CM_A(t) = \bigwedge_{t\in(-x)+(-y)} CM_A(t)\\ &\geq CM_A(-x)\wedge CM_A(-y) = CM_{-A}(x)\wedge CM_{-A}(y).\end{aligned}$$

Also $CM_{-A}(-x) = CM_A(-(-x)) \geq CM_A(-x) = CM_{-A}(x)$. Moreover, by Theorem 2.5(2), we have

$$\begin{aligned}CM_{-A}(x.y) &= CM_A(-(x.y)) = CM_A((-x).y) \geq CM_A(-x)\wedge CM_A(y)\\ &= CM_A(-x)\wedge CM_A(-y) = CM_{-A}(x)\wedge CM_{-A}(y).\end{aligned}$$

Therefore $-A$ is a fuzzy multi-hyperring over *X*. The proofs of *(ii)* and *(iii)* are straightforward. □

Definition 2.8. Let *A* and *B* be two fuzzy multi-hyperrings over *X*. Then, we define $A \oplus B$ by

$$CM_{A\oplus B}(x) = \bigvee\{CM_A(y)\wedge CM_B(z); x, y, z \in X, x \in y+z\},$$

and $A \odot B$ by

$$CM_{A\odot B}(x) = \bigvee\{CM_A(y)\wedge CM_B(z); x, y, z \in X, x \in y.z\}.$$

Example 2.9. Let $(X, +, .)$ be the Krasner hyperring defined in Example 2.2(iv). It is clear that $A = \{\langle\frac{(1,0.8,0.5)}{0}\rangle, \langle\frac{(0.9,0.5,0.1)}{1}\rangle\}$ and $B = \{\langle\frac{(1,0.9,0.8)}{0}\rangle, \langle\frac{(1,0.8,0.6)}{1}\rangle\}$ are two fuzzy multi-hyperrings over *X* and we have $A \oplus B = \{\langle\frac{(1,0.8,0.5)}{0}\rangle, \langle\frac{(1,0.8,0.6)}{1}\rangle\}$ and $A \odot B = \{\langle\frac{(1,0.8,0.5)}{0}\rangle, \langle\frac{(0.9,0.5,0.1)}{1}\rangle\}$.

Theorem 2.10. *Let A be a fuzzy multi-hyperring over X. Then*

(i) $A \oplus A = A$;

(ii) $-A = A$.

Proof. *(i)* Let $x \in X$. Since *A* is a fuzzy multi-hyperring over *X*, then for all $y, z \in X$,

$$\bigwedge_{x \in y+z} CM_A(x) \geq CM_A(y) \wedge CM_A(z).$$

So $CM_A(x) \geq CM_A(y) \wedge CM_A(z)$. Hence

$$CM_A(x) \geq \bigvee\{CM_A(y) \wedge CM_A(z); y, z \in X, x \in y+z\} = CM_{A \oplus A}(x).$$

Thus $A \oplus A \subseteq A$. Again

$$\begin{aligned} CM_{A \oplus A}(x) &= \bigvee\{CM_A(y) \wedge CM_A(z); y, z \in X, x \in y+z\} \\ &\geq CM_A(x) \wedge CM_A(0) = CM_A(x). \end{aligned}$$

So $A \subseteq A \oplus A$. Therefore $A \oplus A = A$.

(ii) Since $CM_{-A}(x) = CM_A(-x) = CM_A(x)$, it follows that $-A = A$. □

Definition 2.11. Let *A* be a fuzzy multi-hyperring over *X*. Then *A* is called commutative if $CM_A(x.y) = CM_A(y.x)$, for all $x, y \in X$.

Example 2.12. let $X = \{0, 1, 2\}$ be the Krasner hyperring with the hyperoperation "+" and the binary operation "." defined in Example 2.2(ii). Then *X* is a non-commutative Krasner hyperring and $A = \{< \frac{(0.6,0.5,0.3)}{0} >, < \frac{(0.5,0.3,0.2)}{1} >, < \frac{(0.5,0.3,0.2)}{2} >\}$ is a commutative fuzzy multi-hyperring over *X*.

Theorem 2.13. *Let A be a fuzzy multiset drawn from X and* $\alpha \in [0,1]$. *Consider* $A_{[\alpha]} = \{x \in X; CM_A(x) \geq \alpha\}$. *Then A is a fuzzy multi-hyperring over X if and only if* $A_{[\alpha]}$ *is a sub-hyperrings of X.*

Proof. Let $x, y \in A_{[\alpha]}$. Then $CM_A(x) \geq \alpha$ and $CM_A(y) \geq \alpha$. Since *A* is a fuzzy multi-hyperring over *X*, then $\bigwedge_{z \in x+y} CM_A(z) \geq CM_A(x) \wedge CM_A(y) \geq \alpha$. Thus $x+y \subseteq A_{[\alpha]}$ because for all $z \in x+y$ we get $CM_A(z) \geq \alpha$ and so $z \in A_{[\alpha]}$. Moreover $CM_A(-x) \geq CM_A(x) \geq \alpha$. Thus $-x \in A_{[\alpha]}$. Also, we have $CM_A(x.y) \geq CM_A(x) \wedge CM_A(y) \geq \alpha$. Hence $x.y \in A_{[\alpha]}$. Therefore for all $\alpha \in [0,1]$, $A_{[\alpha]}$ is a sub-hyperring of *X*.

Conversely, let $x, y \in X$. Assume that $CM_A(x) = \alpha, CM_A(y) = \beta$. Then $\alpha, \beta \in [0,1]$ and so $x \in A_{[\alpha]}$ and $y \in A_{[\beta]}$. Without loss of generality, let $\alpha \wedge \beta = \alpha$ and so $y \in A_{[\alpha]}$. Since $A_{[\alpha]}$ is a sub-hyperrings of *X*, then $x + y \subseteq A_{[\alpha]}$ and $x.y \in A_{[\alpha]}$ and $-x \in A_{[\alpha]}$. Therefore $z \in A_{[\alpha]}$, for all $z \in x + y$. Hence

$$\bigwedge_{z \in x+y} CM_A(z) \geq \alpha = \alpha \wedge \beta = CM_A(x) \wedge CM_A(y).$$

Moreover $CM_A(-x) \geq \alpha = CM_A(x)$. Also

$$CM_A(x.y) \geq \alpha = \alpha \wedge \beta = CM_A(x) \wedge CM_A(y).$$

Thus *A* is a fuzzy multi-hyperring over *X*. □

Example 2.14. Let $(X, +, .)$ be the Krasner hyperring defined in Example 2.2(iii). It is clear that $A = \{\langle \frac{(1,0.9,0.6)}{0} \rangle, \langle \frac{(0.4,0.2)}{1} \rangle, \langle \frac{(0.9,0.8,0.5)}{2} \rangle, \langle \frac{(0.4,0.3)\rangle}{3} \}$ is a fuzzy multi-hyperring over *X* and $A_{[0.5]} = \{0, 2\}$ is a sub-hyperrings of *X*.

