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Isotropy group and Normalizer of a $\mathcal{S}$-Topological Transformation Group

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Abstract. The isotropy group and normalizer of a $\mathcal{S}$-topological transformation group is explored in this work. For every $\mathtt{x} \in X$, it is ascertained that the intersection of all isotropy groups equals the kernel of semi totally continuous action and the isotropy group $G_{\mathtt{x}}$ is clopen for a Frechet space X. Subsequently, it is proved that the quotient topological group together with a compact Hausdorff space X constitutes a $\mathcal{S}$-topological transformation group. Also, it is proved that the kernel of semi totally continuous action is the largest normal subgroup of G and some basic properties of the isotropy group and normalizer of a $\mathcal{S}$-topological transformation group are exhibited.

1 Introduction

The work of Montogomery and Zippin on Hilbert's fifth problem [20] provided the foundation for the study of topological transformation group. A topological transformation group is a transformation group that sustains the topological structure of a space. A fundamental aspect of comprehending the symmetries of geometric object is group action. Group actions provide a solid foundation for exploring the symmetries and transformations of mathematical objects across different mathematical disciplines. The hereditary group rings of groups acting on trees was discussed by R. M. S. Mahmood and M. M. Almahameed [11] in 2020. The isotropy group of a transformation group is a subgroup that retains the given point. In the exploration of Lie groups and their representations, the isotropy group of a transformation group is significant. The isotropy group is critical in comprehending the symmetries and transformations of mathematical objects. The isotropy group is a fundamental notion in the study of transformation groups, giving an approach to describe the symmetries and invariance features of spaces under group actions. It is used to study the structure of the space being utilized and to determine the orbits of the group action. Isotropy group is crucial in studying equivariant maps, equivalence of transformation groups, and homogeneous spaces. In a transformation group, a set of transformations that, when conjugated with subgroup elements, provide transformations that stay within the subgroup is called the normalizer of a subgroup. This notion is used in many areas of mathematics, such as algebra and geometry, and it is essential in the exploration of group theory. The normalizer is critical for interpreting the structure and symmetries of a transformation group. It facilitates identifying transformations that stabilize the subgroup in consideration. H. Ishi [9] proved that the dimension of the isotropy group of a bounded homogeneous domain is atleast one. In 2020, L. N. Bertoncello and D. Levcovitz [3] discussed that the isotropy group of a simple Shamsuddin derivation of the polynomial ring is trivial. L. G. Mendes and I. pan [12] examined that the Shamsuddin derivation is simple if and only if its isotropy group is trivial in 2017. In 2021, R. Baltazer and M. Veloso [1] determined the isotropy group of a locally nilpotent derivation on Danielewski surfaces. S. S. Coutinho and L. F. G. Jales [5] decribed a family of foliations of even degree with one singularity and cyclic isotropy group in 2021. J. Funk, P. Hofstra and B. Steinberg [7] defined and studied the idea of isotropy from a topos theoretic way. In 1965, J. Nagasawa [13, 14] discussed linear isotropy group of a Riemannian manifold and infinitesimal linear isotropy group of an affinely connected manifold and investigated the sufficient conditions for linear isotropy group and infinitesimal linear isotropy group to be equal. G. Tsagas [19] examined the linear isotropy group of an s-manifold in 1977. Linear isotropy group of an affine symmetric space was discussed by J. Nagasawa [15] in 1972. The transitivity of the action of the isotropy group on the Silov boundary was given by J. E. D'Atri, J. Dorfmeister and Y. D. Zhao [6]. J. Tomiyama [18] elucidated the connections among normalizers in the transformation group C*-algebra and those homeomorphisms in topological full group.

