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Matrix Equation and its Four Smaller Equations

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Abstract In this work, we investigate the inclusion relationships between two sets, S_1 and S_2, where S_1 is the set of least-rank solutions of the matrix equation $AXB = C$, while S_2 is the set of solutions of the form $\Gamma = \dfrac{X_{11} + X_{22} + X_{33} + X_{44}}{4}$, where X_{11}, X_{22}, X_{33} and X_{44} are the least-rank solutions of the four smaller equations derived from the original equation $AXB = C$. Then, we deduce the necessary and sufficient conditions for the following relations to hold: $S_1 \cap S_2 \neq \varnothing$, $S_1 \subseteq S_2$ and $S_1 \supseteq S_2$.

1 Introduction

In this work, $\mathbb{C}^{n\times m}$ represents the set of all $n \times m$ complex matrices. In addition, we denote A^* and $r(A)$ as the conjugate transpose and the rank of matrix A, respectively. The Moore-Penrose inverse of matrix $A \in \mathbb{C}^{n\times m}$ is defined as the unique $m \times n$ complex matrix denoted by A^+ satisfying the following four equations:

$$AA^+A = A,\ A^+AA^+ = A^+,\ \left(AA^+\right)^* = AA^+,\ \left(A^+A\right)^* = A^+A.$$

Extensive studies and results regarding matrix inversion and generalized inverses see e.g. ([1, 2, 3]).

Additionally, we introduce two orthogonal projectors induced by $A \in \mathbb{C}^{m\times n}$, namely $F_A = I_n - A^+A$ and $E_A = I_m - AA^+$.

Consider the matrix equation:

$$AXB = C \tag{1.1}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$ are given matrices, and $X \in \mathbb{C}^{n\times p}$ is an unknown matrix.

Linear matrix equations have been examined in various situations. For example, in [16], the author presented necessary and sufficient conditions for the existence of Hermitian nonnegative definite or positive-definite solutions to (1.1) and representations of these solutions. In [9], Tian studied the relations between two approximate solutions of (1.1), namely, least-squares and least-rank solutions, in [6] the authors studied the problem of finding solutions to a system of linear quaternion or octonion equations. For further related works one may refer to ([4, 10, 12, 11]).

Tian proposed the notion of least-rank solutions to matrix equations in [13, 14] based on the minimal rank of the linear matrix function $A - BXC$. The least-rank solutions have since been investigated by many researchers. For instance, in [5], the authors derived the necessary and sufficient conditions for the systems $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ to have a common least-rank solution. In [15], Xu et al. used the Moore-Penrose inverse to deduce the necessary and sufficient conditions for the existence of Hermitian (skew-Hermitian), Re-nonnegative (Re-positive) definite, and Re-nonnegative (Re-positive) definite least-rank solutions to (1.1) and presented explicit representations of the general solutions in cases for which the solvability conditions were satisfied.

To elucidate more properties of the least-rank solutions of (1.1), we can express the matrices A, B, and C with the following partitioned forms:

$$A = \begin{bmatrix} A_{11} \\ A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{22} \\ C_{33} & C_{44} \end{bmatrix}.$$

By comparing both sides of Equation (1.1), we obtain four individual equations:

$$A_{11}XB_{11} = C_{11}, \; A_{11}XB_{22} = C_{22}, \; A_{22}XB_{11} = C_{33}, A_{22}XB_{22} = C_{44}. \tag{1.2}$$

We can consider Equation (1.1) as a combination of these four smaller equations. However, notably, the fourth equation in (1.2) may not have a common solution. In this case, we can rewrite (1.2) as four independent matrix equations:

$$A_{11}X_{11}B_{11} = C_{11}, \; A_{11}X_{22}B_{22} = C_{22}, \; A_{22}X_{33}B_{11} = C_{33}, \; A_{22}X_{44}B_{22} = C_{44}. \tag{1.3}$$

The conditions for these four matrix equations to be consistent may not be the same as those for Equation (1.1). Hence, the possible relationships between the four equations in (1.1) and (1.3) should be investigated for general cases.

Based on the results of Li and Tian in [7], in this paper, we decompose the least-rank solution X of Equation (1.1) into the sum of the least-rank solutions of the equations in (1.3) as follows:

$$\Gamma = \frac{X_{11} + X_{22} + X_{33} + X_{44}}{4}. \tag{1.4}$$

We aim to determine the existence of additional solutions in the combined set by investigating these connections. The results of this study improve our understanding of the properties of solutions to (1.1).