Theorem 2.15. *Let A and B be two fuzzy multi-hyperrings over X. Then*
$-(A \odot B)_{[\alpha]} = (A \odot B)_{[\alpha]}$.

Proof. Let $x \in X$. We have $x \in (A \odot B)_{[\alpha]} \Longleftrightarrow CM_{-(A \odot B)}(x) \geq \alpha \Longleftrightarrow CM_{A \odot B}(-x) \geq \alpha$ $\Longleftrightarrow \bigvee\{CM_A(y) \wedge CM_B(z) \mid y, z \in X, -x = y.z\} \geq \alpha \Longleftrightarrow \bigvee\{CM_A(-y) \wedge CM_B(z) \mid y, z \in X, x = (-y).z\} \geq \alpha \Longleftrightarrow x \in (A \oplus B)_{[\alpha]}$. □

Definition 2.16. Let *A* be a fuzzy multi-hyperring over *X* and $\emptyset \neq B \subseteq A$. Then *B* is said to be a fuzzy sub-multi-hyperring of *A* if *B* itself is a fuzzy multi-hyperring over *X*. The fuzzy sub-multi-hyperring *B* of *A* is proper if $A \neq B$.

Example 2.17. Let $(X, +, .)$ be the Krasner hyperring defined in Example 2.2(i). Consider

$$A = \{\langle \frac{(1,0.8,0.4)}{\bar{0}H} \rangle, \langle \frac{(0.6,0.4,0.1)}{\bar{2}H} \rangle, \langle \frac{(0.8,0.5,0.3)}{\bar{4}H} \rangle, \langle \frac{(0.6,0.4,0.1)}{\bar{6}H} \rangle\}$$

and

$$B = \{\langle \frac{(1,0.9,0.8)}{\bar{0}H} \rangle, \langle \frac{(0.8,0.3)}{\bar{2}H} \rangle, \langle \frac{(0.8,0.4,0.3)}{\bar{4}H} \rangle, \langle \frac{(0.8,0.3)}{\bar{6}H} \rangle\}.$$

It is clear that $A, B \in FMHR(X)$ with $A \subseteq B$ and $A \neq B$. Therefore *A* is a fuzzy (proper) sub-multi-hyperring over *B*.

Proposition 2.18. *Let A be a fuzzy multi-hyperring over X, B be a fuzzy sub-multi-hyperring of A, C be a fuzzy multiset of X and $C \subseteq B$. Then C is a fuzzy sub-multi-hyperring of A if and only if C is a fuzzy sub-multi-hyperring of B. Also, A is a fuzzy submulti-hyperring of B if and only if -A is a fuzzy submulti-hyperring of -B.*

Proof. The proof is straightforward. □

3 Intersection and Union of Fuzzy multi-hyperrings

In the sequel, we define some operations on fuzzy multi-hyperrings and study their properties.

Theorem 3.1. *Let A and B be two fuzzy multi-hyperrings over X. For all $x \in X$, consider $CM_{A \cap B}(x) = CM_A(x) \wedge CM_B(x)$. Then $A \cap B$ is a fuzzy multi-hyperring over X.*

Proof. Let $x, y \in X$. Since A and B are fuzzy multi-hyperrings over *X*, then

$$\begin{aligned}
\bigwedge_{z \in x+y} CM_{A \cap B}(z) &= \bigwedge_{z \in x+y} (CM_A(z) \wedge CM_B(z)) \\
&= (\bigwedge_{z \in x+y} CM_A(z)) \wedge (\bigwedge_{z \in x+y} CM_B(z)) \\
&\geq (CM_A(x) \wedge CM_A(y)) \wedge (CM_B(x) \wedge CM_B(y)) \\
&= (CM_A(x) \wedge CM_B(x)) \wedge (CM_A(y) \wedge CM_B(y)) \\
&= CM_{A \cap B}(x) \wedge CM_{A \cap B}(y).
\end{aligned}$$

Moreover,

$$CM_{A \cap B}(-x) = CM_A(-x) \wedge CM_B(-x) \geq CM_A(x) \wedge CM_B(x) = CM_{A \cap B}(x).$$

Also

$$\begin{aligned}
CM_{A \cap B}(x.y) &= CM_A(x.y) \wedge CM_B(x.y) \\
&\geq (CM_A(x) \wedge CM_A(y)) \wedge (CM_B(x) \wedge CM_B(y)) \\
&= (CM_A(x) \wedge CM_B(x)) \wedge (CM_A(y) \wedge CM_B(y)) \\
&= CM_{A \cap B}(x) \wedge CM_{A \cap B}(y).
\end{aligned}$$

Hence $A \cap B$ is a fuzzy multi-hyperring over *X*. □

Remark 3.2. Let A and B be two fuzzy multi-hyperrings over X and for all $x \in X$, consider $CM_{A \cup B}(x) = CM_A(x) \vee CM_B(x)$. Then $A \cup B$ is not a fuzzy multi-hyperring over X in general.

