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On the Projective special unitary groups $\mathrm{PSU_3(q)}$ characterized by a size of a conjugacy class

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Abstract. One of the important problem in finite groups theory is group characterization by specific property. Properties, such as element order, the set of element with the same order, the largest elements order, graphs, etc. In this paper, we prove that the projective special unitary group $PSU_3(q)$, where $p = \frac{q^2-q+1}{(3,q+1)}$ is a prime number and $q = 5k \pm 2$, $(k \in \mathbb{Z})$ can be uniquely determined by its order and one conjugacy class size.

1 Introduction

One of the important problems in finite group theory is a characterization of a group by specific property. Such properties often involve element orders and their graphs. We say that a group G is characterized by property M if every group fulfilling M is isomorphic to G.
Let G be a finite group. The set of conjugacy classes of G will be denoted by $N(G)$. Let $m_p(G)$ be the number from $N(G)$ which is not divisible by p. For every integer n denote by $\pi(n)$ the set of all prime divisors of n. The prime graph $\pi(G)$ of G is constructed upon the vertex set $\pi(|G|)$ in such a way that two distinct primes p and q are joined by an edge if and only if G has an element of order pq.
Let $t(G)$ be the number of connected components of $\pi(G)$. These components will be denoted by π_1, π_2,...,$\pi_{t(G)}$. If G is of even order, then π_1 is chosen to be the component in which 2 is a vertex. We denote m_1, m_2,...,$m_t(G)$ to be the integers such that $|G| = m_1, ..., m_t(G)$ and $\pi(m_i)$ is the vertex set of π_i. If m_i is odd, call π_i an odd order component [12].
The starting point for our discussion is from a conjecture of J. G. Thompson, which is Problem 12.38 in the Kourovka notebook [23] is as follows:
Thompson's conjecture. Let G be a group with trivial center. If M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$. Next, for example the authors in([2, 3, 4, 5, 6, 9, 10, 15, 26, 27]), proved that the sporadic simple groups, Alt_{10}, $PSL(4,4)$ and $PSL(2,p)$, $PSL(n,2)$, ${}^2D_n(2)$,${}^2D_{n+1}(2)$, $C_n(2)$, alternating group of degree p, $p+1$, $p+2$ and symmetric group of degree p, where p is a prime number and $PSL(5,q)$ are characterizable by using the order of the group and the conjugacy class of size. The group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively. In this paper, we prove that the projective special unitary groups $PSU_3(q)$, where $p = \frac{q^2-q+1}{(3,q+1)}$ is a prime number and $q = 5k \pm 2$, $(k \in \mathbb{Z})$ can be uniquely determined by its order and one conjugacy class of size. For easily we denote conjugacy class of size p by CCS_p. In fact, we prove the following main theorem.
Main Theorem. Let G be a group such that $|G| = |PSU_3(q)|$. If $p = \frac{q^2-q+1}{(3,q+1)}$ is a prime, then $G \cong PSU_3(q)$ if and only if G has a conjugacy class of size $\frac{|PSU_3(q)|}{p}$.

2 Notation and Preliminaries

Lemma 2.1. *[18] Let G be a Frobenius group of even order with kernel K and complement H. Then*

(i) $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;

(ii) $|H|$ divides $|K| - 1$;

(iii) K is nilpotent.

Lemma 2.2. *[8] Let G be a 2-Frobenius group of even order. Then*

(i) $t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;

(ii) G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.

Lemma 2.3. *[30] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(i) G is a Frobenius group;

(ii) G is a 2-Frobenius group;

(iii) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.

Lemma 2.4. *[28] Let G be a non-abelian simple group such that $(5, |G|) = 1$. Then G is isomorphic to one of the following groups:*

(i) $PSL_n(q')$, $n = 2, 3$, $q' \equiv \pm 2 \pmod 5$;

(ii) $G_2(q')$, $q' \equiv \pm 2 \pmod 5$;

(iii) $PSU_3(q')$, $q' \equiv \pm 2 \pmod 5$;

(iv) ${}^3D_4(q')$,$q' \equiv \pm 2 \pmod 5$;

(v) ${}^2G_2(q')$, $q' = 3^{2m+1}$, $m \geq 1$.

Lemma 2.5. *[31] Let q, k, l be natural numbers. Then*

(i) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.

