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q-SEQUENCE SPACE OF NON-ABSOLUTE TYPE AND TOEPLITZ DUALS

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Abstract: *In the present paper, we introduce a method of q-analogue of A^r-matrix of order r. Using this method, we obtain topological properties and some inclusion relations. Additionally, the alpha dual the beta dual and the gamma dual of the newly defined sequence spaces are calculated and their basis have been determined. Finally, the necessary and sufficient conditions on an infinite matrix from newly defined sequence spaces to any of the spaces c, c_0, ℓ_∞ and ℓ_p have been examined.*

1 Introduction

Let s represents the space of all complex valued sequences. The well-known classical sequence spaces are the set of all bounded sequences ℓ_∞, the null sequences c_0, the convergent sequences c and the p-absolutely summable sequences ℓ_p, where $1 \leq p < \infty$. We also indicate the spaces of all convergent and bounded series by cs and bs respectively. A Banach sequence space with continuous co-ordinates is a BK-space. For instance, the space ℓ_p is a BK-space furnished by the norm $\|u\|_{\ell_p} = (\sum_k |u_k|^p)^{1/p}$. We assume throughout that for $1 \leq p < \infty$ and p' is conjugate number of p such that $p^{-1} + p'^{-1} = 1$.
For any sequence space μ and infinite matrix $\mathcal{A}$, the matrix domain of $\mathcal{A}$ is defined as

$$\mu_{\mathcal{A}} = \{u \in s : \mathcal{A}u \in \mu\}. \tag{1.1}$$

Throughout the text, $\mathbb{N}$ is the set of natural numbers including zero.
If a normed linear space $\mathcal{U}$ contains a sequence (b_m), then for every $u \in \mathcal{U}$ there is a unique sequence of scalars (α_m) such that

$$\|u - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_m b_m)\| \to 0 \; as \; m \to \infty,$$

then (b_m) is known as the Schauder basis for $\mathcal{U}$. The series $\sum_{m=0}^{\infty} \alpha_m b_m$ has the sum u known as the expansion of u about the basis (b_m), and we write $u = \sum_{m=0}^{\infty} \alpha_m b_m$, (see, [8]).

Let $\mathcal{U}$ and $\mathcal{V}$ be any two sequence spaces. Then, the multiplier space $\mathcal{M}(\mathcal{U}, \mathcal{V})$ is given as

$$\mathcal{M}(\mathcal{U}, \mathcal{V}) = \{(a_m) \in s : av = (a_m v_m) \in \mathcal{V}, \; for \;\; every \; v \in \mathcal{U}\}.$$

Thus, the α-dual, the β-dual and the γ-dual of $\mathcal{U}$ are denoted as

$$\mathcal{U}^\alpha = \mathcal{M}(\mathcal{U}, \ell_1), \; \mathcal{U}^\beta = \mathcal{M}(\mathcal{U}, cs), \; \mathcal{U}^\gamma = \mathcal{M}(\mathcal{U}, bs).$$

An infinite matrix can be observed as the linear operator from a sequence space into another sequence space. For this, let $\mathcal{U}$ and $\mathcal{V}$ be any arbitrary subsets of s. Let $\mathcal{A} = (a_{mk})$ is an infinite *matrix with complex entries* (a_{mk}). *By* $\mathcal{A}(u) = (\mathcal{A}_m(u)) = (\mathcal{A}u)_m$, *we write the* $\mathcal{A}-$*transforms of a sequence* $u = (u_k)$, *if the series* $\mathcal{A}_m(u) = \sum_k a_{mk}u_k$ *is convergent for* $m \geq 0$.

