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SOME COMB RELATED MEAN GRAPHS

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Abstract A graph $Z(V, E)$ with p vertices and q edges is categorized as mean graph if there is an injective function δ: $V(Z) \rightarrow \{0, 1, 2, ..., q\}$ such that the weights $\{1, 2, ..., q\}$ of all edges are distinct. Weight of each edge uv can be calculated by equation $w_t(uv) = \lceil \frac{\delta(u)+\delta(v)}{2} \rceil$, $u, v \in V(Z)$. In this paper, we prove that different families of comb graph are mean graphs.

1 Introduction

Throughout this paper, we consider finite, simple and undirected graphs. In graph theory, graph labeling refers to the assignment of labels or values to the vertex set $V(G)$ or edge set $E(G)$ or both under some certain conditions. These labels or values provide additional information about the properties and structure of graph. Labeled graphs have vast applications in different models such as X-ray crystallography, coding theory, circuit design, astronomy, radar and communication network addressing. For a summary of various graph labelings and their applications, refer to the latest dynamic survey on graph labeling by Gallian [4].
Mean labeling, introduced by Somasundaram et al. in 2003 [12], was subsequently investigated by the same authors in [13] during the same year, where it was found that mean labeling does not exist for some wheel-related graphs. In 2010, Vaidya et al. [18] showed that the graphs obtained by the composition of paths P_m and P_2, the square of path P_n and the middle graph of path P_n admit mean labeling. Additionally, in their study presented in [20], Vaidya et al. investigated that the graph obtained by two new operations called mutual duplication of a pair of vertices each from each copy of cycle C_n as well as mutual duplication of a pair of edges each from each copy of cycle C_n, shadow graphs of star and bistar graphs admit mean labeling. Furthermore, in [19], they proved that some cycle related graphs are mean graphs, they also discussed mean labeling in the context of arbitrary super subdivision of path P_n. In 2010, Vasuki et al. gave further results on mean graphs in their paper [16]. In 2011, Vaidya et al. [17] proved that step ladder graph, total graph of path P_n and two copies of cycle graph C_n sharing a common edge are all mean graphs. For some related study see [2], [5] and [9]. In 2011, Lourdusamy et al. [7] proved that kC_n-snakes, generalised kC_n-snakes and super subdivisions of cycles are all mean graphs. Two years later, Lourdusamy et al. demonstrated that edge linked cyclic snakes and generalised edge linked cyclic snakes are also mean graphs in their work [8]. In 2014, Avadayappan et al. [1] proved that double triangular snake graph, balloon of the triangular snake graph, quadrilateral snake graph, double and triple quadrilateral snake graphs, cycle snake graph and many other families of graphs are mean graphs. In the same year, Gayathri et al. [3] investigated that different cycle related graphs admit mean labeling. Further insights into the mean labeling of various graphs are presented in the works [10], [11], [14] and [15] of other authors.
Our paper proves that various families of comb graphs, including Ca_m, Cd_m, Ce_m, Cf_m, Ct_m, and Ch_m, are mean graphs. For further details on the above-mentioned notations of comb graphs, refer to the relevant papers [6] and [21].

2 Main Results

Theorem 2.1. *Comb graph Ca_m is mean graph.*

Proof: Comb graph Ca_m is obtained by vertex set $V(Ca_m) = \{b_g^k; 1 \le g \le k+1, 1 \le k \le m\}$ and edge set $E(Ca_m) = \{b_1^k b_1^{k+1}; 1 \le k \le m-1\} \bigcup \{b_g^k b_{g+1}^k; 1 \le k \le m, 1 \le g \le k\}$, [6, 21]. We have to show that comb graph Ca_m is mean graph, for this, define a function δ: $V(Ca_m) \rightarrow \{0, 1, 2, ..., q\}$ such that

$$\delta(b_g^k) = \begin{cases} \frac{k^2+3k}{2} - g, & \text{if } 1 \le g \le k+1; 1 \le k \le m, odd \\ \frac{k^2+k-4}{2} + g, & \text{if } 1 \le g \le k+1; 2 \le k \le m, even. \end{cases}$$

Now we evaluate weights for all edges as:

$$w_t(b_g^k b_{g+1}^k) = \begin{cases} \lceil \frac{k^2+3k-2g-1}{2} \rceil, & \text{if } 1 \le g \le k;\ 1 \le k \le m, odd \\ \lceil \frac{k^2+k+2g-3}{2} \rceil, & \text{if } 1 \le g \le k;\ 2 \le k \le m, even. \end{cases}$$

$$w_t(b_1^k b_1^{k+1}) = \frac{k^2+3k}{2}, \text{if } 1 \le k \le m-1.$$

It is evident that the function defined above by δ is injective, and all edges possess distinct weights $\{1, 2, \ldots, q\}$. Therefore, we can conclude that comb graph Ca_m is a mean graph. Thus, the proof is complete. □