Example 3.3. Let $(X, +, .)$ be the Krasner hyperring defined in Example 2.2(vii). Consider $A = \{\langle\frac{(1,0.5,0.4)}{0}\rangle, \langle\frac{(0.4,0.3)}{2}\rangle\}$ and $B = \{\langle\frac{(1,0.6,0.5)}{0}\rangle, \langle\frac{0.3,0.1}{3}\rangle\}$. It is clear that *A*, *B* and $A \cap B = \{\langle\frac{(1,0.5,0.4)}{0}\rangle\}$ are fuzzy multi-hyperrings over *X*, but $A \cup B = \{\langle\frac{(1,0.6,0.5)}{0}\rangle, \langle\frac{(0.4,0.3)}{2}\rangle, \langle\frac{(0.3,0.1)}{3}\rangle\}$ is not a fuzzy multi hyperrong over *X* because $\bigwedge_{z\in 3+2} C_{A\cup B}(z) \ngeq C_{A\cup B}(3) \wedge C_{A\cup B}(2)$.

Proposition 3.4. *Let $A, B, C \in FMHR(X)$. Then*

(i) if $CM_A(0) = CM_B(0)$, then $A \cap B, A \cup B \subseteq A \oplus B$;

(ii) if $A \subseteq B$ or $B \subseteq A$, then $A \cup B$ is a fuzzy multi-hyperring over X;

(iii) if $A \subseteq B \subseteq C$, then $A \cap B$ and $A \cup B$ are fuzzy sub-multi-hyperrings of C.

Proof. *(i)* Let $x \in X$.

$$\begin{aligned} CM_{A\oplus B}(x) &= \bigvee\{CM_A(y) \wedge CM_B(z); y, z \in X, x \in y + z\} \\ &\geq CM_A(x) \wedge CM_A(0) = CM_A(x). \end{aligned}$$

Similarly, we can show that $CM_{A\oplus B}(x) \geq CM_B(x)$. Therefore

$$CM_{A\oplus B}(x) \geq CM_A(x) \vee CM_B(x) = CM_{A\cup B}(x).$$

Hence $A \oplus B \supseteq A \cup B$. Clearly, $A \cap B \subseteq A \cup B$. Therefore $A \cap B, A \cup B \subseteq A \oplus B$.
The proof of *(ii)* and *(iii)* are straightforward. □

Theorem 3.5. *Let A and B be two fuzzy multi-hyperrings over X. Then $A_{[\alpha]} \cap B_{[\alpha]} = (A \cap B)_{[\alpha]}$ and $A_{[\alpha]} \cup B_{[\alpha]} = (A \cup B)_{[\alpha]}$, for all $\alpha \in [0, 1]$.*

Proof. Let $x \in X$. Then $x \in A_{[\alpha]} \cap B_{[\alpha]} \iff x \in A_{[\alpha]}$ and $x \in B_{[\alpha]} \iff CM_A(x) \geq \alpha$ and $CM_B(x) \geq \alpha \iff CM_A(x) \wedge CM_B(x) \geq \alpha \iff CM_{A\cap B}(x) \geq \alpha \iff x \in (A \cap B)_{[\alpha]}$.
Therefore $A_{[\alpha]} \cap B_{[\alpha]} = (A \cap B)_{[\alpha]}$.
Similarly, we can prove that $A_{[\alpha]} \cup B_{[\alpha]} = (A \cup B)_{[\alpha]}$.

□

Definition 3.6. Let $\{A_i; i \in I\}$ be an arbitrary family of fuzzy multi-hyperrings over *X*, where $I = \{1, 2, \cdots\}$ is an indexed set. Then their intersection, defined by $CM_{\cap_{i\in I}A_i}(x) = \wedge_{i\in I}CM_{A_i}(x)$, for all $x \in X$ and their union, defined by $CM_{\cup_{i\in I}A_i}(x) = \bigvee_{i\in I} CM_{A_i}(x)$, for all $x \in X$. Moreover, we say that this family satisfies in ascending (resp. descending) chain condition if for every chain of fuzzy multi-hyperrings $A_1 \subseteq A_2 \subseteq \cdots$ (resp. $A_1 \supseteq A_2 \supseteq \cdots$) there exist $n \in \mathbb{N}$ such that $A_n = A_m$, for all $m \geq n$.

Theorem 3.7. *Let $\{A_i : i \in I\}$ be an arbitrary family of fuzzy multi-hyperrings over X. Then*

(i) The intersection of A_is, for all $i \in I$ is a fuzzy multi-hyperring over X;

(ii) The union of A_is, for all $i \in I$ is a fuzzy multi-hyperring over X if this family satisfies in either ascending or descending chain condition.

Proof. *(i)* Put $A = \cap_{i\in I}A_i$. So $CM_A(t) = \bigwedge_{i\in I} CM_{A_i}(t)$, for all $t \in X$. Since for all $i \in I$, A_i is a fuzzy multi-hyperring over *X*, then $\bigwedge_{z\in x+y} CM_{A_i}(z) \geq CM_{A_i}(x) \wedge CM_{A_i}(y)$. Thus

$$\begin{aligned} \bigwedge_{z\in x+y} CM_A(z) &= \bigwedge_{z\in x+y} (\bigwedge_{i\in I} CM_{A_i}(z)) = \bigwedge_{i\in I}(\bigwedge_{z\in x+y} CM_{A_i}(z)) \\ &\geq \bigwedge_{i\in I}(CM_{A_i}(x) \wedge CM_{A_i}(y)) = (\bigwedge_{i\in I} CM_{A_i}(x)) \wedge (\bigwedge_{i\in I} CM_{A_i}(y)) \\ &= CM_A(x) \wedge CM_A(y). \end{aligned}$$

Also $CM_A(-x) = \bigwedge_{i\in I} CM_{A_i}(-x) \geq \bigwedge_{i\in I} CM_{A_i}(-x) = CM_A(-x)$. Moreover

$$\begin{aligned} CM_A(x.y) &= \bigwedge_{i\in I} CM_{A_i}(x.y) \geq \bigwedge_{i\in I}(CM_{A_i}(x) \wedge CM_{A_i}(y)) \\ &= (\bigwedge_{i\in I} CM_{A_i}(x)) \wedge (\bigwedge_{i\in I} CM_{A_i}(y)) = CM_A(x) \wedge CM_A(x). \end{aligned}$$

Therefore *A* is a fuzzy multi-hyperring over *X*.