We first introduce the following important lemmas:

Lemma 1.1. *[8, 12] Let $A \in \mathbb{C}^{s\times r}$, $B \in \mathbb{C}^{s\times k}$, $C \in \mathbb{C}^{l\times r}$, $D \in \mathbb{C}^{l\times k}$. Then,*

$$r\begin{bmatrix} A, & B \end{bmatrix} = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{1.5}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.6}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{1.7}$$

The following formulas are derived from (1.5), (1.6) and (1.7):

$$r\begin{bmatrix} A & BF_N \\ E_R C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & R \\ 0 & N & 0 \end{bmatrix} - r(N) - r(R),$$

$$r\begin{bmatrix} M & L \\ E_R A & E_R B \end{bmatrix} = r\begin{bmatrix} M & L & 0 \\ A & B & R \end{bmatrix} - r(R),$$

$$r\begin{bmatrix} M & AF_N \\ L & BF_N \end{bmatrix} = r\begin{bmatrix} M & A \\ L & B \\ O & N \end{bmatrix} - r(N).$$

Lemma 1.2. *[11] Let $D \in \mathbb{C}^{s\times r}$, $E \in \mathbb{C}^{s\times k}$ and $H \in \mathbb{C}^{l\times r}$ be given matrices. Then,*

$$\min_{\substack{X\in\mathbb{C}^{k\times r}\\ Y\in\mathbb{C}^{s\times l}}} r(D - EX - YH) = r\begin{bmatrix} D & E \\ H & 0 \end{bmatrix} - r(E) - r(H). \tag{1.8}$$

Lemma 1.3. *[10] Let $D \in \mathbb{C}^{s\times r}$, $E_1 \in \mathbb{C}^{s\times l_1}$, $E_2 \in \mathbb{C}^{s\times l_2}$, $K_1 \in \mathbb{C}^{k_1\times r}$, $K_2 \in \mathbb{C}^{k_2\times r}$ be matrices, such that $R(E_1) \subset R(E_2)$ and $R(K_2^*) \subset R(K_1^*)$. Then,*

$$\max_{\substack{X_1\in\mathbb{C}^{l_1\times k_1}\\ X_2\in\mathbb{C}^{l_2\times k_2}}} r(D - E_1X_1K_1 - E_2X_2K_2) = \min\left\{ r\begin{bmatrix} D & E_2 \end{bmatrix}, r\begin{bmatrix} D \\ K_1 \end{bmatrix}, r\begin{bmatrix} D & E_1 \\ K_2 & 0 \end{bmatrix} \right\}. \tag{1.9}$$

2 Relationships between least-rank solutions of the matrix equation $AXB = C$ and its four smaller equations

Lemma 2.1. *[9] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$ be given matrices. The least-rank solution of (1.1) can be expressed as follows:*

$$X = -TM^+S + \widehat{T}U + V\widehat{S}, \tag{2.1}$$

where $M = \begin{bmatrix} C & A \\ B & 0 \end{bmatrix}$, $T = \begin{bmatrix} 0 & I_n \end{bmatrix}$, $S = \begin{bmatrix} 0 \\ I_p \end{bmatrix}$, $\widehat{T} = TF_M$, $\widehat{S} = E_MS$, $U \in \mathbb{C}^{(q+n)\times p}$ and $V \in \mathbb{C}^{n\times(m+p)}$ are arbitrary matrices.

The following notations are adopted for the sets of least-rank solutions to the equations in (1.1) and (1.3):

$$S_1 = \left\{X \in \mathbb{C}^{n\times p} \;/\; r(AXB - C) = \min\right\}, \tag{2.2}$$

$$S_2 = \left\{ \Gamma = \frac{(X_{11} + X_{22} + X_{33} + X_{44})}{4} \in \mathbb{C}^{n\times p} \;/\; \begin{matrix} r(A_{11}X_{11}B_{11} - C_{11}) = \min \\ r(A_{11}X_{22}B_{22} - C_{22}) = \min \\ r(A_{22}X_{33}B_{11} - C_{33}) = \min \\ r(A_{22}X_{44}B_{22} - C_{44}) = \min \end{matrix} \right\}. \tag{2.3}$$