In this paper, the main section is focused on the study of isotropy group and normalizer of a $\mathcal{S}$-topological transformation group. It is shown that the intersection of all isotropy group is equal to the kernel of semi totally continuous action at any point $\mathrm{x} \in X$. Later, the isotropy group G_{x} is both open and closed when X is a Frechet space and if X is a finite Frechet space, the subgroup kernel of semi totally continuous action will be both open and closed. Eventually, the quotient space $G\big/\mathrm{Ker}\Psi$ creates a topological group whenever kernel of semi totally continuous action is a normal subgroup of G. Also, it is proved that the compact Hausdorff space X combined with a topological group $G\big/\mathrm{Ker}\Psi$ forms a $\mathcal{S}$-topological transformation group. Finally, it is proved that the kernel of semi totally continuous action is the largest subgroup among the normal subgroups of G and some basic properties of isotropy group and normalizer are discussed. The paper is sectioned as follows. The first section pertains to the introduction. Preliminaries were discussed in the second section. The section three and four is examined with the basic properties of isotropy group and normalizer of a $\mathcal{S}$-topological transformation group and followed by the conclusion.

2 Preliminaries

Definition 2.1. [16] A topological group is a nonempty set G that satisfies the following conditions:
(1) G forms a group.
(2) G is a topological space.
(3) The maps $\varphi : G \times G \to G$ and $\alpha : G \to G$ defined by $\varphi(g, h) = gh$ and $\alpha(g) = g^{-1}$ are both continuous.

Definition 2.2. [4] A topological transformation group is a triplet (G, X, ζ), in which G is a topological group, X is a topological space, the map $\zeta : G \times X \to X$ is continuous, and satisfies the following conditions,
(1) $\zeta(e, \mathrm{x}) = \mathrm{x}$, for every $\mathrm{x} \in X$, where e is the identity element of G.
(2) $\zeta(h_2, \zeta(h_1, \mathrm{x})) = \zeta(h_2h_1, \mathrm{x})$, for all $h_1, h_2 \in G$ and $\mathrm{x} \in X$.

Definition 2.3. [2] A semi totally continuous function f is a map from a topological space X to a topological space Y in which the inverse image of all semi open subsets of Y is clopen in X.

Definition 2.4. [10] A subset of a topological space which is both open and closed is called a clopen set.

Definition 2.5. [2] A semi totally open function f is a map from a topological space X to a topological space Y in which the image of all semi open subsets of X is clopen in Y.

Definition 2.6. [17] A triplet (G, X, Ψ) is called a $\mathcal{S}$-topological transformation group when G is a topological group, X be a topological space, and $\Psi : G \times X \to X$, a semi totally continuous map, and it satisfies the following conditions,
(1) $\Psi(e, x) = x$, for every $x \in X$, where e represents the identity element of G.
(2) $\Psi(h_2, \Psi(h_1, x)) = \Psi(h_2h_1, x)$, for every $h_1, h_2 \in G$ and $x \in X$.

Theorem 2.7. [17] For a $\mathcal{S}$-topological transformation group (G, X, Ψ), $h \in G$, let a map Ψ_h : $X \to X$ be defined by $\Psi_h(x) = \Psi(h, x)$. Then Ψ_h and its inverse are semi totally continuous.

Definition 2.8. [17] The kernel of semi totally continuous action Ψ is given by, $\mathrm{Ker}\Psi = \{\mathrm{h} \in \mathrm{G} \,|\, \mathrm{hx=x}, \forall\, \mathrm{x} \in \mathrm{X}\}$.

Theorem 2.9. [17] KerΨ is a normal subgroup of G.

Theorem 2.10. [17] The kernel of the homomorphism $\Phi : G \to \text{STC}_G(X)$ will be called the kernel of the action $\Psi : G \times X \to X$.

Theorem 2.11. [8] For any subset B and an open subset O of a topological group G, BO and OB are open.

3 Isotropy group of a $\mathcal{S}$-topological transformation group

In this section, the isotropy group of a $\mathcal{S}$-topological transformation group is defined and explored some fundamental properties of isotropy group of a $\mathcal{S}$-topological transformation group.

Definition 3.1. The isotropy group at $\mathtt{x}$ for a $\mathcal{S}$-topological transformation group $\big(G, X, \Psi\big)$ is the set $G_{\mathtt{x}} = \{h \in G \big| h\mathtt{x} = \mathtt{x}\}$, for a fixed $\mathtt{x} \in X$.