(ii) Put $B = \bigcup_{i\in I} A_i$. Then $CM_B(k) = \bigvee_{i\in I} CM_{A_i}(k)$, for all $k \in X$. If $\{A_i; i \in I\}$ satisfies in the ascending chain condition then there exists $i_n \in I$ such that $CM_{A_{i_n}}(x) = \bigvee_{i\in I} CM_{A_i}(x) = CM_B(x)$, for all $x \in X$. Thus

$$\bigwedge_{z\in x+y} CM_B(z) = \bigwedge_{z\in x+y} CM_{A_{i_n}}(z) \geq CM_{A_{i_n}}(x) \wedge CM_{A_{i_n}}(y) = CM_B(x) \wedge CM_B(y).$$

Also $CM_B(-x) = CM_{A_{i_n}}(-x) \geq CM_{A_{i_n}}(x) = CM_B(x)$. Moreover,

$$CM_B(x.y) = CM_{A_{i_n}}(x.y) \geq CM_{A_{i_n}}(x) \wedge CM_{A_{i_n}}(y) = CM_B(x) \wedge CM_B(y).$$

If $\{A_i; i \in I\}$ satisfies in the descending chain condition, then $CM_{A_{i_1}}(x) = \bigvee_{i\in I} CM_{A_i}(x) = CM_B(x)$ and so by the same procedure, we can arrive at the result. □

Proposition 3.8. *Let $\{A_i; i \in I\}$ be an arbitrary family of fuzzy multi-hyperrings over X. Then $-(\bigcap_{i\in I} A_i) = \bigcap_{i\in I}(-A_i)$ and $-(\bigcup_{i\in I} A_i) = \bigcup_{i\in I}(-A_i)$.*

Proof. For all $x \in X$,

$$\begin{aligned} CM_{(-\bigcup_{i\in I} A_i)}(x) &= CM_{\bigcup_{i\in I} A_i}(-x) = \bigvee_{i\in I} CM_{A_i}(-x) \\ &= \bigvee_{i\in I} CM_{-A_i}(x) = CM_{\bigcup_{i\in I}(-A_i)}(x). \end{aligned}$$

Hence $-(\bigcup_{i\in I} A_i) = \bigcup_{i\in I}(-A_i)$. Similarly, we can prove that $-(\bigcap_{i\in I} A_i) = \bigcap_{i\in I}(-A_i)$. □

Definition 3.9. Let *A* be a fuzzy multi-hyperring over *X*. If $A = \langle a\rangle$, for some $a \in X$, then *a* is called the generator of *A*. The fuzzy multi-hyperring generated by *A*, denoted by $\langle A\rangle$, is the intersection of all fuzzy multi-hyperrings containing *A*, and so $\langle A\rangle = \bigcap\{B; B \in FMHR(X), A \subseteq B\}$, that implies $\langle A\rangle$ is the smallest fuzzy multi-hyperring containing *A*.

Theorem 3.10. *Let A and B be two fuzzy multi-hyperrings over X such that $CM_A(0) = CM_B(0)$. If $A \oplus B$ is a fuzzy multi-hyperring over X, then $A \oplus B = \langle A \cup B\rangle$.*

Proof. Let $x \in X$. Since $CM_A(0) = CM_B(0)$, then

$$\begin{aligned} CM_{A\oplus B}(x) &= \bigvee\{CM_A(y) \wedge CM_B(z); y, z \in X, x \in y + z\} \\ &\geq CM_A(x) \wedge CM_B(0) = CM_A(x) \wedge CM_A(0) = CM_A(x). \end{aligned}$$

Thus $A \subseteq A \oplus B$. Similarly, $B \subseteq A \oplus B$ and so $A \cup B \subseteq A \oplus B$. Now, let *C* be any fuzzy multi-hyperring over *X* containing $A \cup B$. Since for all $y, z \in X$, $CM_C(y) \geq CM_A(y)$ and $CM_C(z) \geq CM_B(z)$, then

$$\begin{aligned} CM_{C\oplus C}(x) &= \bigvee\{CM_C(y) \wedge CM_C(z); y, z \in X, x \in y + z\} \\ &\geq \bigvee\{CM_A(y) \wedge CM_B(z); y, z \in X, x \in y + z\} = CM_{A\oplus B}(x). \end{aligned}$$

Thus $A \oplus B \subseteq C \oplus C$. Now, since *C* is a fuzzy multi-hyperring over *X*, then

$$\bigwedge_{x\in y+z} CM_C(x) \geq CM_C(y) \wedge CM_C(z), \text{ for all } y, z \in X.$$

Therefore $CM_C(x) \geq \{CM_C(y) \wedge CM_C(z); y, z \in X, x \in y + z\}$. Hence

$$CM_C(x) \geq \bigvee\{CM_C(y) \wedge CM_C(z); y, z \in X, x \in y + z\} = CM_{C\oplus C}(x).$$

Thus $C \oplus C \subseteq C$ and $A \oplus B \subseteq C$. Therefore $A \oplus B$ is generated by $A \cup B$. □

4 Homomorphism and Direct Product of Fuzzy multi-hyperrings

In the following, we study some homomorphic properties of fuzzy multi-hyperrings.

Definition 4.1. Let *X* and *Y* be two Krasner hyperrings and $f : X \longrightarrow Y$ be a homomorphism. Suppose that *A* and *B* are fuzzy multi-hyperrings over *X* and *Y*, respectively. If $(f(A) \subseteq B)$ $f(A) = B$, then *A* is called (weakly) homomorphic to *B*, denoted by $(A \sim B)$ $A \approx B$ and if f is an isomorphism with $(f(A) \subseteq B)$ $f(A) = B$ then *A* is called (weakly) isomorphic to *B*, denoted by $(A \simeq B)$ $A \cong B$. The image of *A* under *f*, denoted by *f(A)*, is a fuzzy multiset of *Y* defined by $CM_{f(A)}(y) = \vee_{x\in f^{-1}(y)} CM_A(x)$, if $f^{-1}(y) \neq \emptyset$ and $CM_{f(A)}(y) = 0$, otherwise. Moreover, the inverse image of *B* under *f*, denoted by $f^{-1}(B)$, is a fuzzy multiset of *X* defined by $CM_{f^{-1}(B)}(x) = CM_B(f(x))$, for all $x \in X$.