Based on these notations, the results can be obtained as follows:

Theorem 2.2. *Let X and X_{11}, X_{22}, X_{33} and X_{44} be the least-rank solutions to (1.1) and (1.3) respectively, define S_1 and S_2 as in (2.2) and (2.3) respectively. Denote*

$$H_1 = \begin{bmatrix} C & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ -B & 0 & B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ -B & 0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ -B & 0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ -B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix},$$

$$H_2=\begin{bmatrix} C & 0 & -A & 0 & -A & 0 & -A & 0 & -A \\ B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix},$$

$$L=\begin{bmatrix} -C & -A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}C_{11} & \frac{1}{4}A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4}C_{22} & \frac{1}{4}A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4}B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}C_{33} & \frac{1}{4}A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}C_{44} & \frac{1}{4}A_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}B_{22} & 0 \end{bmatrix},$$

$$D_1=\begin{bmatrix} C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ -B_{11} & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ -B_{11} & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ -B_{11} & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix},$$

$$D_2=\begin{bmatrix} C_{11} & 0 & -A_{11} & 0 & -A_{11} & 0 & -A_{11} \\ B_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix}$$

Then, the following hold.
a) $S_1 \cap S_2 \neq \emptyset$ *if and only if*

$$r\begin{bmatrix} 0 & H_1 \\ H_2 & L \end{bmatrix} = r(H_1) + r(H_2).$$

b) $S_2 \subseteq S_1$ *if and only if*

$$r(M) = r\begin{bmatrix} C \\ B \end{bmatrix}, \text{ or } r(M) = r\begin{bmatrix} C & A \end{bmatrix},$$

$$\text{or } r\begin{bmatrix} 0 & H_1 \\ H_2 & L \end{bmatrix} = r\begin{bmatrix} C \\ B \end{bmatrix} + r\begin{bmatrix} C & A \end{bmatrix} + 2\sum_{i=1}^{4} r(M_i).$$

c) $S_1 \subseteq S_2$ *if and only if*

$$r(D_1) = \sum_{i=1}^{4} r(M_i), \text{ or } r(D_2) = \sum_{i=1}^{4} r(M_i),$$

$$\text{or } r \begin{bmatrix} 0 & H_1 \\ H_2 & L \end{bmatrix} = r(D_2) + r(D_1) + 2r(M).$$

Proof. a) The intersection $S_1 \cap S_2 \neq \emptyset$ implies that the minimum rank of the matrix expression $X - \Gamma$ is zero, that is:

$$\min_{\Gamma \in S2,\, X \in S_1} r(X - \Gamma) = 0. \tag{2.4}$$

According to (2.1), the general expressions for the least-rank solutions of the four matrix equations in (1.3) can be written as follows:

$$\begin{aligned} X_{11} &= -T_1 M_1^+ S_1 + T_{11} U_1 + V_1 S_{11}, \\ X_{22} &= -T_2 M_2^+ S_2 + T_{22} U_2 + V_2 S_{22}, \\ X_{33} &= -T_3 M_3^+ S_3 + T_{33} U_3 + V_3 S_{33}, \\ X_{44} &= -T_4 M_4^+ S_4 + T_{44} U_4 + V_4 S_{44}. \end{aligned}$$

where

$$M_1 = \begin{bmatrix} C_{11} & A_{11} \\ B_{11} & 0 \end{bmatrix}, M_2 = \begin{bmatrix} C_{22} & A_{11} \\ B_{22} & 0 \end{bmatrix}, M_3 = \begin{bmatrix} C_{33} & A_{22} \\ B_{11} & 0 \end{bmatrix}, M_4 = \begin{bmatrix} C_{44} & A_{22} \\ B_{22} & 0 \end{bmatrix},$$

$$T_i = \begin{bmatrix} 0 & I_n \end{bmatrix}, S_i = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, T_{ii} = T_i F_{M_i}, S_{ii} = E_{M_i} S_i, \text{ for } i = 1, 2, 3, 4.$$

