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r-Dowling polynomials via a differential operator

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Abstract In this paper, by using a differential operator we give some properties on the r-Dowling polynomials, also we derive new congruences on these polynomials.

1 Introduction

The theory of differential operators studies algebraic properties of various sequences of numbers, polynomials and functions. Specially, the differential operators can be applied to solve ordinary differential equations, to generalize some known identities and to obtain congruences modulo a prime number, see for instance [1, 8, 9, 10]. In this paper we use a differential operator to present some properties and congruences concerning the r-Dowling polynomials. To start, let us give a short introduction on these polynomials. Recall that the Dowling lattices introduced by Dowling [7] are denoted by $\mathcal{Q}_n(G)$, where n is a positive integer and G is a finite group of order $m > 0$. The r-Dowling polynomials are defined by

$$D_{m,r}(n,x) = \sum_{k=0}^{n} W_{m,r}(n,k)\,x^k, \tag{1.1}$$

where $W_{m,r}(n,k)$ is the r- Whitney numbers of the second kind introduced by Cheon and Jung [5]. When $x = 1$, the number

$$D_{m,r}(n,1) := D_{m,r}(n) = \sum_{k=0}^{n} W_{m,r}(n,k), \tag{1.2}$$

is the n-th r-Dowling number. In particular, for $r = 1$ or $r = 1$ and $x = 1$, the polynomial $D_{m,1}(n,x) := D_m(n,x)$ and the number $D_{m,1}(n,1) := D_m(n)$ are, respectively, the n-th Dowling polynomial and the n-th Dowling number, see [2, 3].

Recall the r-Dowling polynomials and the r- Whitney numbers of the second kind satisfy the following relations, see [5]

$$\sum_{n\geq 0} D_{m,r}(n,x)\frac{t^n}{n!} = \exp\left(rt + x\frac{e^{mt}-1}{m}\right), \tag{1.3}$$

$$D_{m,r}(n+1,x) = rD_{m,r}(n,x) + x\sum_{i=0}^{n}\binom{n}{i}m^{n-i}D_{m,r}(i,x), \tag{1.4}$$

$$W_{m,r}(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(m(k-j)+r)^n, \tag{1.5}$$

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (r+mk)\,W_{m,r}(n-1,k), \tag{1.6}$$

$$W_{m,r}(n,k) = \sum_{i=k}^{n}\binom{n}{i}m^{i-k}r^{n-i}\left\{{i \atop k}\right\}, \tag{1.7}$$

where ${n \brace k}$ is the (n,k)-th Stirling number of the second kind which counts the number of partition of the set $[n] := \{1, \cdots, n\}$ into k non-empty subsets, see [4].
Also, recall that it is known that [6] for any prime p, we have

$${p \brace k} \equiv 0 \,(\text{mod}\, p)\,,\ 1 < k < p, \text{ and } {p \brace 0} = 0,\ {p \brace p} = {0 \brace 0} = {p \brace 1} = 1,$$

which give for p not dividing mr :

$$\begin{aligned} W_{m,r}(p,k) &\equiv 0 \,(\text{mod}\, p)\,, \text{for } 2 \leq k \leq p-1, && (1.8)\\ W_{m,r}(p,0) &\equiv r \,(\text{mod}\, p)\,, && (1.9)\\ W_{m,r}(p,1) &\equiv W_{m,r}(p,p) \equiv 1 \,(\text{mod}\, p)\,, && (1.10)\\ W_{m,r}(p-1,k) &\equiv 0 \,(\text{mod}\, p)\,, \text{ for } 1 \leq k \leq p-2. && (1.11) \end{aligned}$$

The first few r-Dowling polynomials are

$$\begin{aligned} D_{m,r}(0,x) &= 1,\\ D_{m,r}(1,x) &= x + r,\\ D_{m,r}(2,x) &= x^2 + (m+2r)\,x + r^2,\\ D_{m,r}(3,x) &= x^3 + (3m+3r)\,x^2 + \left(m^2 + 3mr + 3r^2\right)x + r^3. \end{aligned}$$

2 Some properties involving r-Dowling polynomials

In this section, we use a differential operator to establish some properties of the r-Dowling polynomials.