If $\mathcal{A}u \in \mathcal{V}$ *with* $u \in \mathcal{U}$, *then* $\mathcal{A}$ *defines a matrix mapping from* $\mathcal{U}$ *into* $\mathcal{V}$. *Further,* $(\mathcal{U}, \mathcal{V})$ *indicates the family of all infinite matrices that maps* $\mathcal{U}$ *into* $\mathcal{V}$. *Thus,* $\mathcal{A}$ *is in* $(\mathcal{U}, \mathcal{V})$ *if and only if* $\mathcal{A}u = ((\mathcal{A}u)_m) \in \mathcal{V}$, $\forall\ u \in \mathcal{U}$, *that is,* $\mathcal{A} \in (\mathcal{U}, \mathcal{V})$ *if and only if* $\mathcal{A}_m \in \mathcal{U}^{\beta}$, $\forall\ m$ *(see, [20]). In the literature, several authors have constructed the sequence spaces via the domain of some special matrices. One may refer to ([1], [2], [5], [11], [14], [16], [18]) and references therein.*

2 q-sequence spaces

The purpose of this paper is primarily focuses on the q*-analogue of* $\mathcal{A}^r$ *matrices and to acquire the new results akin to the* q*-analogue. Let us embark with* q*-integer definition, following [12].*

Definition 2.1. A q-integer is defined as

$$[m]_q = \begin{cases} \frac{1-q^m}{1-q}, & q \in \mathbb{R}^+ \setminus \{1\} \\ m, & q = 1 \end{cases}$$

This is known as the q-analogue of m. It is trivial that if $q \to 1^-$, then $[m]_q \to m$. We indicate $[m]_q$ briefly by $[m]$.

Given $q > 0$, *define* $\mathbb{N}_q = \{[m] : m \in \mathbb{N}\}$. *Then it can be seen from* $[m]_q$ *that*

$$\mathbb{N}_q = \left\{0, 1, 1+q, 1+q+q^2, 1+q+q^2+q^3, \cdots\right\}. \tag{2.1}$$

It is clear that when we put $q = 1$ *in (2.1), then* $\mathbb{N}_q$ *reduces to be the set of natural numbers and the set of all non-negative integers* $\mathbb{N}$.

Definition 2.2. For any integer m and n with $m \geq n \geq 0$, the q-binomial coefficient is defined as

$$\begin{bmatrix} m \\ n \end{bmatrix} = \begin{cases} \frac{m!}{[m-n]![n]!}, & 0 \leq n \leq m \\ 0, & otherwise \end{cases}$$

where q-factorial of $[m]!$ of m is defined as

$$[m]! = \begin{cases} \prod_{k=1}^{m} [k], & m = 1, 2, 3, \cdots \\ 1, & m = 0 \end{cases}$$

Proposition 2.3. *There are two types of Pascal rules namely*

$$\begin{bmatrix} m \\ j \end{bmatrix} = \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} m-1 \\ j \end{bmatrix}$$

and

$$\begin{bmatrix} m \\ j \end{bmatrix} = q^{m-j} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} + \begin{bmatrix} m-1 \\ j \end{bmatrix}$$

where, $1 \leq j \leq n-1$.

It is found that studies including q*-integers and its applications have become an functioning research areas. Several authors published many research papers on the* q*-analogs and the existing theories, following ([7], [9], [15], [17]). The* q*-analogue of Cesàro sequence spaces were defined and studied by Demiriz and Şahin [6] and Yaying et al. [21] . There are several ways to define the Cesàro matrices via* q*-analogs. However, the following Theorem provides a suitable* q*-analogs of the Cesàro matrices of order one.*

Theorem 2.4. *[3] $C^1(q^k) = (c^1_{mk}(q^k))$ such that*

$$c^1_{mk}(q^k) = \begin{cases} \frac{q^k}{[m+1]_q}, & 0 \leq k \leq m \\ 0, & k > m \end{cases}$$

for all $m, k \in \mathbb{N}$.

In [4], Başar introduced the new class $\mathcal{A}^r = (a^r_{mk})$ of Toeplitz matrices given as

$$a^r_{mk} = \begin{cases} \frac{1+r^k}{m+1}, & 0 \leq k \leq m \\ 0, & k > m \end{cases}$$

for all $m, k \in \mathbb{N}$. It is known that the matrices $\mathcal{A}^r$ is regular for $0 < r < 1$ and is stronger than the Cesàro matrices of order one. In the rest of the paper, our main target is on the $\mathcal{A}^r$-matrices by means of q-analogs which has explicitly the following form

$$\mathcal{A}^r(q^k) = \begin{bmatrix} [2]_q & 0 & 0 & 0 & \cdots & 0 \\ \frac{[2]_q}{[2]_q} & \frac{[1+r]_q q}{[2]_q} & 0 & 0 & \cdots & 0 \\ \frac{[2]_q}{[3]_q} & \frac{[1+r]_q q}{[3]_q} & \frac{[1+r^2]_q q^2}{[3]_q} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{[2]_q}{[m+1]_q} & \frac{[1+r]_q q}{[m+1]_q} & \frac{[1+r^2]_q q^2}{[m+1]_q} & \cdots & \frac{[1+r^m]_q q^m}{[m+1]_q} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