Example 3.2. Let $GL_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ and $\mathbb{Z}_2^2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$. Now, let $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, r = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, s = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, u = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then the map $\Psi : GL_2(\mathbb{Z}_2) \times \mathbb{Z}_2^2 \to \mathbb{Z}_2^2$ given by $\Psi(\mathtt{x}, \mathtt{y}) = \mathtt{xy}$ is a semi totally continuous action with a discrete topology under matrix multiplication. Therefore $\big(GL_2(\mathbb{Z}_2), \mathbb{Z}_2^2, \Psi\big)$is a $\mathcal{S}$-topological transformation group. Then $G_a = \{p, q\}, G_b = \{p, t\}, G_c = \{p, s\}$ and $G_d = \{p, q, r, s, t, u\}$ are the isotropy groups of $\big(GL_2(\mathbb{Z}_2), \mathbb{Z}_2^2, \Psi\big)$.

Example 3.3. Let $G = \big\{e, i, j, ij \big| i^2 = e, j^2 = e, ij = ji\big\}$ be a group, $K = \big\{e, i\big\}$ be a subgroup of G and $G/K = h_1K = \big\{eK, jK\big\}$. The map $\Psi : G \times G/K \to G/K$ given by $\Psi(h', h_1K) = h'h_1K$ is a semi totally continuous action and $\big(G, G/K, \Psi\big)$ forms a $\mathcal{S}$-topological transformation group. Then $G_{eK} = \{e, i\}$ and $G_{jK} = \{e, i\}$ are the isotropy groups of $\big(G, G/K, \Psi\big)$.

Example 3.4. Let G be a group acting on itself by left multiplication equipped with discrete topology and $\Psi : G \times G \to G$, $(h, \mathtt{x}) \mapsto h\mathtt{x}$ is a semi totally continuous action. Then $\big(G, G, \Psi\big)$ is a $\mathcal{S}$-topological transformation group. Therefore $G_{\mathtt{x}} = \{e\}$ is the isotropy group of $\big(G, G, \Psi\big)$.

Remark 3.5. The isotropy group $G_{\mathtt{x}}$ is a subgroup of G.

Remark 3.6. For any $\mathtt{x} \in X$ and $h \in G$, $G_{h\mathtt{x}} = hG_{\mathtt{x}}h^{-1}$.

Definition 3.7. Let $\big(G, X, \Psi\big)$ be a $\mathcal{S}$-topological transformation group,

(i) if $G_{\mathtt{x}} = G$ for all $\mathtt{x} \in X$, then $\big(G, X, \Psi\big)$ is trivial.

(ii) if $G_{\mathtt{x}} = \{e\}$ for all $\mathtt{x} \in X$, then $\big(G, X, \Psi\big)$ is free.

(iii) if $G_{\mathtt{x}} = G$ or $\{e\}$ for all $\mathtt{x} \in X$, then $\big(G, X, \Psi\big)$ is semifree.

(iv) if $G_{\mathtt{x}}$ is finite for all $\mathtt{x} \in X$, then $\big(G, X, \Psi\big)$ is almost free.

(v) if $\bigcap_{\mathtt{x} \in X} G_{\mathtt{x}} = \{e\}$, then $\big(G, X, \Psi\big)$ is effective.

(vi) if ker Ψ is finite, then $\big(G, X, \Psi\big)$ is almost effective.

Example 3.8. The semi totally continuous action in Example 3.4 is free, since $G_{\mathtt{x}} = \{e\}$ for all $\mathtt{x} \in G$. Since every free action is semifree, effective, so the action is semifree, effective and the isotropy group $G_{\mathtt{x}}$ is finite for all $\mathtt{x} \in G$, the action is almost free.

Proposition 3.9. Ker$\Psi = \bigcap_{x \in X} G_x$.