Lemma 2.1. *The r-Dowling polynomials satisfy the following identity*

$$D_{m,r}(n+1,x) = (x+r)\,D_{m,r}(n,x) + mx\frac{d}{dx}D_{m,r}(n,x)\,. \quad (2.1)$$

Proof. By (1.6), we have

$$\begin{aligned} D_{m,r}(n+1,x) &= \sum_{k=0}^{n+1} W_{m,r}(n+1,k)\,x^k\\ &= \sum_{k=0}^{n+1} \left[W_{m,r}(n,k-1) + (r+mk)\,W_{m,r}(n,k)\right]x^k\\ &= x\sum_{k=0}^{n} W_{m,r}(n,k)\,x^k + r\sum_{k=0}^{n} W_{m,r}(n,k)\,x^k + mx\sum_{k=0}^{n} kW_{m,r}(n,k)\,x^{k-1}. \end{aligned}$$

□

Let $\mathbf{D} = \frac{d}{dx}$ be the differential operator and let $\mathbf{Q}$ be the differential operator defined by

$$\mathbf{Q} = x + r + mx\mathbf{D}. \quad (2.2)$$

.

Proposition 2.2. *For any non-negative integers n, s, the following identities hold*

$$\begin{aligned} \mathbf{Q}^n 1 &= D_{m,r}(n,x)\,, && (2.3)\\ \mathbf{Q}^s\left(D_{m,r}(n-s,x)\right) &= D_{m,r}(n,x)\,,\ 0 \leq s \leq n, && (2.4)\\ \mathbf{Q}^{n+1} 1 &= (x+r)\,\mathbf{Q}^n 1 + mx\frac{d}{dx}\mathbf{Q}^n 1, && (2.5)\\ \mathbf{Q}^{n+1} 1 &= r\mathbf{Q}^n 1 + x\,(\mathbf{Q}+m)^n\,1. && (2.6) \end{aligned}$$

Proof. For (2.3), we proceed by induction on n. Indeed, for $n=0$, we have

$$\mathbf{Q}^0 1 = 1 = D_{m,r}(0,x),$$

and for $n=1$ or $n=2$, we get

$$\begin{aligned}\mathbf{Q}^1 1 &= ((x+r)+mx\mathbf{D})\,1 = x+r = D_{m,r}(1,x),\\ \mathbf{Q}^2 1 &= \mathbf{Q}\left(\mathbf{Q}^1 1\right) = \mathbf{Q}(x+r) = (x+r)^2 + mx = D_{m,r}(2,x).\end{aligned}$$

Assume that $\mathbf{Q}^n 1 = D_{m,r}(n,x)$. Then

$$\begin{aligned}\mathbf{Q}^{n+1}1 &= \mathbf{Q}\left(\mathbf{Q}^n 1\right)\\ &= ((x+r)+mx\mathbf{D})\left(\sum_{k=0}^{n} W_{m,r}(n,k)\,x^k\right)\\ &= \sum_{k=0}^{n+1} W_{m,r}(n,k-1)\,x^k + r\sum_{k=0}^{n} W_{m,r}(n,k)\,x^k + \sum_{k=0}^{n} mkW_{m,r}(n,k)\,x^k\\ &= \sum_{k=0}^{n+1}\left[W_{m,r}(n,k-1) + (r+mk)\,W_{m,r}(n,k)\right]x^k\\ &= \sum_{k=0}^{n+1} W_{m,r}(n+1,k)\,x^k\\ &= D_{m,r}(n+1,x),\end{aligned}$$

which completes the step induction. For (2.4), we have

$$D_{m,r}(n,x) = \mathbf{Q}^n 1 = \mathbf{Q}\left(\mathbf{Q}^{n-1}1\right) = \mathbf{Q}D_{m,r}(n-1,x) = \cdots = \mathbf{Q}^s D_{m,r}(n-s,x).$$

The identities (2.5) and (2.6) can be obtained easily from (2.1) and (1.4). □

Lemma 2.3. *The r-Dowling polynomials satisfy the following identity*

$$D_{m,r}(n,x) = e^{-\frac{x}{m}}\sum_{k\geq 0}\frac{x^k}{k!m^k}(mk+r)^n. \tag{2.7}$$