3 The q-sequence spaces $[a^r_p]$, $[a^r_0]$, $[a^r_c]$ and $[a^r_\infty]$

This section involves the introduction and completeness of $A^r(q^k)$ space via q-analogue. Additionally, we also give linearly isomorphism, inclusion relations and construct the Schauder basis for the newly identified spaces. Let us now define the sequence spaces $[a^r_p]$, $[a^r_0]$, $[a^r_c]$ and $[a^r_\infty]$ as the set of all sequences such that $\mathcal{A}^r(q^k)$-transforms of them are in the spaces a^r_p, a^r_0, a^r_c and a^r_∞ respectively, i. e,

$$[a^r_p] = \left\{ u = (u_k) \in s : \sum_m \left| \frac{1}{[m+1]_q} \sum_{k=0}^{m} [1+r^k]_q q^k u_k \right|^p < \infty \right\}, (0 < p < \infty)$$

$$[a^r_0] = \left\{ u = (u_k) \in s : \lim_{m\to\infty} \frac{1}{[m+1]_q} \sum_{k=0}^{m} [1+r^k]_q q^k u_k = 0 \right\},$$

$$[a^r_c] = \left\{ u = (u_k) \in s : \lim_{m\to\infty} \frac{1}{[m+1]_q} \sum_{k=0}^{m} [1+r^k]_q q^k u_k \text{ exists} \right\},$$

$$[a^r_\infty] = \left\{ u = (u_k) \in s : \sup_{m\in\mathbb{N}} \left| \frac{1}{[m+1]_q} \sum_{k=0}^{m} [1+r^k]_q q^k u_k \right| < \infty \right\},$$

where $\mathcal{A}^r(q^k)$ is the method of q-sequence space of A^r-matrix of order r. Let us redefine the sequence spaces with definition of matrix domain (1.1) by $[a^r_p] = \{a^r_p\}_{\mathcal{A}^r(q^k)}$, $[a^r_0] = \{a^r_0\}_{\mathcal{A}^r(q^k)}$, $[a^r_c] = \{a^r_c\}_{\mathcal{A}^r(q^k)}$ and $[a^r_\infty] = \{a^r_\infty\}_{\mathcal{A}^r(q^k)}$.
If $\mathcal{U}$ is any normed sequence space, then we say that the matrix domain $\{\mathcal{U}\}_{\mathcal{A}^r(q^k)}$ as the q-analogs of $\mathcal{A}^r$ sequence space. Let us define the sequence $v = (v_k)$ by the $\mathcal{A}^r(q^k)$ transform of a sequence $u = (u_k)$, i.e.,

$$v^q_k = \frac{1}{[k+1]_q} \sum_{i=0}^{k} [1+r^i]_q u_i \quad (k \in \mathbb{N}). \tag{3.1}$$

Now we may have the following result:

Theorem 3.1. *The sets* $[a_p^r], [a_0^r], [a_c^r]$ *and* $[a_\infty^r]$ *are linear spaces with the coordinate wise addition and scalar multiplication which are* BK*-spaces with the norm* $\|u\|_{[a_p^r]} = \|\mathcal{A}^r u\|_p$ *and* $\|u\|_{[a_0^r]} = \|u\|_{[a_c^r]} = \|u\|_{[a_\infty^r]}$.

Proof. The proof of the first part is trivial. Since a_p^r, a_0^r, a_c^r and a_∞^r are BK-spaces accompanied by natural norm, and $a_{mm}^r(q^k) \neq 0$ and $a_{mm}^r(q^k) = 0, k > m, \forall\, k, m \in \mathbb{N}$. Therefore by Wilansky ([20], Theorem 4.3.2) it gives us to the fact that the spaces $[a_p^r], [a_0^r], [a_c^r]$ and $[a_\infty^r]$ are BK-spaces. □

As a consequence the absolute property does not hold on the spaces $[a_p^r], [a_0^r], [a_c^r]$ *and* $[a_\infty^r]$, *they are the spaces of the non-absolute type.*

Theorem 3.2. *The spaces* $[a_p^r], [a_0^r], [a_c^r]$ *and* $[a_\infty^r]$ *are linearly isomorphic to the spaces* ℓ_p, c_0, c *and* ℓ_∞.