(ii) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

(iii) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$ the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

3 Proof of the Main Theorem

In this section, we prove the main theorem in the following lemmas. For this purpose, we denote the projective special unitary groups $PSU_3(q)$ and prime number $\frac{q^2-q+1}{(3,q+1)}$ by U and p respectively. Furthermore by [29], $PSU_3(q)$ has conjugacy class of size $\frac{|PSU_3(q)|}{p}$. First we denote that if $G \cong PSU_3(q)$, then $CCS_p(G) = CCS_p(PSU_3(q))$ and $|G| = |PSU_3(q)|$. Now, assume $CCS_p(G) = CCS_p(PSU_3(q))$ and $|G| = |PSU_3(q)|$. The aim is to prove $G \cong PSU_3(q)$. By the assumption on q, there exists an element α of order p in G such that $C_G(\alpha) =< \alpha >$ and $C_G(\alpha)$ is a Sylow p-subgroup of G. By the Sylow's theorem, we have that $C_G(\beta) =< \beta >$ for any element β in G of order p. In the following we prove p is an isolated vertex in $\Gamma(G)$. We note that $|PSU_3(q)| = \frac{q^3(q^3+1)(q^2-1)}{(3,q+1)}$ and $CCS_p(PSU_3(q)) = \frac{|PSU_3(q)|}{p}$.

Lemma 3.1. *p is an isolated vertex in $\Gamma(G)$.*
Proof. *We shall prove that p is an isolated vertex of $\Gamma(G)$. Suppose to contrary. Then there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $tp \geq 2p = 2(\frac{q^2-q+1}{(3,q+1)}) > \frac{q^2+q}{(3,q+1)}$, thus $k(G) > \frac{q^2+q}{(3,q+1)}$. As a result $t(G) \geq 2$.*

So by lemma 2.3 we have the following lemmas.

Lemma 3.2. *The group G is neither a Frobenius group and a 2-Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 3.2, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now, by Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Because of that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K| - 1$.
Case (ii) is impossible. So $|H| = p$ and $|K| = \frac{|G|}{p}$. Hence $\frac{q^2-q+1}{(3,q+1)} \mid \frac{q^3(q^3+1)(q^2-1)}{q^2-q+1} - 1$. First if $(3, q+1) = 1$, then $q^2 - q + 1 \mid (q^2 - q + 1)(q^4 + 2q^3 - 3q - 3) + 2$. Thus $p \mid 2$ which is impossible. Now assume $(3, q+1) = 3$, so $\frac{q^2-q+1}{3} \mid \frac{q^3(q^3+1)(q^2-1)}{q^2-q+1} - 1$, it follows that $q^2 - q + 1 \mid 3q^6 + 3q^5 - 3q^4 - 3q^3 - 3$, so $q^2 - q + 1 \mid (q^2 - q + 1)(3q^4 + 6q^3 - 9q - 9) + 6$ where this is contradiction. Now, we prove that G is not a 2-Frobenius group. Start from the opposite and assume that G be a 2-Frobenius group, so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively. Set $|G/K| = x$. Since p is an isolated vertex of $\Gamma(G)$, we have $|K/H| = p$ and by lemma 2.2, $|G/K|$ divides $|Aut(K/H)|$. Thus $|G/K| \mid p - 1$. First, if $(3, q+1) = 1$ then $|G/K|$ divides $q^2 - q$. At the same time by lemma 2.5, $(q^2 - q, q^2 - q + 1) = 1$. Because of that $q^2 - q + 1 \mid |H|$. Therefore $H_t \rtimes K/H$ is a Frobenius group with kernel H_t and complement K/H, where $t = q^2 - q + 1$. So $|K/H|$ divides $|H_t| - 1$. It implies that $p \mid p - 1$, but this is a contradiction. For the other case as $(3, q+1) = 3$, we have a contradiction. □