Proof. By (1.3), we have

$$\begin{aligned}\sum_{n\geq 0} D_{m,r}(n,x)\frac{t^n}{n!} &= \exp\left(rt + x\frac{e^{mt}-1}{m}\right)\\ &= e^{-\frac{x}{m}}\exp(rt)\exp\left(\frac{xe^{mt}}{m}\right)\\ &= e^{-\frac{x}{m}}\sum_{s\geq 0} r^s\frac{t^s}{s!}\sum_{k\geq 0}\frac{x^k}{m^k k!}\sum_{n\geq 0}(mk)^n\frac{t^n}{n!}\\ &= e^{-\frac{x}{m}}\sum_{k\geq 0}\frac{x^k}{m^k k!}\sum_{n\geq 0}\left(\sum_{s=0}^{n}\frac{(mk)^s}{s!}\frac{r^{n-s}}{(n-s)!}\right)t^n\\ &= \sum_{n\geq 0}\left(e^{-\frac{x}{m}}\sum_{k\geq 0}\frac{x^k}{m^k k!}(mk+r)^n\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficient of t^n on both sides, we get the desired result. □

Proposition 2.4. *For any polynomial f and any non-negative integer s, there holds*

$$f(\mathbf{Q}+s)\,1 = e^{-\frac{x}{m}}\sum_{i\geq 0} f(mi+r+s)\frac{(x/m)^i}{i!}. \tag{2.8}$$

Proof. Let $f=\sum_{k=0}^{n}a_kx^k$ be a polynomial. Then by Lemma 2.3, we have

$$\begin{aligned} f\left(\mathbf{Q}+s\right)1 &= \sum_{k=0}^{n}a_k\sum_{j=0}^{k}\binom{k}{j}s^{k-j}\mathbf{Q}^j 1 \\ &= \sum_{k=0}^{n}a_k\sum_{j=0}^{k}\binom{k}{j}s^{k-j}D_{m,r}\left(j,x\right) \\ &= e^{-\frac{x}{m}}\sum_{i\geq 0}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}\frac{x^i}{i!m^i}a_k\left(mi+r\right)^j \\ &= e^{-\frac{x}{m}}\sum_{i\geq 0}\sum_{k=0}^{n}a_k\left(mi+r+s\right)^k\frac{x^i}{i!m^i} \\ &= e^{-\frac{x}{m}}\sum_{i\geq 0}f\left(mi+r+s\right)\frac{\left(x/m\right)^i}{i!}. \end{aligned}$$

□

Proposition 2.5. *For any polynomial f and any non-negative integer s, we have*

$$\sum_{i=0}^{s}\binom{s}{i}m^i\frac{d^i}{dx^i}f\left(\mathbf{Q}\right)1 = e^{-\frac{x}{m}}\sum_{i\geq 0}f\left(mi+r+sm\right)\frac{\left(x/m\right)^i}{i!}. \tag{2.9}$$

In particular, for $f=x^n$ or $f=x^n$ and $s=1$, we get

$$\sum_{i=0}^{s}\binom{s}{i}m^i\frac{d^i}{dx^i}D_{m,r}\left(n,x\right) = D_{m,r+sm}\left(n,x\right), \tag{2.10}$$

$$D_{m,r}\left(n,x\right)+m\frac{d}{dx}D_{m,r}\left(n,x\right) = D_{m,r+m}\left(n,x\right). \tag{2.11}$$

Proof. By (2.8), we have

$$e^{\frac{x}{m}}f\left(\mathbf{Q}\right)1=\sum_{i\geq 0}f\left(mi+r\right)\frac{\left(x/m\right)^i}{i!}.$$

Then

$$\begin{aligned} \frac{d}{dx}\left(e^{\frac{x}{m}}f\left(\mathbf{Q}\right)1\right) &= \frac{d}{dx}\sum_{i\geq 0}f\left(mi+r\right)\frac{\left(x/m\right)^i}{i!} \\ &= \frac{1}{m}\sum_{i\geq 0}f\left(mi+r+m\right)\frac{\left(x/m\right)^i}{i!}, \end{aligned}$$

and by induction on $s\geq 0$, we get

$$\frac{d^s}{dx^s}\left(e^{\frac{x}{m}}f\left(\mathbf{Q}\right)1\right)=\frac{1}{m^s}\sum_{i\geq 0}f\left(mi+r+sm\right)\frac{\left(x/m\right)^i}{i!}.$$