Proof. Since the transformation ψ defined by the condition (3.1) from $[a_p^r]$ to ℓ_p by $u \to v = \psi u = \mathcal{A}^r u$ is a linear bijection and norm preserving. Therefore it shows the spaces $[a_p^r], [a_0^r], [a_c^r]$ and $[a_\infty^r]$ are linearly isomorphic, as desired. □

Now, we may give the following results on the inclusion relations involving the spaces $[a_p^r], [a_0^r], [a_c^r]$ *and* $[a_\infty^r]$.

Theorem 3.3. *The inclusions* $c_0 \subset [a_0^r]$ *and* $c \subset [a_c^r]$ *strictly hold for every* $q \in \mathbb{R}^+$.

Proof. We consider the case $c_0 \subset [a_0^r]$, let us take $v \in c_0$. Since $\mathcal{A}^r(q^k)$ is regular, we immediately see that $\mathcal{A}^r(q^k)v \in c_0$ which shows that $v \in [a_0^r]$. Hence $c_0 \subset [a_0^r]$ hold. Now, consider the sequence $u = \{u_k(q)\}$ defined as

$$u_k(q) = \begin{cases} \frac{\sqrt[3]{k}}{q[1+r^k]_q}, & k = n^3, n \in \mathbb{N} \\ 0, & k \neq n^3 \in \mathbb{N} \end{cases}$$

for each $q \in \mathbb{R}^+$. Then we have

$$\begin{aligned} \{\mathcal{A}^r(q^k)u\}_m &= \frac{1}{[m+1]_q} \sum_{k=0}^{m} [1+r^k]_q q^k u_k(q) \\ &= \frac{1}{[m+1]_q} \sum_{k=0}^{n} q^k (k) \frac{1}{q} = \frac{n(n+1)}{2[m+1]_q} q^{n-1}, (n^3 \leq m < (n+1)^3, n \in \mathbb{N}), \end{aligned}$$

which shows $\mathcal{A}^r(q^k)u \to 0$, as $m \to \infty$ and so $\mathcal{A}^r(q^k)u \in c_0$ which implies that $u \in [a_0^r]$ but not in c_0.
The inclusion $c \subset [a_c^r]$ can be proved in the similar lines. □

We now state the following Lemma which is required in proving the inclusion relation involving the space $[a_p^r]$.

Lemma 3.4. *[10] Let* (a_m) *is a sequence of non-negative term, and* $\mathcal{A}_m = a_0 + a_1 + \cdots + a_m$, $\forall m \in \mathbb{N}$. *Then the following inequality*

$$\sum_m \left(\frac{\mathcal{A}_m}{m+1}\right)^p < \left(\frac{p}{p-1}\right) \sum_m a_m^p$$

holds, for $p > 1$.

Theorem 3.5. *The inclusions* $\ell_p \subset [a_p^r]$ *strictly holds for* $1 \leq p < \infty$.

Proof. In order to prove the validity of the inclusion $\ell_p \subset [a_p^r]$ for $1 \leq p < \infty$, it is sufficient to show the existence of a number $M > 0$ with $\|u\|_{[a_p^r]} \leq M \|u\|_p$ for all $u \in \ell_p$. For this, let $u \in \ell_p$ for $1 < p < \infty$. Then considering the inequality and using Lemma 3.4

$$\left| \frac{1}{[k+1]_q} \sum_{i=0}^{k} [1+r^i]_q q^i u_i \right|^p \leq \left(2 \sum_{i=0}^{k} \frac{q^i |u_i|}{[k+1]_q} \right)^p$$

$$\sum_k \left| \frac{1}{[k+1]_q} \sum_{i=0}^{k} [1+r^i]_q q^i u_i \right|^p \leq \left(\frac{2p}{p-1} \right)^p \sum_k q^i |u_i|^p$$

which gives the fact that $\|u\|_{[a_p^r]} \leq 2p \|u\|_p$, and so $\|u\|_{[a_p^r]} \leq M \|u\|_p$ for $M = 2p$ for $1 < p < \infty$. Therefore, the inclusion $\ell_p \subset [a_p^r]$ holds.