Theorem 2.2. *Comb graph Cd_m is mean graph.*

Proof: Comb graph Cd_m is obtained by vertex set $V(Cd_m) = \{b_g^k; 1 \le k \le m, 1 \le g \le \lfloor\frac{k+3}{2}\rfloor\}$ and edge set $E(Cd_m) = \{b_g^k b_{g+1}^k; 1 \le k \le m\ , 1 \le g \le \lfloor\frac{k+1}{2}\rfloor\} \bigcup \{b_1^k b_1^{k+1}; 1 \le k \le m-1\}$, [6, 21]. We have to show that comb graph Cd_m is mean graph, for this, define a function δ: $V(Cd_m) \rightarrow \{0, 1, 2, ..., q\}$ such that

$$\delta(b_g^k) = \begin{cases} \frac{k^2+6k+1}{4} - g, & \text{if } 1 \le g \le \lfloor\frac{k+3}{2}\rfloor; 1 \le k \le m, odd \\ \frac{k^2+4k-12}{4} + (g+1), & \text{if } 1 \le g \le \lfloor\frac{k+3}{2}\rfloor; 2 \le k \le m, even. \end{cases}$$

Now we evaluate weights for all edges as follows:

$$w_t(b_g^k b_{g+1}^k) = \begin{cases} \lceil \frac{k^2+6k-4g-1}{4} \rceil, & \text{if } 1 \le g \le \lfloor\frac{k+1}{2}\rfloor; 1 \le k \le m, odd \\ \lceil \frac{k^2+4k+4g-6}{4} \rceil, & \text{if } 1 \le g \le \lfloor\frac{k+1}{2}\rfloor; 2 \le k \le m, even. \end{cases}$$

$$w_t(b_1^k b_1^{k+1}) = \lceil \frac{k^2+6k-1}{4} \rceil, 1 \le k \le m-1.$$

It is evident that the function defined above by δ is injective, and all edges possess distinct weights $\{1, 2, \ldots, q\}$. Therefore, we can conclude that comb graph Cd_m is a mean graph. This brings us to the end of the proof. □

Theorem 2.3. *Comb graph Ce_m is mean graph.*

Proof: Comb graph Ce_m is obtained by vertex set $V(Ce_m) = \{b_g^k; 1 \le k \le m, 1 \le g \le 2k+1\}$ and edge set $E(Ce_m) = \{b_g^k b_{g+1}^{k+1}; 1 \le k \le m-1, g = k+1\} \bigcup \{b_g^k b_{g+1}^k; 1 \le k \le m, 1 \le g \le 2k\}$, [6]. We have to show that comb graph Ce_m is mean graph, for this, define a function δ: $V(Ce_m) \rightarrow \{0, 1, 2, ..., q\}$ such that

$$\delta(b_g^k) = \begin{cases} k^2 + g - 2, & \text{if } 1 \le g \le 2k+1;\ 1 \le k \le m, odd \\ k^2 + 2k - g, & \text{if } 1 \le g \le 2k+1;\ 2 \le k \le m, even. \end{cases}$$

Now we evaluate weights for all edges as follows:

$$w_t(b_g^k b_{g+1}^k) = \begin{cases} \lceil \frac{2k^2+2g-3}{2} \rceil, & \text{if } 1 \le g \le 2k;\ 1 \le k \le m, odd \\ \lceil \frac{2k^2+4k-2g-1}{2} \rceil, & \text{if } 1 \le g \le 2k;\ 2 \le k \le m, even. \end{cases}$$

$$w_t(b_g^k b_{g+1}^{k+1}) = \lceil \frac{2k^2+2k+2g-3}{2} \rceil, \text{if } g = k+1; 1 \le k \le m-1.$$

It is evident that the function defined above by δ is injective, and all edges possess distinct weights $\{1, 2, \ldots, q\}$. Therefore, we can conclude that comb graph Ce_m is a mean graph. This concludes the demonstration. □

Theorem 2.4. *Comb graph Cf_m is mean graph.*

Proof: Comb graph Cf_m is obtained by vertex set $V(Cf_m) = \{b_g^k; 1 \le g \le 7, 1 \le k \le m\}$ and edge set $E(Cf_m) = \{b_4^k b_4^{k+1}; 1 \le k \le m-1\} \bigcup \{b_g^k b_{g+1}^k; 1 \le k \le m, 1 \le g \le 6\}$, [6, 21]. We have to show that comb graph Cf_m is mean graph, for this, define a function δ: $V(Cf_m) \rightarrow \{0, 1, 2, ..., q\}$ such that

$$\delta(b_g^k) = \begin{cases} 7k + g - 8, & \text{if } 1 \le g \le 7;\ 1 \le k \le m, odd \\ 7k - g, & \text{if } 1 \le g \le 7;\ 2 \le k \le m, even. \end{cases}$$