We can rewrite the expression $X - \Gamma$ as follows:

$$\begin{aligned} X - \Gamma &= -TM^+S + \frac{T_1 M_1^+ S_1}{4} + \frac{T_2 M_2^+ S_2}{4} + \frac{T_3 M_3^+ S_3}{4} + \frac{T_4 M_4^+ S_4}{4} + \widehat{T} U + V \widehat{S} \\ &\quad - T_{11} U_1 - V_1 S_{11} - T_{22} U_2 - V_2 S_{22} - T_{33} U_3 - V_3 S_{33} - T_{44} U_4 - V_4 S_{44} \\ &= -TM^+S + \frac{T_1 M_1^+ S_1}{4} + \frac{T_2 M_2^+ S_2}{4} + \frac{T_3 M_3^+ S_3}{4} + \frac{T_4 M_4^+ S_4}{4} \\ &\quad + \begin{bmatrix} \widehat{T} & T_{11} & T_{22} & T_{33} & T_{44} \end{bmatrix} \begin{bmatrix} U \\ -U_1 \\ -U_2 \\ -U_3 \\ -U_4 \end{bmatrix} + \begin{bmatrix} V & -V_1 & -V_2 & -V_3 & -V_4 \end{bmatrix} \begin{bmatrix} \widehat{S} \\ S_{11} \\ S_{22} \\ S_{33} \\ S_{44} \end{bmatrix} \\ &= G + NZ + WK, \end{aligned} \tag{2.5}$$

where

$$G = -TM^+S + \frac{T_1 M_1^+ S_1}{4} + \frac{T_2 M_2^+ S_2}{4} + \frac{T_3 M_3^+ S_3}{4} + \frac{T_4 M_4^+ S_4}{4},$$

$$N = \begin{bmatrix} \widehat{T} & T_{11} & T_{22} & T_{33} & T_{44} \end{bmatrix}, K = \begin{bmatrix} \widehat{S} \\ S_{11} \\ S_{22} \\ S_{33} \\ S_{44} \end{bmatrix},$$

and $Z = \begin{bmatrix} U^* & -U_1^* & -U_2^* & -U_3^* & -U_4^* \end{bmatrix}^*$, $W = \begin{bmatrix} V & -V_1 & -V_2 & -V_3 & -V_4 \end{bmatrix}$ are arbitrary with appropriate sizes.
By applying (1.8) in Lemma (1.2) to Equation (2.5), we can deduce the following:

$$\min_{X\in S_1,\Gamma\in S2} r\left(X-\Gamma\right) = \min_{Z,W} r\left(G+NZ+WK\right) = r\begin{bmatrix} G & N \\ K & 0 \end{bmatrix} - r\left(N\right) - r\left(K\right). \quad (2.6)$$

By applying Lemma (1.1) and three elementary block matrix operations, we obtain

$$r\begin{bmatrix} G & N \\ K & 0 \end{bmatrix} = r\begin{bmatrix} -TM^+S+\frac{T_1M_1^+S_1}{4}+\frac{T_2M_2^+S_2}{4}+\frac{T_3M_3^+S_3}{4}+\frac{T_4M_4^+S_4}{4} & \widehat{T} & T_{11} & T_{22} & T_{33} & T_{44} \\ \widehat{S} & 0 & 0 & 0 & 0 & 0 \\ S_{11} & 0 & 0 & 0 & 0 & 0 \\ S_{22} & 0 & 0 & 0 & 0 & 0 \\ S_{33} & 0 & 0 & 0 & 0 & 0 \\ S_{44} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} -TM^+S+\frac{T_1M_1^+S_1}{4}+\frac{T_2M_2^+S_2}{4}+\frac{T_3M_3^+S_3}{4}+\frac{T_4M_4^+S_4}{4} & TF_M & T_1F_{M_1} & T_2F_{M_2} & T_3F_{M_3} & T_4F_{M_4} \\ E_MS & 0 & 0 & 0 & 0 & 0 \\ E_{M_1}S_1 & 0 & 0 & 0 & 0 & 0 \\ E_{M_2}S_2 & 0 & 0 & 0 & 0 & 0 \\ E_{M_3}S_3 & 0 & 0 & 0 & 0 & 0 \\ E_{M_4}S_4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} -TM^+S+\frac{T_1M_1^+S_1}{4}+\frac{T_2M_2^+S_2}{4}+\frac{T_3M_3^+S_3}{4}+\frac{T_4M_4^+S_4}{4} & T & T_1 & T_2 & T_3 & T_4 & 0 & 0 & 0 & 0 & 0 \\ S & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 \\ S_1 & 0 & 0 & 0 & 0 & 0 & 0 & M_1 & 0 & 0 & 0 \\ S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_2 & 0 & 0 \\ S_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3 & 0 \\ S_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_4 \\ 0 & M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-2\sum_{i=1}^{4} r\left(M_i\right) - 2r\left(M\right)$$