Now, let us define the sequence $u_k(q) = \frac{(-1)^k}{[1+r^k]_q q}$ for every $k \in \mathbb{N}$. Then

$$\sum_m \left| \sum_{k=0}^{m} \frac{[1+r^k]_q}{[m+1]_q} q^k u_k \right|^p = \sum_m \left| \sum_{k=0}^{m} \frac{q^k (-1)^k}{q[m+1]_q} \right|^p$$
$$= \sum_m \left| \frac{1-(-1)^{m+1}}{2[m+1]_q} \right|^p < \infty$$

which yields us that $u \in [a_p^r]$ but not in ℓ_p. That is to say u is in $[a_p^r] \setminus \ell_p$, thus the inclusion $\ell_p \subset [a_p^r]$ is strict.

Similarly, it is easy to prove the inclusion relation $\ell_1 \subset [a_1^r]$ is also strict. □

Theorem 3.6. *If* $1 \leq p < t$, *then* $[a_p^r] \subset [a_t^r]$.

Proof. Suppose $u \in [a_p^r]$ and $1 \leq p < t$. Using the relation (3.1) and since $[a_p^r] \cong \ell_p$ yields the fact that $v \in \ell_p$. Also $\ell_p \subset \ell_t$ implies that $v \in \ell_t$. This gives that $u \in [a_t^r]$ and thus $[a_p^r] \subset [a_t^r]$ holds. □

Theorem 3.7. *The inclusions* $\ell_\infty \subset [a_\infty^r]$ *and* $[a_p^r] \subset [a_\infty^r]$ *are strictly hold.*

Proof. Let $u \in \ell_p$. Then

$$\begin{aligned}
\|u\|_{[a_\infty^r]} &= \sup_{m \in \mathbb{N}} \left| \sum_{k=0}^{m} \frac{[1+r^k]_q}{[m+1]_q} q^k u_k \right| \\
&\leq \|qu\|_\infty \left(\sup_{m \in \mathbb{N}} \sum_k \frac{[1+r^k]_q}{[m+1]_q} \right) \\
&= \|qu\|_\infty \left\{ \sup_{m \in \mathbb{N}} \left[1 + \frac{[1-r^{m+1}]_q}{[m+1]_q [1-r]_q} \right] \right\} \\
&\leq 2 \|qu\|_\infty
\end{aligned}$$

which gives us that $u \in [a_\infty^r]$ and thus $\ell_\infty \subset [a_\infty^r]$.

Further, let us define the sequence $u = u_k(q)$

$$u_k(q) = (-1)^k \frac{2k+1}{[1+r^k]_q q}, \ (k \in \mathbb{N})$$

is in $[a_\infty^r]$, but not in ℓ_∞. Therefore, the space $[a_\infty^r]$ strictly includes the space ℓ_∞.

Next, let u is in $[a_p^r]$ for $1 \leq p < \infty$. Then clearly $v = [A^r u] \in \ell_p \subset \ell_\infty$ which gives us the fact that u is in $[a_\infty^r]$. Hence $[a_p^r] \subset [a_\infty^r]$ holds. Also, take the sequence $u = e = (1, 1, 1, ...)$ belongs to the set $[a_\infty^r] \setminus [a_p^r]$, then $[a_p^r] \subset [a_\infty^r]$ also strictly holds. □

As a consequence of isomorphism ψ *in Theorem 3.2 is onto and inverse image of the basis of those spaces,* ℓ_p, c_0 *and* c, *are the basis of new spaces* $[a_p^r], [a_0^r], [a_c^r]$, *respectively.*