Now we evaluate weights for all edges as follows:

$$w_t(b_g^k b_{g+1}^k) = \begin{cases} \lceil \frac{14k+2g-15}{2} \rceil, & \text{if } 1 \le g \le 6;\ 1 \le k \le m, odd \\ \lceil \frac{14k-2g-1}{2} \rceil, & \text{if } 1 \le g \le 6;\ 2 \le k \le m, even. \end{cases}$$

$$w_t(b_4^k b_4^{k+1}) = \lceil \frac{14k-1}{2} \rceil, 1 \le k \le m-1.$$

It is evident that the function defined above by δ is injective, and all edges possess distinct weights $\{1, 2, \ldots, q\}$. Therefore, we can conclude that comb graph Cf_m is a mean graph. Hence the required result is proved. □

Theorem 2.5. *Comb graph Ct_m is mean graph.*

Proof: Comb graph Ct_m is obtained by vertex set $V(Ct_m) = \{b_g^k; 1 \le k \le m, 1 \le g \le 6\}$ and edge set $E(Ct_m) = \{b_g^k b_{g+1}^k; 1 \le k \le m, 1 \le g \le 5\} \bigcup \{b_4^k b_3^{k+1}; 1 \le k \le m-1, odd\} \bigcup \{b_3^k b_4^{k+1}; 2 \le k \le m-1, even\}$, [21]. We have to show that comb graph Ct_m is mean graph, for this, define a function δ: $V(Ct_m) \rightarrow \{0, 1, 2, ..., q\}$such that

$$\delta(b_g^k) = \begin{cases} g + 6k - 7, & \text{if } 1 \le g \le 6;\ 1 \le k \le m, odd \\ 6k - g, & \text{if } 1 \le g \le 6;\ 2 \le k \le m, even. \end{cases}$$

Now we evaluate weights for all edges as

$$w_t(b_g^k a_{g+1}^k) = \begin{cases} \lceil \frac{2g+12k-13}{2} \rceil, & \text{if } 1 \le g \le 5;\ 1 \le k \le m, odd \\ \lceil \frac{12k-2g-1}{2} \rceil, & \text{if } 1 \le g \le 5;\ 2 \le k \le m, even. \end{cases}$$

$$w_t(b_4^k b_3^{k+1}) = \lceil \frac{12k-1}{2} \rceil, \text{if } 1 \le k \le m-1, odd.$$

$$w_t(b_3^k b_4^{k+1}) = 6k, \text{if } 2 \le k \le m-1, even.$$

It is evident that the function defined above by δ is injective, and all edges possess distinct weights $\{1, 2, \ldots, q\}$. Therefore, we can conclude that comb graph Ct_m is a mean graph. Hence the result is established. □

Theorem 2.6. *Comb graph Ch_m is mean graph.*

Proof: Comb graph Ch_m is obtained by vertex set $V(Ch_m) = \{b_g^k; 1 \leq g \leq 3, 1 \leq k \leq m, odd\} \bigcup \{b_g^k; 2 \leq k \leq m, even, 1 \leq g \leq 4\}$ and edge set $E(Ch_m) = \{b_g^k b_{g+1}^k; 1 \leq k \leq m, odd, 1 \leq g \leq 2\} \bigcup \{b_g^k b_{g+1}^k; 2 \leq k \leq m, even, 1 \leq g \leq 3\} \bigcup \{b_1^k b_1^{k+1}; 1 \leq k \leq m-1\}$, [21].
We have to show that comb graph Ch_m is mean graph, for this, define a function δ: $V(Ch_m) \rightarrow \{0, 1, 2, ..., q\}$ such that

$$\delta(b_g^k) = \begin{cases} \lceil \frac{2g+7k-9}{2} \rceil, & \text{if } 1 \leq g \leq 3;\ 1 \leq k \leq m, odd \\ \lceil \frac{7k-2g}{2} \rceil, & \text{if } 1 \leq g \leq 4;\ 2 \leq k \leq m, even. \end{cases}$$

Now we evaluate weights for all edges as

$$w_t(b_g^k b_{g+1}^k) = \begin{cases} \lceil \frac{2g+7k-8}{2} \rceil, & \text{if } 1 \leq g \leq 2;\ 1 \leq k \leq m, odd \\ \lceil \frac{7k-2g-1}{2} \rceil, & \text{if } 1 \leq g \leq 3;\ 2 \leq k \leq m, even. \end{cases}$$

$$w_t(b_1^k b_1^{k+1}) = \lceil \frac{14k-2}{4} \rceil, 1 \leq k \leq m-1.$$

It is evident that the function defined above by δ is injective, and all edges possess distinct weights $\{1, 2, \ldots, q\}$. Therefore, we can conclude that comb graph Ch_m is a mean graph. This completes the proof. □

3 Conclusion remarks

In this work, we demonstrate that various families of comb graph, including Ca_m, Cd_m, Ce_m, Cf_m, Ct_m, and Ch_m, are mean graphs as they all admit mean labeling. These findings have significant implications for various applications, such as network design and optimization. The demonstration of the mean graphs mentioned above has the potential to provide valuable insights in the field of graph theory.

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