$$= r\begin{bmatrix} 0 & T & T_1 & T_2 & T_3 & T_4 & 0 & 0 & 0 & 0 & 0 \\ S & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 \\ S_1 & 0 & 0 & 0 & 0 & 0 & 0 & M_1 & 0 & 0 & 0 \\ S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_2 & 0 & 0 \\ S_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3 & 0 \\ S_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_4 \\ 0 & M & 0 & 0 & 0 & 0 & -M & 0 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 & 0 & 0 & 0 & \frac{1}{4}M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0 & \frac{1}{4}M_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_3 & 0 & 0 & 0 & 0 & \frac{1}{4}M_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_4 & 0 & 0 & 0 & 0 & \frac{1}{4}M_4 \end{bmatrix} - 2\sum_{i=1}^{4} r\left(M_i\right) - 2r\left(M\right)$$

$$
\begin{aligned}
&= p+n+r\left[\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C & 0 & -A & 0 & -A & 0 & -A & 0 & -A \\
B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0
\end{array}\right. \\
&\qquad\left.\begin{array}{cccccccccc}
C & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
-B & 0 & B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\
-B & 0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\
-B & 0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\
-B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \\
-C & -A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4}C_{11} & \frac{1}{4}A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4}B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4}C_{22} & \frac{1}{4}A_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4}B_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}C_{33} & \frac{1}{4}A_{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}B_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}C_{44} & \frac{1}{4}A_{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}B_{22} & 0
\end{array}\right] \\
&- 2\sum_{i=1}^{4} r\left(M_i\right) - 2r\left(M\right) \\
&= p+n+r\begin{bmatrix} 0 & H_1 \\ H_2 & L \end{bmatrix} - 2\sum_{i=1}^{4} r\left(M_i\right) - 2r\left(M\right). \qquad (2.7)
\end{aligned}
$$

$$r(K) = r\begin{bmatrix} \widehat{S} \\ S_{11} \\ S_{22} \\ S_{33} \\ S_{44} \end{bmatrix} = r\begin{bmatrix} S & M & 0 & 0 & 0 & 0 \\ S_1 & 0 & M_1 & 0 & 0 & 0 \\ S_2 & 0 & 0 & M_2 & 0 & 0 \\ S_3 & 0 & 0 & 0 & M_3 & 0 \\ S_4 & 0 & 0 & 0 & 0 & M_4 \end{bmatrix} - \sum_{i=1}^{4} r(M_i) - r(M)$$

$$= r\begin{bmatrix} 0 & C & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_p & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ I_p & 0 & 0 & B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ I_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix} - \sum_{i=1}^{4} r(M_i) - r(M)$$

$$= p + r\begin{bmatrix} C & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ -B & 0 & B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ -B & 0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ -B & 0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ -B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix} - \sum_{i=1}^{4} r(M_i) - r(M)$$

$$= p + r(H_1) - \sum_{i=1}^{4} r(M_i) - r(M). \tag{2.8}$$

$$r(N) = r\begin{bmatrix} \widehat{T} & T_{11} & T_{22} & T_{33} & T_{44} \end{bmatrix}$$

$$= r\begin{bmatrix} 0 & I_n & 0 & I_n & 0 & I_n & 0 & I_n & 0 & I_n \\ C & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix} - \sum_{i=1}^{4} r(M_i) - r(M)$$

$$= n + r \begin{bmatrix} C & 0 & -A & 0 & -A & 0 & -A & 0 & -A \\ B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix} - \sum_{i=1}^{4} r(M_i) - r(M)$$

$$= n + r(H_2) - \sum_{i=1}^{4} r(M_i) - r(M). \tag{2.9}$$

By substituting (2.7)-(2.9) into (2.6), we obtain

$$\min_{\substack{X \in S_1 \\ \Gamma \in S_2}} r(X - \Gamma) = r \begin{bmatrix} 0 & H_1 \\ H_2 & L \end{bmatrix} - r(H_2) - r(H_1). \tag{2.10}$$