Theorem 3.8. *Define the sequence $b^{(k)}(q) = \{b_m^{(k)}(q)\}_{m\in\mathbb{N}}$ of the space $[a_0^r]$ given as*

$$b_m^{(k)}(q) = \begin{cases} (-1)^{m-k}\frac{[m+1]_q}{q^k[1+r^k]_q}, & k \leq m \leq k+1 \\ 0, & k > m \text{ or } m > k+1 \end{cases} \tag{3.2}$$

for every fixed $k \in \mathbb{N}$.
(i) The sequence $\{b^{(k)}(q)\}_{k\in\mathbb{N}}$ be a basis for $[a_p^r]$ and $[a_0^r]$ and any $u \in [a_p^r]$ or $[a_0^r]$ is uniquely expressed as $u = \sum_k \gamma_k(q)b^{(k)}(q)$, where $\gamma_k(q) = \{\mathcal{A}^r(q^k)u\}_k$, $\forall k \in \mathbb{N}$.
(ii) The set $\{t, b^{(k)}(q)\}_{k\in\mathbb{N}}$ be a basis for the space $[a_c^r]$ and any $u \in [a_c^r]$ is uniquely expressed as $u = \ell t + \sum_k [\gamma_k(q) - \ell]\, b^{(k)}(q)$, where $t = \frac{1}{q^k[1+r^k]_q}$ and $\ell = \lim_{k\to\infty}\{\mathcal{A}^r(q^k)u\}_k$.

4 Duals of the q-spaces $[a_p^r]$, $[a_0^r]$, $[a_c^r]$ and $[a_\infty^r]$

In the current segment, we deal with portrayal of the α-dual, the β-dual and the γ-dual of the spaces $[a_p^r], [a_0^r], [a_c^r]$ and $[a_\infty^r]$ of non-absolute type. Firstly, let us state the following Lemmas.

Lemma 4.1. *[19] $\mathcal{A} = (a_{mk}) \in (c_0 : \ell_1) = (c : \ell_1)$ if and only if*

$$\sup_{K,N\in\mathcal{G}} \left| \sum_{n\in N}\sum_{k\in K} a_{mk} \right| < \infty,$$

where $\mathcal{G}$ indicates the family of all finite subsets of $\mathbb{N}$.

Lemma 4.2. *[19] $\mathcal{A} = (a_{mk}) \in (c : c)$ if and only if $\exists\ \alpha_k,\ \alpha \in \mathbb{C}$ such that*

$$\lim_{m\to\infty} a_{mk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{4.1}$$

$$\sup_{m\in\mathbb{N}} \sum_k |a_{mk}| < \infty, \tag{4.2}$$

$$\lim_{m\to\infty} \sum_k a_{mk} = \alpha. \tag{4.3}$$

Lemma 4.3. *[13] Let $\mathcal{X}$ be a sequence space . Then*
(i) $\mathcal{X}$ is monotone if and only, if $t_0\mathcal{X} \subset \mathcal{X}$, where t_0 is the span of the set of all sequences of zeroes and ones, and $t_0\mathcal{X} = \{au = (a_k u_k) : a \in t_0, u \in \mathcal{X}\}$.
(ii) $\mathcal{X}$ is perfect $\Rightarrow$ $\mathcal{X}$ is normal $\Rightarrow$ $\mathcal{X}$ is monotone.
(iii) $\mathcal{X}$ is normal $\Rightarrow \mathcal{X}^\alpha = x^\gamma$.
(iv) $\mathcal{X}$ is monotone $\Rightarrow \mathcal{X}^\alpha = x^\beta$.

Theorem 4.4. *Define the following sets*

$$\phi_1^r(q) = \left\{ a = a_k \in s : \sup_{k\in\mathbb{N}} \left| \frac{[k+1]_q}{q^k[1+r^k]_q} a_k \right| < \infty \right\},$$

$$\phi_2^r(q) = \left\{ a = a_k \in s : \sum_k \left| \frac{[k+1]_q}{q^k[1+r^k]_q} a_k \right| < \infty \right\},$$

$$\phi_3^r(q) = \left\{ a = a_k \in s : \sup_{K\in\mathcal{G}} \sum_m \left| \sum_{k\in K} (-1)^{m-k} \frac{[k+1]_q}{q^m[1+r^m]_q} a_m \right| < \infty \right\}.$$

Then
(i) $\{[a_0^r]\}^\alpha = \{[a_c^r]\}^\alpha = \phi_3^r(q)$.
(ii) $\{[a_p^r]\}^\alpha = \phi_1^r(q)$ for $0 < p \leq 1$.
(iii) $\{[a_p^r]\}^\alpha = \phi_2^r(q)$ for $1 < p < \infty$.