On the other hand, we have

$$\begin{aligned} \frac{d^s}{dx^s}\left(e^{\frac{x}{m}}f\left(\mathbf{Q}\right)1\right) &= \sum_{i=0}^{s}\binom{s}{i}\frac{d^{s-i}}{dx^{s-i}}e^{x/m}\frac{d^i}{dx^i}f\left(\mathbf{Q}\right)1 \\ &= \frac{e^{\frac{x}{m}}}{m^s}\sum_{i=0}^{s}\binom{s}{i}m^i\frac{d^i}{dx^i}f\left(\mathbf{Q}\right)1. \end{aligned}$$

This completes the proof. □

Hence, by setting r = 1 in Proposition 2.5, we get:

Corollary 2.6. *For any non-negative integer s, there holds*

$$\sum_{i=0}^{s}\binom{s}{i}m^{i}\frac{d^{i}}{dx^{i}}D_{m}\left(n,x\right) = D_{m,1+sm}\left(n,x\right), \tag{2.12}$$

$$D_{m}\left(n,x\right)+m\frac{d}{dx}D_{m}\left(n,x\right) = D_{m,1+m}\left(n,x\right). \tag{2.13}$$

Proposition 2.7. *Let f be a polynomial with real coefficients . If the polynomial $f\left(\mathbf{Q}\right)1$ has only real zeros, then the polynomial $f\left(\mathbf{Q}+m\right)1$ also has only real zeros.*

Proof. From (2.8), we have

$$\begin{aligned}\frac{d}{dx}\left(e^{\frac{x}{m}}f\left(\mathbf{Q}\right)1\right) &= \frac{d}{dx}\left(\sum_{k\geq 0}f\left(mk+r\right)\frac{\left(x/m\right)^{k}}{k!}\right)\\ &= \frac{1}{m}\sum_{k\geq 0}f\left(mk+r+m\right)\frac{\left(x/m\right)^{k}}{k!}\\ &= \frac{e^{x/m}}{m}f\left(\mathbf{Q}+m\right)1.\end{aligned}$$

So the proof can be obtained by application of Rolle's theorem on the function $e^{\frac{x}{m}}f\left(\mathbf{Q}\right)1$. □

Example 2.8. For $f=x^{n}$, $f\left(\mathbf{Q}\right)1=D_{m,r}\left(n,x\right)$ has only real zeros [5], hence the polynomial $\sum_{k=0}^{n}\binom{n}{k}\left(m\right)^{n-k}D_{m,r}\left(k,x\right)$ has only real zeros.

3 Congruences on the r-Dowling polynomials

In this section, we give some congruences on the r-Dowling polynomials.

Lemma 3.1. *For any polynomial f in $\mathbb{Z}\left[x\right]$ and any prime number p not dividing mr, there holds*

$$\left(\mathbf{Q}^{p}-\mathbf{Q}\right)f\left(\mathbf{Q}\right)1\equiv x^{p}f\left(\mathbf{Q}\right)1\ \ (mod\ p\mathbb{Z}\left[x\right]). \tag{3.1}$$

More generally, for any integer $s\geq 1$, we have

$$\left(\mathbf{Q}^{p^{s}}-\mathbf{Q}\right)f\left(\mathbf{Q}\right)1\equiv\left(x^{p}+\cdots+x^{p^{s}}\right)f\left(\mathbf{Q}\right)1\ \ (mod\ p\mathbb{Z}\left[x\right]). \tag{3.2}$$