Theorem 4.2. *Let X and Y be two Krasner hyperrings and $f : X \longrightarrow Y$ be a homomorphism of Krasner hyperrings. Then*

(i) if A is a fuzzy multi-hyperring over X, then $f(A)$ is a fuzzy multi-hyperring over Y;

(ii) if B is a fuzzy multi-hyperring over Y, then $f^{-1}(B)$ is a fuzzy multi-hyperring over X.

Proof. (i) Let $x, y \in Y$ and $z \in x + y$. If $f^{-1}(x) = \emptyset$ or $f^{-1}(y) = \emptyset$, then $CM_{f(A)}(x) = 0$ or $CM_{f(A)}(y) = 0$. Thus $\bigwedge_{z\in x-y} CM_{f(A)}(z) \geq 0 = CM_{f(A)}(x) \wedge CM_{f(A)}(y)$. Moreover, we have $CM_{f(A)}(x.y) \geq 0 = CM_{f(A)}(x) \wedge CM_{f(A)}(y)$. Hence, the result holds by Theorem 2.6. If $f^{-1}(x) \neq \emptyset$ and $f^{-1}(y) \neq \emptyset$, then $\exists r, s \in X$ such that $f(r) = x$ and $f(s) = y$, then

$$CM_A(r) = \bigvee_{f(t)=x} CM_A(t) = CM_{f(A)}(x) \text{ and } CM_A(s) = \bigvee_{f(t)=y} CM_A(t) = CM_{f(A)}(y).$$

Since f is a homomorphism, then $z \in f(r) + f(s) = f(r + s)$. Hence $\exists k \in r + s$; $z = f(k)$. Since *A* is a fuzzy multi-hyperring over *X*, then

$$\begin{aligned} CM_{f(A)}(z) &= \bigvee_{f(t)=z} CM_A(t) \geq CM_A(k) \geq \bigwedge_{z\in r+s} CM_A(z) \\ &\geq CM_A(r) \wedge CM_A(s) = CM_{f(A)}(x) \wedge CM_{f(A)}(y). \end{aligned}$$

Thus $\bigwedge_{z\in x+y} CM_{f(A)}(z) \geq CM_{f(A)}(x) \wedge CM_{f(A)}(y)$. Also

$$CM_{f(A)}(-x) = \bigvee_{f(t)=-x} CM_A(t) \geq CM_A(-r) \geq CM_A(r) = CM_{f(A)}(x).$$

Moreover,

$$\begin{aligned} CM_{f(A)}(x.y) &= \bigvee_{f(t)=x.y} CM_A(t) \geq CM_A(r.s) \geq CM_A(r) \wedge CM_A(s) \\ &= CM_{f(A)}(x) \wedge CM_{f(A)}(y). \end{aligned}$$

Therefore $f(A)$ is a fuzzy multi-hyperring over *Y*.

(ii) Let *B* be a fuzzy multi-hyperring over *Y*. Since *f* is a homomorphism, then for all $x, y \in X$,

$$\begin{aligned} \bigwedge_{z\in x+y} CM_{f^{-1}(B)}(z) &= \bigwedge_{z\in x+y} CM_B(f(z)) = \bigwedge_{t\in f(x)+f(y)} CM_B(t) \\ &\geq CM_B(f(x)) \wedge CM_B(f(y)) \\ &= CM_{f^{-1}(B)}(x) \wedge CM_{f^{-1}(B)}(y). \end{aligned}$$

Also $CM_{f^{-1}(B)}(-x) = CM_B(f(-x)) = CM_B(-f(x)) \geq CM_B(f(x)) = CM_{f^{-1}(B)}(x)$, and

$$\begin{aligned} CM_{f^{-1}(B)}(x.y) &= CM_B(f(x.y)) = CM_B(f(x).f(y)) \geq CM_B(f(x)) \wedge CM_B(f(x)) \\ &= CM_{f^{-1}(B)}(x) \wedge CM_{f^{-1}(B)}(y). \end{aligned}$$

Therefore $f^{-1}(B)$ is a fuzzy multi-hyperring over *X*. □

Proposition 4.3. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two homomorphisms of Krasner hyperrings and A, B be two fuzzy multi-hyperrings over X, Z, respectively. Then*

(i) $(gf)(A) = g(f(A))$;

(ii) $(gf)^{-1}(B) = f^{-1}(g^{-1}(B))$.

Proof. *(i)* Let $z \in Z$. If $g^{-1}(z) = \emptyset$, then clearly the result holds. If $g^{-1}(z) \neq \emptyset$, then

$$\begin{aligned} CM_{g(f(A))}(z) &= \bigvee_{y\in g^{-1}(z)} CM_{f(A)}(y) = \bigvee_{y\in g^{-1}(z)} (\bigvee_{x\in f^{-1}(y)} CM_A(x)) \\ &= \bigvee_{x\in (gf)^{-1}(z)} CM_A(x) = CM_{(gf)(A)}(z). \end{aligned}$$

Therefore $g(f(A)) = (gf)(A)$.

(ii) Let $x \in X$. Then

$$\begin{aligned} CM_{(gf)^{-1}(B)}(x) &= CM_B((gf)(x)) = CM_B(g(f(x))) = CM_{g^{-1}(B)}(f(x)) \\ &= CM_{f^{-1}(g^{-1}(B))}(x). \end{aligned}$$

Hence $(gf)^{-1}(B) = f^{-1}(g^{-1}(B))$. □

Theorem 4.4. *Let $f : X \longrightarrow Y$ be a homomorphism of Krasner hyperrings and $\{A_i; i \in I\}$ and $\{B_j; j \in J\}$ be two arbitrary families of fuzzy multi-hyperrings over X and Y, respectively. Then*

(i) $f(-A_i) = -f(A_i)$ *and* $f^{-1}(-B_j) = -f^{-1}(B_j)$, *for all* $i \in I, j \in J$;

(ii) $f(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f(A_i)$, $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$ *and* $f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j)$, $f(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f(B_j)$.

(iii) $f(A_{[\alpha]}) = (f(A))_{[\alpha]}$ *(if $\{A_i; i \in I\}$ satisfies in ascending chain condition) and* $f^{-1}(B_{[\alpha]}) = (f^{-1}(B))_{[\alpha]}$.

Proof. Let $x \in X$ and $y \in Y$.