Proof. Let $h \in$ KerΨ. Since Ker$\Psi = \{h \in G | h\mathrm{p} = \mathrm{p}, \forall\, \mathrm{p} \in X\}$, $h \in G_{\mathrm{p}}, \forall\, \mathrm{p} \in X$. Thus $h \in \bigcap_{\mathrm{p} \in X} G_{\mathrm{p}}$ and so Ker$\Psi \subseteq \bigcap_{x \in X} G_x$. On the other hand, let $k \in \bigcap_{x \in X} G_x$, which implies $k \in G_{\mathrm{p}}, \forall\, \mathrm{p} \in X$. Thus $h\mathrm{p} = \mathrm{p}, \forall\, \mathrm{p} \in X$. Hence $k \in$ KerΨ. Therefore the other inclusion is also true. □

Remark 3.10. Ker$\Psi \trianglelefteq G$, then by Proposition 3.9, $\bigcap_{x \in X} G_x \trianglelefteq G$.

Remark 3.11. The semi totally continuous action in the Example 3.8 is almost effective by the Proposition 3.9.

Proposition 3.12. For a map $i : G \to G \times X$ defined by $i(h) = (h, \mathrm{p})$, i is continuous and $G - G_{\mathrm{p}} = i^{-1}\big(\Psi^{-1}\big(X - \{\mathrm{p}\}\big)\big)$, for a fixed point $\mathrm{p} \in X$.

Proof. By the definition of product topology, i is continuous. For any $h \in G - G_{\mathrm{p}}$, $\Psi \circ i(h) = \Psi(h, \mathrm{p}) = h\mathrm{p} \neq \mathrm{p}$. Thus, $h \in i^{-1}\big(\Psi^{-1}\big(X - \{\mathrm{p}\}\big)\big)$. Therefore $G - G_{\mathrm{p}} \subset i^{-1}\big(\Psi^{-1}\big(X - \{\mathrm{p}\}\big)\big)$. Now, for any $h \in i^{-1}\big(\Psi^{-1}\big(X - \{\mathrm{p}\}\big)\big)$, $\Psi \circ i(h) = \Psi(h, \mathrm{p}) \in X - \{\mathrm{p}\}$ and $\Psi(h, \mathrm{p}) \neq \mathrm{p}$. Thus $h \in G - G_{\mathrm{p}}$. Therefore $i^{-1}\big(\Psi^{-1}\big(X - \{\mathrm{p}\}\big)\big) \subset G - G_{\mathrm{p}}$. Hence $G - G_{\mathrm{p}} = i^{-1}\big(\Psi^{-1}\big(X - \{\mathrm{p}\}\big)\big)$. □

Theorem 3.13. If X is a Frechet space, then G_x is clopen for all $x \in X$.

Proof. By the Proposition 3.12, $G - G_x = i^{-1}\big(\Psi^{-1}\big(X - \{x\}\big)\big)$. Assume that $X - \{x\}$ is open. Since Ψ is semi totally continuous, $\Psi^{-1}\big(X - \{x\}\big)$ is clopen. Since i is continuous, $i^{-1}\big(\Psi^{-1}\big(X - \{x\}\big)\big)$ is clopen. This implies $G - G_x$ is clopen. Hence G_x is clopen. □

Proposition 3.14. If X is a Frechet space, then the map $\zeta : X \to G_x$ given by $\zeta(x) = hx$, for $x \in X$ and $h \in H$ is semi totally open.

Proof. Since X is a Frechet space, then by Theorem 3.13, G_x is clopen. Clearly, ζ is bijective. So, the image of every semiopen in X is clopen in G_x. This implies ζ is semi totally open. □

Proposition 3.15. For a free and effective $\mathcal{S}$-topological transformation group, the topological group G is disconnected.

Proof. Since G_x is a subgroup of G and by the Theorem 3.13, G_x is clopen. Then by the Definition 3.7, G is disconnected. □

Corollary 3.16. If X is a finite Frechet space, then the subgroup KerΨ is both open and closed.