Proof. For (3.1) it suffices to take $f\left(x\right)=x^{n}$ and we proceed by induction on n. For $n=0$ or $n=1$, by (1.8), (1.9) and (1.10), we have

$$\begin{aligned}\mathbf{Q}^{p}1 &= \sum_{k=0}^{p}W_{m,r}\left(p,k\right)x^{k}\\ &\equiv W_{m,r}\left(p,0\right)x^{0}+W_{m,r}\left(p,1\right)x+W_{m,r}\left(p,p\right)x^{p}\\ &\equiv r+x+x^{p}\\ &= \mathbf{Q}^{1}1+x^{p}\mathbf{Q}^{0}1\ \ (\mathrm{mod}\ p\mathbb{Z}\left[x\right]),\end{aligned}$$

and by (1.4), we have

$$\begin{aligned}\mathbf{Q}^{1+p}1 &= D_{m,r}\left(1+p,x\right)\\ &= rD_{m,r}\left(p,x\right)+x\sum_{i=0}^{p}\binom{p}{i}m^{p-i}D_{m,r}\left(i,x\right)\\ &\equiv r\left(r+x+x^{p}\right)+x\left(m+r+x+x^{p}\right)\\ &= x^{2}+\left(m+2r\right)x+r^{2}+\left(x+r\right)x^{p}\\ &= \mathbf{Q}^{2}1+x^{p}\mathbf{Q}^{1}1\ \ (\mathrm{mod}\ p\mathbb{Z}\left[x\right]).\end{aligned}$$

Assume that $\mathbf{Q}^n(\mathbf{Q}^p-\mathbf{Q})1\equiv x^p\mathbf{Q}^n1 \pmod{p\mathbb{Z}[x]}$. Then by (2.3) and (2.5), we have

$$\begin{aligned}\mathbf{Q}^{n+1}(\mathbf{Q}^p-\mathbf{Q})1 &= \mathbf{Q}[\mathbf{Q}^n(\mathbf{Q}^p-\mathbf{Q})1]\\ &= \left(x+r+mx\frac{d}{dx}\right)(\mathbf{Q}^n(\mathbf{Q}^p-\mathbf{Q})1)\\ &= (x+r)\mathbf{Q}^n(\mathbf{Q}^p-\mathbf{Q})1+mx\frac{d}{dx}\mathbf{Q}^n(\mathbf{Q}^p-\mathbf{Q})1\\ &\equiv (x+r)x^p\mathbf{Q}^n1+mx\frac{d}{dx}(x^p\mathbf{Q}^n1)\\ &\equiv x^p\left[(x+r)\mathbf{Q}^n1+mx\frac{d}{dx}\mathbf{Q}^n1\right]\\ &= x^p\left[(x+r)\mathbf{Q}^n1+\mathbf{Q}^{n+1}1-(x+r)\mathbf{Q}^n1\right]\\ &= x^p\mathbf{Q}^{n+1}1 \pmod{p\mathbb{Z}[x]}.\end{aligned}$$

For (3.2) it suffices to take $f(x)=x^n$ and we proceed by induction on $s\geq 1$. By (3.1) clearly the property is true for $s=1$. Assume that

$$\left(\mathbf{Q}^{p^s}-\mathbf{Q}\right)\mathbf{Q}^n1\equiv\left(x^p+\cdots+x^{p^s}\right)\mathbf{Q}^n1 \pmod{p\mathbb{Z}[x]}.$$

Then

$$\begin{aligned}\left(\mathbf{Q}^{p^{s+1}}-\mathbf{Q}\right)\mathbf{Q}^n1 &= \left[\left(\mathbf{Q}^{p^s}-\mathbf{Q}+\mathbf{Q}\right)^p-\mathbf{Q}\right]\mathbf{Q}^n1\\ &= \left(\mathbf{Q}^{p^s}-\mathbf{Q}+\mathbf{Q}\right)^p\mathbf{Q}^n1-\mathbf{Q}\mathbf{Q}^n1\\ &\equiv \left(\mathbf{Q}^{p^s}-\mathbf{Q}\right)^p\mathbf{Q}^n1+(\mathbf{Q}^p-\mathbf{Q})\mathbf{Q}^n1\\ &\equiv \left(x^p+\cdots+x^{p^s}\right)\left(\mathbf{Q}^{p^s}-\mathbf{Q}\right)^{p-1}\mathbf{Q}^n1+x^p\mathbf{Q}^n1\\ &\vdots\\ &\equiv \left(x^p+\cdots+x^{p^s}\right)^p\mathbf{Q}^n1+x^p\mathbf{Q}^n1\\ &\equiv \left(x^{p^2}+\cdots+x^{p^{s+1}}\right)\mathbf{Q}^n1+x^p\mathbf{Q}^n1\\ &= \left(x^p+x^{p^2}+\cdots+x^{p^{s+1}}\right)\mathbf{Q}^n1 \pmod{p\mathbb{Z}[x]},\end{aligned}$$

hence, the proof is completed. □

So, by choosing $f(x)=x^n$ in Lemma 3.1 we may state the following theorem.