(i) By Theorem 2.5*(ii)*, we have

$$\begin{aligned} CM_{f(-A_i)}(y) &= \bigvee_{x\in f^{-1}(y)} CM_{-A_i}(x) = \bigvee_{x\in f^{-1}(y)} CM_{A_i}(-x) = \bigvee_{x\in f^{-1}(y)} CM_{A_i}(x) \\ &= CM_{f(A_i)}(y) = CM_{f(A_i)}(-y) = CM_{(-f(A_i))}(y). \end{aligned}$$

Hence $f(-A_i) = -f(A_i)$, for all $i \in I$. Moreover,

$$\begin{aligned} CM_{-f^{-1}(B_j)}(x) &= CM_{f^{-1}(B_j)}(-x) = CM_{B_j}(f(-x)) = CM_{B_j}(-f(x)) \\ &= CM_{-B_j}(f(x)) = CM_{f^{-1}(-B_j)}(x). \end{aligned}$$

Thus $f^{-1}(-B_j) = -f^{-1}(B_j)$, for all $j \in J$.

(ii) We have

$$\begin{aligned} CM_{f(\bigcup_{i\in I} A_i)}(y) &= \bigvee_{x\in f^{-1}(y)} CM_{\bigcup_{i\in I} A_i}(x) = \bigvee_{x\in f^{-1}(y)} (\bigvee_{i\in I} CM_{A_i})(x) \\ &= \bigvee_{i\in I}(\bigvee_{x\in f^{-1}(y)} CM_{A_i}(x)) = \bigvee_{i\in I} CM_{f(A_i)}(y) = CM_{\bigcup_{i\in I} f(A_i)}(y). \end{aligned}$$

So $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$. Similarly, $f(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f(A_i)$. Also

$$\begin{aligned} CM_{f^{-1}(\bigcup_{j\in J} B_j)}(x) &= CM_{\bigcup_{j\in J} B_j}(f(x)) = \bigvee_{j\in J} CM_{B_j}(f(x)) = \bigvee_{j\in J} CM_{f^{-1}(B_j)}(x) \\ &= CM_{\bigcup_{j\in J} f^{-1}(B_j)}(x). \end{aligned}$$

Hence $f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$. Similarly, $f(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f(B_j)$.

(iii) If $x \in f(A_{[\alpha]})$, then $x = f(t)$, for some $t \in A_{[\alpha]}$ and so $CM_A(t) \geq \alpha$, that implies $CM_{f(A)}(x) = \vee_{z\in f^{-1}(x)} CM_A(z) \geq CM_A(t) \geq \alpha$. Thus $x \in (f(A))_{[\alpha]}$ and so $f(A_{[\alpha]}) \subseteq (f(A))_{[\alpha]}$. On the other hand, if $y \in (f(A))_{[\alpha]}$, then $CM_{f(A)}(y) = \vee_{k\in f^{-1}(y)} CM_A(k) \geq \alpha$ and by hypothesis, $CM_A(k) \geq \alpha$, for some $k \in f^{-1}(y)$. Therefore $k \in A_{[\alpha]}$ and so $y = f(k) \in f(A_{[\alpha]})$. Thus $(f(A))_{[\alpha]} \subseteq f(A_{[\alpha]})$. Therefore equality holds. Now, we have $x \in (f^{-1}(B))_{[\alpha]} \Longleftrightarrow CM_{f^{-1}(B)}(x) \geq \alpha \Longleftrightarrow CM_B(f(x)) \geq \alpha \Longleftrightarrow f(x) \in B_{[\alpha]} \Longleftrightarrow x \in f^{-1}(B_{[\alpha]})$. Hence the result holds. □

In the following, we study some properties of the direct product of fuzzy multi-hyperrings.

Theorem 4.5. *Let X, Y be two Krasner hyperrings and A, B be two fuzzy multisets drawn from X, Y respectively such that $CM_A(0) = CM_B(0')$, where $0 \in X$ and $0' \in Y$. Consider the direct product of A and B by $CM_{A\times B}(x,y) = CM_A(x) \wedge CM_B(y)$, for all $x \in X$, $y \in Y$. Then A and B are (commutative) fuzzy multi-hyperrings over X and Y, respectively, if and only if $A \times B$ is a (commutative) fuzzy multi-hyperring over $X \times Y$.*

Proof. Let $(x_1,x_2),(y_1,y_2) \in X \times Y$. Since *A* and *B* are two fuzzy multi-hyperrings over *X* and *Y* respectively, then

$$\begin{aligned}
\bigwedge_{(x,y)\in(x_1,x_2)+(y_1,y_2)} CM_{A\times B}(x,y) &= \bigwedge_{(x,y)\in(x_1+y_1,x_2+y_2)} (CM_A(x) \wedge CM_B(y)) \\
&= (\bigwedge_{x\in x_1+y_1} CM_A(x)) \wedge (\bigwedge_{y\in x_2+y_2} CM_B(y)) \\
&\geq (CM_A(x_1) \wedge CM_A(y_1)) \wedge (CM_B(x_2) \wedge CM_B(y_2)) \\
&= (CM_A(x_1) \wedge CM_B(x_2)) \wedge (CM_A(y_1) \wedge CM_B(y_2)) \\
&= CM_{A\times B}(x_1,x_2) \wedge CM_{A\times B}(y_1,y_2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
CM_{A\times B}(-(x_1,x_2)) &= CM_{A\times B}(-x_1,-x_2) = CM_A(-x_1) \wedge CM_B(-x_2) \\
&\geq CM_A(x_1) \wedge CM_B(x_2) = CM_{A\times B}(x_1,x_2).
\end{aligned}$$

Also,

$$\begin{aligned}
CM_{A\times B}((x_1,x_2).(y_1,y_2)) &= CM_{A\times B}(x_1.y_1, x_2.y_2) = CM_A(x_1.y_1) \wedge CM_B(x_2.y_2) \\
&\geq (CM_A(x_1) \wedge CM_A(y_1)) \wedge (CM_B(x_2) \wedge CM_B(y_2)) \\
&\geq (CM_A(x_1) \wedge CM_B(x_2)) \wedge (CM_A(y_1) \wedge CM_B(y_2)) \\
&= CM_{A\times B}(x_1,x_2) \wedge CM_{A\times B}(y_1,y_2).
\end{aligned}$$

Hence the direct product of *A* and *B* is a fuzzy multi-hyperring over $X \times Y$. Now, if *A* and *B* are commutative, then

$$\begin{aligned}
CM_{A\times B}((x_1,x_2).(y_1,y_2)) &= CM_{A\times B}(x_1.y_1, x_2.y_2) = CM_A(x_1.y_1) \wedge CM_B(x_2.y_2) \\
&= CM_A(y_1.x_1) \wedge CM_B(y_2.x_2) = CM_{A\times B}(y_1.x_1, y_2.x_2) \\
&= CM_{A\times B}((y_1,y_2).(x_1,x_2)).
\end{aligned}$$

Thus $A \times B$ is commutative.