Proof. By the Theorem 3.13, G_x is both open and closed. For any $x \in X$, $G_x \leq G$ which implies $\bigcap_{x \in X} G_x \leq G$. Also, from the Proposition 3.9, Ker$\Psi = \bigcap_{x \in X} G_x$. It follows that Ker$\Psi$ is both open and closed subgroup of G. □

Remark 3.17. In Corollary 3.16, if X is an infinite Frechet space, then the subgroup KerΨ is not open which is given by the following example.

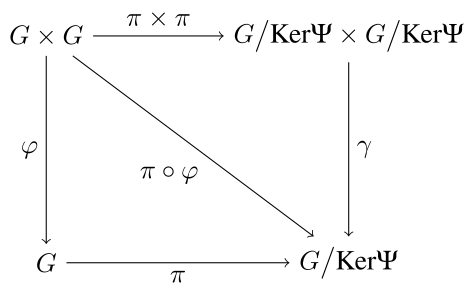
Example 3.18. Let G be a group under addition acting on an infinite Frechet space $\mathbb{Z}$ with finite complement topology and $\Psi : G \times X \to X$, $(h, x) \mapsto h + x$. Now, the isotropy group $G_x = \{e\}$ is closed but not open. Hence, Ker$\Psi = \bigcap_{x \in X} G_x$ is not open.

Proposition 3.19. For a topological group G and its clopen subgroup KerΨ, $G/$KerΨ is a quotient space and the projection map $\pi : G \to G/$KerΨ is surjective, continuous and open.

Proof. Let π from G to quotient set $G/$KerΨ is given by $h \mapsto hK$. Define a topology $\tau_{G/Ker\Psi} = \{\pi(U) | U \in \tau_G\}$, where τ_G is the topology on G. Thus, $G/$KerΨ is a quotient space. Since every coset has an inverse image, π is surjective and clearly, π is continuous. Now, let $\mathtt{U}$ be an open subset of G, then $\pi^{-1}(\pi(\mathtt{U})) = \mathtt{U}K = \bigcup\{\mathtt{U}k, k \in K\}$, which is open. So, $\pi(\mathtt{U})$ is open in $G/$KerΨ. Hence π is an open map. □

Proposition 3.20. $G/\text{Ker}\Psi$ is a topological group, for any clopen normal subgroup $\text{Ker}\Psi$ of G.

Proof. Since $\text{Ker}\Psi \trianglelefteq G$, $G/\text{Ker}\Psi$ forms a quotient group and by the Proposition 3.19, $G/\text{Ker}\Psi$ is a quotient space. Now, for $G/\text{Ker}\Psi$ to form a topological group, it is enough to show that the map $\gamma : G/\text{Ker}\Psi \times G/\text{Ker}\Psi \to G/\text{Ker}\Psi$ defined by $\gamma(h_1K, h_2K) = h_1h_2^{-1}K$ is continuous and this can be verified by the following commutative diagram.



By the Proposition 3.19, π is continuous, surjective and open. Thus $\pi \times \pi$ is surjective and since product of open map is open, so $\pi \times \pi$ is open. Since π and φ are continuous, so is $\pi \circ \varphi$. Since $\pi \times \pi$ is surjective and open, $\pi \circ \varphi$ is continuous, so γ is continuous. □

The next theorem shows that the topological group G/Ker Ψ along with a compact Hausdorff space X forms a $\mathcal{S}$-topological transformation group.

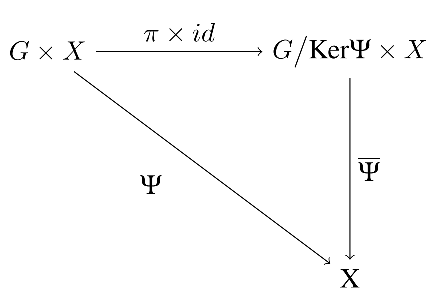
Theorem 3.21. If X is a compact Hausdorff space, G is a compact topological group and a map $\overline{\Psi} : G/\text{Ker}\Psi \times X \to X$ be defined by $\overline{\Psi}(hk, \mathtt{x}) = \Psi(h, \mathtt{x})$. Then $\left(G/\text{Ker}\Psi, X, \overline{\Psi}\,\right)$ forms a $\mathcal{S}$-topological transformation group.