Theorem 3.2. *For any integers $n\geq 0$, $s\geq 1$ and any prime number p not dividing mr, the following congruence holds*

$$D_{m,r}(n+p^s,x) \equiv D_{m,r}(n+1,x)+\left(x^p+\cdots+x^{p^s}\right)D_{m,r}(n,x) \pmod{p\mathbb{Z}[x]}.$$

In particular for $r=1$, we get

$$D_m(n+p^s,x) \equiv D_m(n+1,x)+\left(x^p+\cdots+x^{p^s}\right)D_m(n,x) \pmod{p\mathbb{Z}[x]}.$$

When $x=1$ or $x=1$ and $r=1$, we obtain

$$D_{m,r}(n+p^s) \equiv D_{m,r}(n+1)+sD_{m,r}(n) \pmod{p}, \tag{3.3}$$

$$D_m(n+p^s) \equiv D_m(n+1)+sD_m(n) \pmod{p}. \tag{3.4}$$

Corollary 3.3. *For any integers $n \geq 0,\ v \geq 0,\ s \geq 1$ and any prime number p not dividing mr, we have*

$$D_{m,r}\left(n+vp^{s},x\right) \equiv \sum_{k=0}^{v}\binom{v}{k}\left(x^{p}+\cdots+x^{p^{s}}\right)^{k} D_{m,r}\left(n+v-k,x\right)\ (mod\ p\mathbb{Z}\left[x\right]).$$

In particular for $r=1$, we have

$$D_{m}\left(n+vp^{s},x\right) \equiv \sum_{k=0}^{v}\binom{v}{k}\left(x^{p}+\cdots+x^{p^{s}}\right)^{k} D_{m}\left(n+v-k,x\right)\ (mod\ p\mathbb{Z}\left[x\right]).$$

In the case $x=1$ or $x=1$ and $r=1$, we get

$$D_{m,r}\left(n+vp^{s}\right) \equiv \sum_{k=0}^{v}\binom{v}{k}s^{k}D_{m,r}\left(n+v-k\right)\ (mod\ p), \tag{3.5}$$

$$D_{m}\left(n+vp^{s}\right) \equiv \sum_{k=0}^{v}\binom{v}{k}s^{k}D_{m}\left(n+v-k\right)\ (mod\ p). \tag{3.6}$$

Proof. The above congruence results from the following identity

$$D_{m,r}\left(n+vp^{s},x\right)=\mathbf{Q}^{n}\left(\mathbf{Q}^{p^{s}}-\mathbf{Q}+\mathbf{Q}\right)^{v}1=\sum_{k=0}^{v}\binom{v}{k}\mathbf{Q}^{n+v-k}\left(\mathbf{Q}^{p^{s}}-\mathbf{Q}\right)^{k}1.$$

□

Corollary 3.4. *Let $n \geq 0,\ s \geq 1$ be integers and let p be a prime number not dividing mr. Then*

$$D_{m,r}\left(n+p^{s+1}-p^{s}\right) \equiv \sum_{k=0}^{p-1}(-s)^{k}\left(D_{m,r}\left(n-k\right)+D_{m,r}\left(n-1-k\right)\right)\ (mod\ p),$$

$$D_{m,r}\left(n+p^{s+1}-2p^{s}\right) \equiv \sum_{k=0}^{p-2}(-s)^{k}\left(1+k\right)\left(D_{m,r}\left(n-1-k\right)+D_{m,r}\left(n-2-k\right)\right)\ (mod\ p),$$

$$D_{m,r}\left(n+p\mathcal{N}_{p}\right) \equiv D_{m,r}\left(n+\mathcal{N}_{p}\right)(mod\ p),$$

$$\sum_{k=1}^{\mathcal{N}_{p}}\binom{\mathcal{N}_{p}}{k}D_{m,r}\left(n+\mathcal{N}_{p}-k\right) \equiv 0\,(mod\ p),$$

where

$$\mathcal{N}_{p}=1+p+\cdots+p^{p-1}.$$

Proof. Replace v by $p-1$ or $p-2$ in (3.5) and use (3.3) and the congruences

$$\binom{p-1}{k} \equiv (-1)^{k}\ (\text{mod}\ p),\quad \binom{p-2}{k} \equiv (-1)^{k}(1+k)\ (\text{mod}\ p).$$