Conversely, let $x \in A$, $y \in B$ and *A*, *B* be two fuzzy multisets drawn from *X*, *Y*, respectively.

Since $A \times B$ is a fuzzy multi-hyperring over $X \times Y$, then

$$\begin{aligned}
\bigwedge_{z\in x+y} CM_A(z) &= \bigwedge_{z\in x+y} (CM_A(z)\wedge CM_A(0)) = \bigwedge_{z\in x+y} (CM_A(z)\wedge CM_B(0'))\\
&= \bigwedge_{(z,0')\in(x+y,0'+0')} CM_{A\times B}(z,0') = \bigwedge_{(z,0')\in(x,0')+(y,0')} CM_{A\times B}(z,0')\\
&\geq CM_{A\times B}(x,0')\wedge CM_{A\times B}(y,0')\\
&= (CM_A(x)\wedge CM_B(0'))\wedge(CM_A(y)\wedge CM_B(0'))\\
&= CM_A(x)\wedge CM_A(y)\wedge CM_B(0') = CM_A(x)\wedge CM_A(y)\wedge CM_A(0)\\
&= CM_A(x)\wedge CM_A(y).
\end{aligned}$$

Moreover,

$$\begin{aligned}
CM_A(-x) &= CM_A(-x)\wedge CM_A(0) = CM_A(-x)\wedge CM_B(-0')\\
&= CM_{A\times B}(-x,-0') = CM_{A\times B}(-(x,0')) \geq CM_{A\times B}(x,0')\\
&= CM_A(x)\wedge CM_B(0')\\
&= CM_A(x)\wedge CM_A(0) = CM_A(x).
\end{aligned}$$

Also,

$$\begin{aligned}
CM_A(x.y) &= CM_A(x.y)\wedge CM_A(0.0) = CM_A(x.y)\wedge CM_B(0'.0')\\
&= CM_{A\times B}(x.y,0'.0') = CM_{A\times B}((x,0').(y,0'))\\
&\geq CM_{A\times B}(x,0')\wedge CM_{A\times B}(y,0') \geq (CM_A(x)\wedge CM_B(0'))\wedge(CM_A(y)\wedge CM_B(0'))\\
&= CM_A(x)\wedge CM_A(y)\wedge CM_A(0) = CM_A(x)\wedge CM_A(y).
\end{aligned}$$

Therefore *A* is a fuzzy multi-hyperring over *X*. Similarly, *B* is a fuzzy multi-hyperring over *Y*. Also, if $A \times B$ is commutative, then

$$\begin{aligned}
CM_A(x.y) &= CM_A(x.y)\wedge CM_A(0.0)\\
&= CM_A(x.y)\wedge CM_B(0'.0')\\
&= CM_{A\times B}((x.y,0'.0')) = CM_{A\times B}((x,0').(y,0'))\\
&= CM_{A\times B}((y,0').(x,0')) = CM_{A\times B}(y.x,0'.0')\\
&= CM_A(y.x)\wedge CM_B(0'.0') = CM_A(y.x)\wedge CM_A(0.0) = CM_A(y.x).
\end{aligned}$$

Therefore *A* is commutative on *X*. Similarly, we can prove that *B* is commutative on *Y*. □

Corollary 4.6. *Let X_i be Krasner hyperrings and A_i be (commutative) fuzzy multi-hyperrings over X_i, respectively, for $i = 1, 2, \cdots, n$. Then direct product of A_i defined by*

$$CM_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \cdots, x_n) = CM_{A_1}(x_1)\wedge CM_{A_2}(x_2)\wedge\cdots\wedge CM_{A_n}(x_n),$$

for all $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$, is a (commutative) fuzzy multi-hyperring over $X_1 \times X_2 \times \cdots \times X_n$.

Proof. The proof follows by Theorem 4.5. □

Corollary 4.7. *Let X, Y be Krasner hyperrings and A, B be fuzzy multi-hyperrings over X, Y, respectively. Then $(A \times B)_{[\alpha]} = A_{[\alpha]} \times B_{[\alpha]}$, for all $\alpha \in [0, 1]$.*

Proof. The proof follows by Theorem 2.13 and Theorem 4.5. □

Corollary 4.8. *Let A and B two be two fuzzy multi-hyperrings over two X and Y, respectively. Then*

(i) $-(A \times B) = (-A) \times (-B)$*;*

(ii) $CM_{A\times B}(0,0') \geq CM_{A\times B}(x,y)$,
where $(x,y) \in X \times Y$ *and* 0 *and* $0'$ *are identity elements of X and Y, respectively.*

Proof. The proof follows by Theorem 2.5 and Theorem 4.5. □

Example 4.9. Let $X = \frac{\mathbb{Z}_{12}}{H}$ and $Y = \{0,1,2\}$ be two Krasner hyperrings with hyperoperations and binary operations defined in Example 2.2(i) and Example 2.2(ii), respectively. Consider $A = \{\langle\frac{1,0.6,0.5}{\bar{0}H}\rangle, \langle\frac{0.8,0.4,0.1}{\bar{2}H}\rangle, \langle\frac{0.9,0.5,0.4}{\bar{4}H}\rangle, \langle\frac{0.8,0.4,0.1}{\bar{6}H}\rangle\}$ and $B = \{\langle\frac{(1,0.5,0.2)}{0}\rangle\}$. It is clear that *A* and *B* are fuzzy multi-hyperrings over *X* and *Y*, respectively and we have
$A \times B = \{\langle\frac{(1,0.5,0.2)}{(\bar{0}H,0)}\rangle, \langle\frac{(0.8,0.4,0.1)}{(\bar{2}H,0)}\rangle, \langle\frac{(0.9,0.5,0.2)}{(\bar{4}H,0)}\rangle, \langle\frac{(0.8,0.4,0.1)}{(\bar{6}H,0)}\rangle\}$.