Proof. To prove $\left(G/\text{Ker}\Psi, X, \overline{\Psi}\,\right)$ is a $\mathcal{S}$-topological transformation group. It is enough to show $\overline{\Psi}$ is semi totally continuous and satisfies the conditions of group actions.

$\overline{\Psi}$ is well defined,

$$\begin{aligned} h_1k\mathtt{x} &= h_2k\mathtt{x} \\ h_1\mathtt{x}(h_2\mathtt{x})^{-1} \in K &= Ker\Psi \\ \Psi\big(h_1\mathtt{x}(h_2\mathtt{x})^{-1}\big) &= e \\ h_1\mathtt{x}(h_2\mathtt{x})^{-1} &= e \\ h_1\mathtt{x} &= h_2\mathtt{x} \\ \overline{\Psi}(h_1k, \mathtt{x}) &= \overline{\Psi}(h_2k, \mathtt{x}) \end{aligned}$$

Now, let $e \in G/\text{Ker}\Psi$, then $\overline{\Psi}(e, \mathtt{x}) = \mathtt{x}$. Let $h_1, h_2 \in G$, $k \in \text{Ker}\Psi$. Then $\overline{\Psi}(h_2k, \overline{\Psi}(h_1k, \mathtt{x})) = \overline{\Psi}(h_2k, h_1\mathtt{x}) = h_2h_1\mathtt{x}$, and $\overline{\Psi}(h_2kh_1\mathtt{x}, \mathtt{x}) = h_2h_1\mathtt{x}$. Thus $\overline{\Psi}(h_2k, \overline{\Psi}(h_1k, \mathtt{x})) = \overline{\Psi}(h_2kh_1\mathtt{x}, \mathtt{x})$. Therefore the conditions of group actions are satisfied. Now, the semi totally continuity of $\overline{\Psi}$ follows by the commutative diagram,



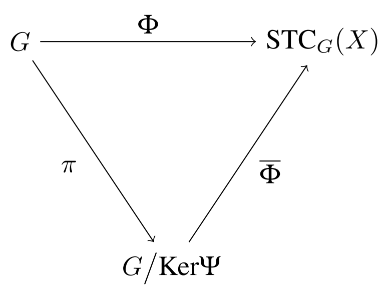
Since the map π is surjective, continuous and open, so is $\pi \times id$. Now, for every semiopen set $V \in X, \Psi^{-1}(V)$ is a clopen set in $G \times X$. Since $G \times X$ is compact and $G/\text{Ker}\Psi \times X$ is Hausdorff, $\pi \times id$ is closed. Thus every clopen set in $G \times X$ maps to a clopen set in $G/\text{Ker}\Psi \times X$, so $\overline{\Psi}^{-1}(V)$ is clopen. Thus, $\overline{\Psi}$ is semi totally continuous. Hence $\left(G/\text{Ker}\Psi, X, \overline{\Psi}\right)$ forms a $\mathcal{S}$-topological transformation group. □

Corollary 3.22. If X is a compact Hausdorff space, then a $\mathcal{S}$- topological transformation group (G, X, Ψ) is an effective $\mathcal{S}$-topological transformation group $\left(G/\text{Ker}\Psi, X, \overline{\Psi}\right)$

Proof. By the Definition 3.7 and Theorem 3.21, the proof is obvious. □

Corollary 3.23. The map $\overline{\Phi} : G/\text{Ker}\Psi \to \text{STC}_G(X)$ is a continuous injective homomorphism for a Frechet space X.

Proof. By Proposition 3.19, π is continuous, surjective and open and since Φ is a injective homomorphism, so is $\overline{\Phi}$. Now, the continuity of $\overline{\Phi}$ follows from the following commutative diagram,



Therefore, $\overline{\Phi}$ is a continuous injective homomorphism. □

4 Normalizer of a $\mathcal{S}$-topological transformation group

The normalizer of a $\mathcal{S}$-topological transformation group is defined and studied in this section.