By substituting (2.10) into (2.4), we obtain (a).
b) Note that $S_1 \supseteq S_2$ is equivalent to

$$\max_{\Gamma \in S_2} \min_{X \in S_1} r(X - \Gamma) = 0. \tag{2.11}$$

Then, we have

$$\min_{X \in S_1} r(X - \Gamma) = \min_{U,V} r\left(-TM^+S - \Gamma + \widehat{T}U + V\widehat{S}\right). \tag{2.12}$$

Applying (1.8) to (2.12) yields

$$\min_{X \in S_1} r(X - \Gamma) = r \begin{bmatrix} -TM^+S - \Gamma & \widehat{T} \\ \widehat{S} & 0 \end{bmatrix} - r\left(\widehat{T}\right) - r\left(\widehat{S}\right). \tag{2.13}$$

According to (1.5) and (1.6), we have

$$r\left(\widehat{T}\right) = r(TF_M) = r \begin{bmatrix} 0 & I_n \\ C & A \\ B & 0 \end{bmatrix} - r(M) = r \begin{bmatrix} C \\ B \end{bmatrix} - r(M) + n, \tag{2.14}$$

$$r\left(\widehat{S}\right) = r(E_M S) = \begin{bmatrix} 0 & C & A \\ I_P & B & 0 \end{bmatrix} - r(M) = r \begin{bmatrix} C & A \end{bmatrix} - r(M) + p. \tag{2.15}$$

The 2×2 block matrix on the right-hand side of (2.13) can be rewritten as

$$\begin{bmatrix} -TM^+S - \Gamma & \widehat{T} \\ \widehat{S} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} G + \widehat{T}U + V\widehat{S} - T_{11}U_1 - V_1S_{11} - T_{22}U_2 - V_2S_{22} - T_{33}U_3 - V_3S_{33} - T_{44}U_4 - V_4S_{44} & \widehat{T} \\ \widehat{S} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} G & \widehat{T} \\ \widehat{S} & 0 \end{bmatrix} - \begin{bmatrix} T_{11} & T_{22} & T_{33} & T_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \begin{bmatrix} I_p & 0 \end{bmatrix}$$

$$- \begin{bmatrix} I_n \\ 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix} \begin{bmatrix} S_{11} & 0 \\ S_{22} & 0 \\ S_{33} & 0 \\ S_{44} & 0 \end{bmatrix}. \tag{2.16}$$

In addition, we have

$$r \begin{bmatrix} I_n & T_{11} & T_{22} & T_{33} & T_{44} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

$$r \begin{bmatrix} I_p & 0 \\ S_{11} & 0 \\ S_{22} & 0 \\ S_{33} & 0 \\ S_{44} & 0 \end{bmatrix} = \begin{bmatrix} I_p & 0 \end{bmatrix}.$$

Thus, $R \begin{bmatrix} T_{11} & T_{22} & T_{33} & T_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} \subseteq R \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ and $R \begin{bmatrix} S_{11}^* & S_{22}^* & S_{33}^* & S_{44}^* \\ 0 & 0 & 0 & 0 \end{bmatrix} \subseteq R \begin{bmatrix} I_p \\ 0 \end{bmatrix}$.
Hence, by applying (1.9) to (2.16), we obtain

$$\max_{\Gamma \in S_2} r \begin{bmatrix} -TM^+S - \Gamma & \widehat{T} \\ \widehat{S} & 0 \end{bmatrix}$$

$$= \min \left\{ r \begin{bmatrix} G & \widehat{T} & I_n \\ \widehat{S} & 0 & 0 \end{bmatrix}, r \begin{bmatrix} G & \widehat{T} \\ \widehat{S} & 0 \\ I_p & 0 \end{bmatrix}, r \begin{bmatrix} G & \widehat{T} & T_{11} & T_{22} & T_{33} & T_{44} \\ \widehat{S} & 0 & 0 & 0 & 0 & 0 \\ S_{11} & 0 & 0 & 0 & 0 & 0 \\ S_{22} & 0 & 0 & 0 & 0 & 0 \\ S_{33} & 0 & 0 & 0 & 0 & 0 \\ S_{44} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

$$= \min \left\{ n + r\left(\widehat{S}\right), p + r\left(\widehat{T}\right), r \begin{bmatrix} G & N \\ K & 0 \end{bmatrix} \right\}. \tag{2.17}$$