Theorem 4.10. *Let* $f : X \longrightarrow Z$ *and* $g : Y \longrightarrow W$ *be two homomorphisms of Krasner hyperrings and A, B, C, D be fuzzy multi-hyperrings over X, Y, Z, W, respectively. Consider* $f \times g : X \times Y \longrightarrow Z \times W$ *by setting* $(f \times g)(x,y) = (f(x), f(y))$. *Then*

(i) $f \times g$ *is a homomorphism of Krasner hyperrings;*

(ii) $(f \times g)(A \times B)$ *is a fuzzy multi-hyperring over* $Z \times W$ *such that* $(f \times g)(A \times B) = f(A) \times g(B)$*;*

(iii) $(f \times g)^{-1}(C \times D)$ *is a fuzzy multi-hyperring over* $X \times Y$ *such that* $(f \times g)^{-1}(C \times D) = f^{-1}(C) \times f^{-1}(D)$.

Proof. *(i)* Let $(a,b),(c,d) \in X \times Y$. Since *f* and *g* are homomorphisms of Krasner hyperrings, then

$$\begin{aligned}(f \times g)((a,b)+(c,d)) &= (f \times g)(\{(x,y);(x,y) \in (a+c,b+d)\}) \\ &= \{(f(x),g(y)); x \in a+c, y \in b+d\} \\ &= (\{f(x); x \in a+c\}, \{g(y); y \in b+d\}) \\ &= (f(a+c), g(b+d)) = (f(a)+f(c), g(b)+g(d)) \\ &= (f(a),g(b)) + (f(c),g(d)) = (f \times g)(a,b) + (f \times g)(c,d).\end{aligned}$$

Moreover

$$\begin{aligned}(f \times g)((a,b).(c,d)) &= (f \times g)(a.c, b.d) = (f(a.c), g(b.d)) \\ &= (f(a).f(c), g(b).g(d)) = (f(a),g(b)).(f(c),g(d)) \\ &= (f \times g)(a,b).(f \times g)(c,d).\end{aligned}$$

Also, we have $(f \times g)(0,0) = (f(0),g(0)) = (0,0)$. Thus $f \times g$ is a homomorphism of Krasner hyperrings.

(ii) The first result is obtained by Theorem 4.5 and Theorem 4.2, so we prove that $(f \times g)(A \times B) = f(A) \times g(B)$. Let $(z,w) \in Z \times W$. If $(f \times g)^{-1}(z,w) = \emptyset$, then clearly the result holds. If $(f \times g)^{-1}(z,w) \neq \emptyset$, then we get $(f \times g)^{-1}(z,w) = (f^{-1}(z), g^{-1}(w))$. Therefore

$$\begin{aligned}C_{(f\times g)(A\times B)}(z,w) &= \bigvee_{(x,y)\in(f\times g)^{-1}(z,w)} C_{A\times B}(x,y) \\ &= \bigvee_{(x,y)\in(f^{-1}(z),g^{-1}(w))} (C_A(x) \wedge C_B(y)) \\ &= (\bigvee_{x\in f^{-1}(z)} C_A(x)) \wedge (\bigvee_{y\in g^{-1}(w)} C_B(y)) \\ &= C_{f(A)}(z) \wedge C_{g(B)}(w) = C_{f(A)\times g(B)}(z,w).\end{aligned}$$

(iii) Similarly, by Theorem 4.5 and Theorem 4.2, we have the first part, and therefore we prove that $(f \times g)^{-1}(C \times D) = f^{-1}(C) \times f^{-1}(D)$. Let $(x,y) \in X \times Y$. Then

$$\begin{aligned}C_{(f\times g)^{-1}(C\times D)}(x,y) &= C_{C\times D}((f \times g)(x,y)) = C_{C\times D}(f(x),g(y)) \\ &= C_C(f(x)) \wedge C_D(g(y)) = C_{f^{-1}(C)}(x) \wedge C_{g^{-1}(D)}(y) \\ &= C_{f^{-1}(C)\times g^{-1}(D)}(x,y).\end{aligned}$$

□

Corollary 4.11. *Let $f_i : X_1 \longrightarrow Y_i$, $i = 1, 2, \cdots, n$ be homomorphisms of Krasner hyperrings and A_1, A_2, $\cdots$, A_n and B_1, B_2, $\cdots$, B_n be fuzzy multi-hyperrings over X_1, X_2, $\cdots$, X_n and Y_1, Y_2, $\cdots$, Y_n, respectively. Consider $f_1 \times f_2 \times \cdots \times f_n : X_1 \times X_2 \times \cdots \times X_n \longrightarrow Y_1 \times Y_2 \times \cdots \times Y_n$ by setting $(f_1 \times f_2 \times \cdots \times f_n)(x_1, x_2, \cdots, x_n) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n))$. Then*

(i) $f_1 \times f_2 \times \cdots \times f_n$ *is a homomorphism of Krasner hyperrings;*

(ii) $(f_1 \times f_2 \times \cdots \times f_n)(A_1 \times A_2 \times \cdots \times A_n)$ *is a fuzzy multi-hyperring over* $Y_1 \times Y_2 \times \cdots \times Y_n$*, such that* $(f_1 \times f_2 \times \cdots \times f_n)(A_1 \times A_2 \times \cdots \times A_n) = f(A_1) \times f(A_2) \times \cdots \times f(A_n)$*;*

(iii) $(f_1 \times f_2 \times \cdots \times f_n)^{-1}(B_1 \times B_2 \times \cdots \times B_n)$ *is a fuzzy multi-hyperring over* $X_1 \times X_2 \times \cdots \times X_n$*, such that* $(f_1 \times f_2 \times \cdots \times f_n)^{-1}(B_1 \times B_2 \times \cdots \times B_n) = f^{-1}(B_1) \times f^{-1}(B_2) \times \cdots \times f^{-1}(B_n)$.

Proof. The proof is straightforward. □

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