Lemma 3.3. *The group G is isomorphic to the group U.*

Proof. By lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of lemma 2.3. At the moment by lemma 3.2 and lemma 2.2 implies that G is neither a Frobenius group and a 2-Frobenius group. Thus only the case (c) of lemma 2.3 occure. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. On the other hand, $5 \nmid |G|$, so K/H is isomorphic one of the groups lemma 2.4.
Step 1. Suppose that $K/H \cong A_1(q')$, $q' \equiv \pm 2 (\bmod 5)$. On the other hand, by[30], $\pi(A_1(q')) = q' \pm 1$ or $\frac{q' \pm 1}{2}$. We know that $|A_1(q')|$ divide $|G|$, $\frac{q'(q'^2-1)}{(2,q'-1)} \mid \frac{q^3(q^3+1)(q^2-1)}{(3,q+1)}$. First, if $(2, q'-1) = 1$ and $(3, q+1) = 1$ then, we consider $p = q' \pm 1$, so $q^2 - q + 1 = q' \pm 1$. As aresult $q^2 - q = q'$ and $q^2 - q + 2 = q'$. Since that $|A_1(q')| \nmid |G|$, where this is a contradiction. Now, if $(2, q'-1) = 2$, $(3, q+1) = 3$ then $\frac{q^2-q+1}{3} = \frac{q' \pm 1}{2}$. It follows that $q' = \frac{2q^2-2q+5}{3}$ and $q' = \frac{2q^2-2q-1}{3}$. Since that $|A_1(q')| \nmid |G|$, where this is a contradiction. For $(2, q'-1) = 2$ and $(3, q+1) = 1$, we have $\frac{q' \pm 1}{2} = q^2 - q + 1$ it follows that $q' = 2q^2 - 2q + 3$ and $q' = 2q^2 - 2q + 1$. Since that $|A_1(q')| \nmid |G|$, where this is a contradiction. For $(2, q'-1) = 1$ and $(3, q+1) = 3$, we have a contradiction.
If $K/H \cong PSL_3(q')$, then we have a contradiction, similarly.
Step 2. Suppose that $K/H \cong G_2(q')$ where $q' \equiv \pm 2 (\bmod 5)$. On the other hand, by [30], $\pi(G_2(q') = q'^2 \pm q' + 1$. We know that $|G_2(q')|$ divide $|G|$, so $q'^6(q'^6 - 1)(q'^2 - 1) \mid \frac{q^3(q^3+1)(q^2-1)}{(3,q+1)}$. Now, we consider $p = q'^2 \pm q' + 1$, so $\frac{q^2-q+1}{(3,q+1)} = q'^2 \pm q' + 1$. First, if $(3, q+1) = 1$ then $q^2 - q + 1 = q'^2 \pm q' + 1$, it follows that $q(q-1) = q'(q' \pm 1)$. Now since that $(q, q-1) = 1$, so $q - 1 = q'$. But $|G_2(q')| \nmid |G|$, where this is a contradiction. Now, if $(3, q+1) = 3$, then $\frac{q^2-q+1}{3} = q'^2 \pm q' + 1$. Hence, $q^2 - q + 1 = 3q'^2 \pm 3q' + 3$. Next, we deduce $q^2 - q - 2 = 3q'(q'+1)$ and $q^2 - q - 2 = 3q'(q'-1)$. It follows that $(q+1)(q-2) = 3q'(q' \pm 1)$. On the other hand $(q+1, q-2) = 1$ or 3, so if $(q+1, q-2) = 1$, then $q - 2 = q' - 1$, $q + 1 = 3q'$. But $|G_2(q')| \nmid |G|$, where this is a contradiction. Now, if $(q+1, q-2) = 3$ then $q + 1 = 3q'$ and $q - 2 = q' \pm 1$. But $|G_2(q')| \nmid |G|$, where this is a contradiction.
Step 3. Suppose that $K/H \cong {}^3D_4(q')$, $q \equiv \pm 2 (\bmod 5)$. On the other hand, by [30], $\pi({}^3D_4(q')) =$

$q'^4 - q'^2 + 1$. We know that $|^3D_4(q')|$ divided $|G|$, so $q'^{12}(q'^8 + q'^4 + 1)(q'^6 - 1)(q'^2 - 1) \mid \frac{q^3(q^3+1)(q^2-1)}{(3,q+1)}$. Now, we consider $p = q'^4 - q'^2 + 1$, so $\frac{q^2-q+1}{(3,q+1)} = q'^4 - q'^2 + 1$. Now, if $(3, q+1) = 1$, then $q^2 - q + 1 = q'^4 - q'^2 + 1$ follows that $q(q-1) = q'^2(q'^2 - 1)$. Since that $(q, q-1) = 1$, so $q = q'^2$. But $|^3D_4(q')| \nmid |G|$, where this is a contradiction.

Step 4. Suppose that $K/H \cong {}^2G_2(q')$, $q \equiv \pm 2 (\mathrm{mod}\, 5)$. On the other hand, by [30], $\pi(^2G_2(q')) = q' \pm \sqrt{3q'} + 1$. We know that $|^2G_2(q')|$ divided $|G|$, so $q'^3(q'^3 + 1)(q' - 1) \mid \frac{q^3(q^3+1)(q^2-1)}{(3,q+1)}$. Now, we consider $p = q' \pm \sqrt{3q'} + 1$, so $\frac{q^2-q+1}{(3,q+1)} = q' \pm \sqrt{3q'} + 1$. Now, if $(3, q+1) = 1$, then $q^2 - q + 1 = q' \pm \sqrt{3q'} + 1$ follows that $q(q-1) = 3^{2m+1}(3^m \pm 1)$. Since that $(q, q-1) = 1$, so $q - 1 = 3^m - 1$ and $q = 3^{m+1}$. So $3^m(3^{m+1} - 1) = 3^{2m+1}(3^m \pm 1)$, where this is a contradiction. Hence, $K/H \cong PSU_3(q)$. Now since that $|K/H| = |U| = |G|$ and also $p \in \pi(K/H)$ so $p = p'$. So $\frac{q^2-q+1}{(3,q+1)} = \frac{q'^2-q'+1}{(3,q'+1)}$. Thus $q = q'$. On the other hand, $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, thus $H = 1$, $G = K \cong U$. □

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