Definition 4.1. Let $\left(G, X, \Psi\right)$ be a $\mathcal{S}$-topological transformation group. The set $N_G(\text{x}) = \{h \in G | h\text{x}h^{-1} = \text{x}\}$ is called the normalizer of x in G.

Definition 4.2. Let $\text{STC}_G(X)$ be a group and $\text{STC}_H(X)$ be a subgroup, then the normalizer of $\text{STC}_H(X)$ in $\text{STC}_G(X)$ is $N_{\text{STC}_G(X)}(\text{STC}_H(X)) = \left\{\Psi_g \in \text{STC}_G(X) \middle| \Psi_{g^{-1}}\text{STC}_H(X)\Psi_g = \text{STC}_H(X)\right\}$

Definition 4.3. Let $\text{STC}_G(X)$ be a group and $\text{STC}_H(X)$ be a subgroup, then the centralizer of $\text{STC}_H(X)$ in $\text{STC}_G(X)$ is $C_{\text{STC}_G(X)}(\text{STC}_H(X)) = \left\{\Psi_g \in \text{STC}_G(X) \middle| \Psi_g\Psi_h = \Psi_h\Psi_g, \text{for all } \Psi_h \in \text{STC}_H(X)\right\}$

Proposition 4.4. Let K be a subgroup of a topological group G. Let $K_\circ = \bigcap_{h \in G} hKh^{-1}$ and $(G, G/K, \Psi)$ be the semi totally continuous action. Then $Ker\Psi = K_\circ$.

Proof. For any $k \in K_\circ$ and $h \in G$,

$$\begin{aligned} k &= hKh^{-1} \\ kh &= hKh^{-1}h \\ khK &= hK. \end{aligned}$$

This implies $K_\circ$ acts trivially on G/H. Therefore $K_\circ \subset \text{Ker}\Psi$. Conversely, assume that $h_\circ hK = hK$, for all $h \in G$. Then $h_\circ \in hKh^{-1}$, for all $h \in G$. Therefore $h_\circ \in K_\circ$ and $\text{Ker}\Psi \subset K_\circ$. Hence $Ker\Psi = K_\circ$. □

Remark 4.5. If $K_\circ$ is a finite Frechet Space, then by Proposition 3.9 $K_\circ$ is both open and closed subgroup of G.

Remark 4.6. If G acts on itself, then by Proposition 3.9, $G_{\mathtt{x}} = N_G(\mathtt{x})$ or $C_G(\mathtt{x})$, the normalizer or centralizer of x in G. Since $Z(G) = \bigcap_{\mathtt{x}\in X} C_G(\mathtt{x})$, $Z(G) = \text{Ker}\Psi$.

Proposition 4.7. KerΨ is the largest subgroup among the normal subgroups of G contained in K.

Proof. Let Ker$\Psi \trianglelefteq G$ and Ker$\Psi \leq K$. Let $N \trianglelefteq G$ contained in K. Then $N = hNh^{-1} \leq hKh^{-1}$, for all $h \in G$. This implies $N \trianglelefteq \bigcap_{h\in G} hKh^{-1}$. Hence Ker$\Psi$ is the largest normal subgroup of G contained in H. □

Corollary 4.8. If G/K is a compact Hausdorff space, then $\left(G/K_\circ, G/K, \overline{\Psi}\right)$ is an effective $\mathcal{S}$-topological transformation group.

Proof. By the Definition 3.7 and Corollary 3.22, the proof is obvious. □

5 Conclusion

The isotropy group and normalizer of a $\mathcal{S}$-topological transformation group has been analyzed and discussed with some basic algebraic and topological properties. Also, it is established that the topological group $G/\text{Ker}\Psi$ along with a compact Hausdorff space X forms a $\mathcal{S}$-topological topological transformation group and the kernel of semi totally continuous action is the largest normal subgroup of G. In future work, we will investigate some additional features of the isotropy group and normalizer of $\mathcal{S}$-topological transformation group. The results stated in this paper have the potential to stimulate further research and pave the way for equivariant maps, orbit spaces and homogeneous spaces.

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