Combining (2.17) and (2.13) yields

$$\max_{\Gamma \in S_2} \min_{X \in S_1} r\left(X - \Gamma\right)$$

$$= \min \left\{ n - r\left(\widehat{T}\right), p - r\left(\widehat{S}\right), r \begin{bmatrix} G & N \\ K & 0 \end{bmatrix} - r\left(\widehat{T}\right) - r\left(\widehat{S}\right) \right\}$$

$$= \min \left\{ \begin{array}{c} r\left(M\right) - r \begin{bmatrix} C \\ B \end{bmatrix}, r\left(M\right) - r \begin{bmatrix} C & A \end{bmatrix}, \\ r \begin{bmatrix} 0 & H_1 \\ H_2 & L \end{bmatrix} - r \begin{bmatrix} C \\ B \end{bmatrix} - r \begin{bmatrix} C & A \end{bmatrix} - 2\sum_{i=1}^{4} r\left(M_i\right) \end{array} \right\}. \tag{2.18}$$

By substituting (2.18) into (2.11), we obtain (b).
(c) The inclusion $S_1 \subseteq S_2$ is equivalent to

$$\max_{X \in S_1} \min_{\Gamma \in S_2} r\left(X - \Gamma\right) = 0. \tag{2.19}$$

Applying (1.8) to the matrix expression $X - \Gamma$ yields

$$\min_{\Gamma \in S_2} r\left(X - \Gamma\right)$$

$$= \min_{\Gamma \in S_2} \left(\begin{array}{c} \left(X + \frac{T_1 M_1^+ S_1}{4} + \frac{T_2 M_2^+ S_2}{4} + \frac{T_3 M_3^+ S_3}{4} + \frac{T_4 M_4^+ S_4}{4}\right) - \\ \begin{bmatrix} T_{11} & T_{22} & T_{33} & T_{44} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} - \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix} \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{44} \end{bmatrix} \end{array} \right)$$

$$= r \begin{bmatrix} X + \frac{T_1 M_1^+ S_1}{4} + \frac{T_2 M_2^+ S_2}{4} + \frac{T_3 M_3^+ S_3}{4} + \frac{T_4 M_4^+ S_4}{4} & T_{11} & T_{22} & T_{33} & T_{44} \\ S_{11} & 0 & 0 & 0 & 0 \\ S_{22} & 0 & 0 & 0 & 0 \\ S_{33} & 0 & 0 & 0 & 0 \\ S_{44} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$- r \begin{bmatrix} T_{11} & T_{22} & T_{33} & T_{44} \end{bmatrix} - r \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{44} \end{bmatrix}. \tag{2.20}$$

The 5×5 block matrix in (2.20) can be rewritten as

$$\begin{bmatrix} X + \frac{T_1 M_1^+ S_1}{4} + \frac{T_2 M_2^+ S_2}{4} + \frac{T_3 M_3^+ S_3}{4} + \frac{T_4 M_4^+ S_4}{4} & T_{11} & T_{22} & T_{33} & T_{44} \\ S_{11} & 0 & 0 & 0 & 0 \\ S_{22} & 0 & 0 & 0 & 0 \\ S_{33} & 0 & 0 & 0 & 0 \\ S_{44} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} G & T_{11} & T_{22} & T_{33} & T_{44} \\ S_{11} & 0 & 0 & 0 & 0 \\ S_{22} & 0 & 0 & 0 & 0 \\ S_{33} & 0 & 0 & 0 & 0 \\ S_{44} & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \widehat{T} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} U \begin{bmatrix} I_p & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} I_n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V \begin{bmatrix} \widehat{S} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.21}$$