Proof. Let $a = (a_m) \in s$ and define the matrix $C = (c^r_{mk})$ via the sequence $a = (a_m)$ as

$$c^r_{mk} = \begin{cases} (-1)^{m-k} \frac{[k+1]_q}{q^m[1+r^m]_q} a_m, & m-k \leq k \leq m \\ 0, & 0 \leq k < m-1 \text{ or } k > m \end{cases}$$

for every $m, k \in \mathbb{N}$. Thus the relation (3.1) bearing in mind, let us immediately see that

$$a_m u_m = \sum_{k=m-1}^{m} (-1)^{m-k} \frac{[k+1]_q}{q^m[1+r^m]_q} a_m v_k = (Cv)_m \tag{4.4}$$

for every $m \in \mathbb{N}$. Therefore, we see by (4.4) that $au = (a_m u_m) \in \ell_1$ whenever $u \in [a^r_0]$ or $[a^r_c]$ if and only if $Cv \in \ell_1$ whenever $v \in c_0$ *or* c. Then using Lemma 4.1, we observe that

$$\sup_{K \in \mathcal{G}} \sum_m \left| \sum_{k \in K} (-1)^{m-k} \frac{[k+1]_q}{q^m[1+r^m]_q} a_m \right| < \infty$$

which implies the fact that $\{[a^r_0]\}^\alpha = \{[a^r_c]\}^\alpha = \phi^r_3(q)$.
By similar fashion, Part (ii) and Part (iii) can be proved. □

Theorem 4.5. *Define the sets $\phi^r_4(q), \phi^r_5(q), \phi^r_6(q)$ and $\phi^r_7(q)$ as follows*

$$\phi^r_4(q) = \left\{ a = a_k \in s : \sum_k \left| \Delta\left(\frac{a_k}{q^k[1+r^k]_q} \right) [k+1]_q \right|^{p'} < \infty \right\},$$

$$\phi^r_5(q) = \left\{ a = a_k \in s : \sum_k \left| \Delta\left(\frac{a_k}{q^k[1+r^k]_q} \right) [k+1]_q \right| < \infty \right\},$$

$$\phi^r_6(q) = \left\{ a = a_k \in s : \left(\frac{a_k}{[1+r^k]_q} \right) \in cs \right\},$$

$$\phi^r_7(q) = \left\{ a = a_k \in s : \left(\frac{[k+1]_q}{[1+r^k]_q} a_k \right) \in \ell_\infty \right\},$$

where

$$\Delta\left(\frac{a_k}{q^k[1+r^k]} \right) = \left(\frac{a_k}{q^k[1+r^k]} - \frac{a_{k+1}}{q^{k+1}[1+r^{k+1}]} \right) \text{ for all } k \in \mathbb{N}.$$

Then
(i) $\{[a^r_0]\}^\beta = \phi^r_5(q) \cap \phi^r_7(q)$.
(ii) $\{[a^r_c]\}^\beta = \phi^r_5(q) \cap \phi^r_6(q)$.
(iii) $\{[a^r_p]\}^\beta = \phi^r_1(q)$ *for* $0 < p \leq 1$.
(iv) $\{[a^r_p]\}^\beta = \phi^r_1(q) \cap \phi^r_4(q)$ *for* $1 < p < \infty$.

Proof. Let us consider the case for the space $[a^r_c]$, the other parts can be obtained in the similar way.
Let $u \in [a^r_c]$ and $a \in s$. Then, consider the equality

$$\sum_{k=0}^{m} a_k u_k = \sum_{k=0}^{m} \left[\frac{[k+1]_q}{[1+r^k]_q} v_k - \frac{[k]_q}{[1+r^k]_q} v_{k-1} \right] \frac{a_k}{q_k} \tag{4.5}$$

$$\sum_{k=0}^{m} a_k u_k = \sum_{k=0}^{m-1} \Delta\left(\frac{a_k}{q^k[1+r^k]_q} \right) [k+1]_q v_k + \frac{[m+1]_q}{[1+r^m]_q} a_m v_m$$
$$= (Dv)_m$$