The last congruence follows by setting $v=\mathcal{N}_{p}$ and $s=1$ in (3.5). □

Corollary 3.5. *For any prime number p not dividing mr, there holds*

$$D_{m,r}\left(n,x\right)+m\frac{d^{p}}{dx^{p}}D_{m,r}\left(n,x\right) \equiv D_{m,r+pm}\left(n,x\right)\ (mod\ p\mathbb{Z}\left[x\right]),$$

$$\sum_{i=0}^{p-1}(-1)^{i}m^{i}\frac{d^{i}}{dx^{i}}D_{m,r}\left(n,x\right) \equiv D_{m,r+(p-1)m}\left(n,x\right)\ (mod\ p\mathbb{Z}\left[x\right]),$$

$$\sum_{i=0}^{p-2}(-1)^{i}(1+i)m^{i}\frac{d^{i}}{dx^{i}}D_{m,r}\left(n,x\right) \equiv D_{m,r+(p-2)m}\left(n,x\right)\ (mod\ p\mathbb{Z}\left[x\right]).$$

Proof. Replace s by $p,\ p-1$ or $p-2$ in (2.10) and use the congruences

$$\binom{p}{k} \equiv 0 \pmod p,\ 1 \leq k \leq p-1,$$
$$\binom{p-1}{k} \equiv (-1)^k \pmod p,\ \binom{p-2}{k} \equiv (-1)^k (1+k) \pmod p.$$

□

Proposition 3.6. *For any prime p not dividing mr, we have*

$$x\sum_{k=0}^{p-1} \frac{D_{m,r}(k,x)}{(-m)^k} \equiv x^p - rx^{p-1} + x \ (mod\ p\mathbb{Z}[x]).$$

In particular for $r=1$ or $r=1$ and $x=1$, we get

$$x\sum_{k=0}^{p-1} \frac{D_m(k,x)}{(-m)^k} \equiv x^p - x^{p-1} + x \ (mod\ p\mathbb{Z}[x]),$$
$$\sum_{k=0}^{p-1} \frac{D_m(k)}{(-m)^k} \equiv 1 \ (mod\ p).$$

Proof. By (1.8) and (1.11), we have

$$D_{m,r}(p,x) \equiv x^p + x + r \pmod{p\mathbb{Z}[x]},\ D_{m,r}(p-1,x) \equiv 1 + x^{p-1} \pmod{p\mathbb{Z}[x]}.$$

On the other hand by using (2.6), we get

$$\begin{aligned} x\sum_{k=0}^{p-1} \frac{D_{m,r}(k,x)}{(-m)^k} &= x\sum_{k=0}^{p-1} (-m)^{-k} D_{m,r}(k,x) \\ &\equiv x\sum_{k=0}^{p-1} \binom{p-1}{k} m^{p-1-k} \mathbf{Q}^k 1 \\ &= x(\mathbf{Q}+m)^{p-1} 1 \\ &= \mathbf{Q}^p 1 - r\mathbf{Q}^{p-1} 1 \\ &= D_{m,r}(p,x) - rD_{m,r}(p-1,x) \\ &\equiv x^p - rx^{p-1} + x \pmod{p\mathbb{Z}[x]}. \end{aligned}$$

□

4 Conclusion

In our present investigation, we studied some properties and congruences involving the r-Dowling polynomials. These polynomials are used in different mathematical frameworks, specially in combinatorics which are linked to the number of colored partitions on a finite set. One way to study such sequence of polynomials, the differential operators make easer such study. As it is shown above, we conclude that the differential operator used here can be considered an interesting mathematical tool and that the results obtained combine existing and new results. Other differential operators can be used similarly to study other sequences of polynomials such the geometric polynomials, Laguerre polynomials, etc.

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