Applying (1.9) to (2.21) yields

$$\max_{X \in S_1} r \begin{bmatrix} X + \frac{T_1 M_1^+ S_1}{4} + \frac{T_2 M_2^+ S_2}{4} + \frac{T_3 M_3^+ S_3}{4} + \frac{T_4 M_4^+ S_4}{4} & T_{11} & T_{22} & T_{33} & T_{44} \\ S_{11} & 0 & 0 & 0 & 0 \\ S_{22} & 0 & 0 & 0 & 0 \\ S_{33} & 0 & 0 & 0 & 0 \\ S_{44} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \min \left\{ \begin{array}{c} r\begin{bmatrix} G & T_{11} & T_{22} & T_{33} & T_{44} & I_n \\ S_{11} & 0 & 0 & 0 & 0 & 0 \\ S_{22} & 0 & 0 & 0 & 0 & 0 \\ S_{33} & 0 & 0 & 0 & 0 & 0 \\ S_{44} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, r\begin{bmatrix} G & T_{11} & T_{22} & T_{33} & T_{44} \\ S_{11} & 0 & 0 & 0 & 0 \\ S_{22} & 0 & 0 & 0 & 0 \\ S_{33} & 0 & 0 & 0 & 0 \\ S_{44} & 0 & 0 & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 \end{bmatrix}, \\ r\begin{bmatrix} G & T_{11} & T_{22} & T_{33} & T_{44} & \widehat{T} \\ S_{11} & 0 & 0 & 0 & 0 & 0 \\ S_{22} & 0 & 0 & 0 & 0 & 0 \\ S_{33} & 0 & 0 & 0 & 0 & 0 \\ S_{44} & 0 & 0 & 0 & 0 & 0 \\ \widehat{S} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \right\}$$

$$= \min \left\{ n + r\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{44} \end{bmatrix}, p + r\begin{bmatrix} T_{11} & T_{22} & T_{33} & T_{44} \end{bmatrix}, r\begin{bmatrix} G & N \\ K & 0 \end{bmatrix} \right\}. \tag{2.22}$$

Furthermore, we have

$$r\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{44} \end{bmatrix} = r\begin{bmatrix} S_1 & M_1 & 0 & 0 & 0 \\ S_2 & 0 & M_2 & 0 & 0 \\ S_3 & 0 & 0 & M_3 & 0 \\ S_4 & 0 & 0 & 0 & M_4 \end{bmatrix} - \sum_{i=1}^{4} r\left(M_i\right)$$

$$= r\begin{bmatrix} 0 & C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ I_p & B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ I_p & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ I_p & 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix} - \sum_{i=1}^{4} r\left(M_i\right)$$

$$= p + r\begin{bmatrix} C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ -B_{11} & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ -B_{11} & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ -B_{11} & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{bmatrix} - \sum_{i=1}^{4} r\left(M_i\right)$$

$$= p + r\left(D_1\right) - \sum_{i=1}^{4} r\left(M_i\right), \tag{2.23}$$

$$
\begin{aligned}
& r\left[\begin{array}{llll} T_{11} & T_{22} & T_{33} & T_{44} \end{array}\right] \\
& = r\left[\begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \\ M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & M_4 \end{array}\right]-\sum_{i=1}^{4} r\left(M_i\right) \\
& = r\left[\begin{array}{cccccccc} 0 & I_n & 0 & I_n & 0 & I_n & 0 & I_n \\ C_{11} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{array}\right]-\sum_{i=1}^{4} r\left(M_i\right) \\
& = n+r\left[\begin{array}{ccccccc} C_{11} & 0 & -A_{11} & 0 & -A_{11} & 0 & -A_{11} \\ B_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{22} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{33} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} & A_{22} \\ 0 & 0 & 0 & 0 & 0 & B_{22} & 0 \end{array}\right]-\sum_{i=1}^{4} r\left(M_i\right) \\
& = n+r\left(D_2\right)-\sum_{i=1}^{4} r\left(M_i\right). \qquad (2.24)
\end{aligned}
$$

Substituting (2.23) and (2.24) into (2.20) and combining (2.22) and (2.20) yields

$$
\begin{aligned}
& \max_{X \in S_1} \min_{\Gamma \in S_2} r(X-\Gamma) \\
& = \min \left\{\sum_{i=1}^{4} r\left(M_i\right)-r\left(D_2\right), \sum_{i=1}^{4} r\left(M_i\right)-r\left(D_1\right), r\left[\begin{array}{cc} 0 & H_1 \\ H_2 & L \end{array}\right]-r\left(D_2\right)-r\left(D_1\right)-2r(M)\right\}. \qquad (2.25)
\end{aligned}
$$

Finally, by substituting (2.25) into (2.19), we obtain the desired results in (c). □

3 Conclusion

In the previous section we studied a problem relating to the relations between the original matrix equation in (1.1) and its four smaller equations in (1.3), by using various well-known formulas concerning rank and Moore-penrose inverses. These results give some profound investigations into the properties of the least-rank solutions of Equation (1.1).

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