for all $m \in \mathbb{N}$, where $D = (d^r_{mk})$ is defined as

$$d^r_{mk} = \begin{cases} \Delta\left(\frac{a_k}{q^k[1+r^k]_q} \right) [k+1]_q, & 0 \leq k \leq m-1 \\ \frac{[m+1]_q}{[1+r^m]_q} a_m, & k = m \\ 0, & k > m \end{cases}$$

for every $m, k \in \mathbb{N}$. Thus by Lemma 4.2 and from (4.5), we deduce that $au = (a_k u_k) \in cs$ whenever $u = (u_k) \in [a_c^r]$ if and only if $Dv \in c$ whenever $v = (v_k) \in c$. Evidently, the columns of the matrix D are in the sequence space c. Therefore, as a consequence we derive from (4.2) and (4.3) that

$$\sum_k \left| \Delta \left(\frac{a_k}{q^k [1 + r^k]_q} \right) [k + 1]_q \right| < \infty \tag{4.6}$$

$$\left(\frac{[k + 1]_q}{[1 + r^k]_q} a_k \right) \in \ell_\infty \tag{4.7}$$

$$\left(\frac{a_k}{[1 + r^k]_q} \right) \in cs \tag{4.8}$$

respectively. However, the equation (4.7) is redundant, and thus by combining equations (4.6) and (4.8) we obtained that $\{[a_0^r]\}^\beta = \phi_5^r(q) \cap \phi_7^r(q)$. □

Theorem 4.6. *The γ-dual of the spaces $[a_p^r], [a_0^r], [a_c^r]$ are as follows*
(i) $\{[a_0^r]\}^\gamma = \{[a_c^r]\}^\gamma = \phi_5^r(q) \cap \phi_6^r(q)$.
(ii) $\{[a_p^r]\}^\gamma = \phi_1^r(q)$ *for* $0 < p \leq 1$.
(iii) $\{[a_p^r]\}^\gamma = \phi_1^r(q) \cap \phi_4^r(q)$ *for* $1 < p < \infty$.

Proof. The proof is in similar lines as the proof of the Theorem 4.5, we leave out the details for the reader. □

By combining Theorem 4.4 and Theorem 4.5 with Lemma 4.3, we derive the following Corollary.

Corollary 4.7. *The set $[a_p^r]$ is not monotone, and hence it is neither perfect nor normal.*

5 Matrix mappings on the q-spaces $[a_p^r]$ and $[a_c^r]$

In this segment, we now desire to characterize some matrix mappings related to q-analogue from the spaces $[a_p^r]$ and $[a_c^r]$, into some of the known sequence spaces. Here in what follows, we shall write for brevity that

$$\tilde{a}_{mk} = \Delta \left(\frac{a_{mk}}{q^k [1 + r^k]} \right) [k + 1]_q = \left(\frac{a_{mk}}{q^k [1 + r^k]} - \frac{a_{m,k+1}}{q^{k+1} [1 + r^{k+1}]} \right) [k + 1]_q$$

and

$$a(m, k) = \sum_{i=0}^{m} a_{ik} \ \ \forall \ \ m, k \in \mathbb{N}.$$

Let us now state the following Lemmas due to Wilansky [20].

Lemma 5.1. *The matrix mappings between the BK-spaces are continuous.*

Lemma 5.2. $\mathcal{A} = (a_{mk}) \in (c : \ell_p)$ *if and only if*

$$\sup_{F \in \mathcal{G}} \sum_m \left| \sum_{k \in F} a_{mk} \right|^p < \infty, \text{ for } 1 \leq p < \infty. \tag{5.1}$$

Theorem 5.3. *Let $1 < p < \infty$. Then, $\mathcal{A} \in ([a_p^r] : \ell_\infty)$ if and only if*

$$\sup_{k \in \mathbb{N}} \left| \frac{[k + 1]_q}{[1 + r^k]_q} a_{mk} \right| < \infty \text{ for each } m \in \mathbb{N}, \tag{5.2}$$

$$\sup_{m \in \mathbb{N}} |\tilde{a}_{mk}|^{p'} < \infty. \tag{5.3}$$

Proof. Suppose the conditions (5.2) and (5.3) hold and let us take any $u \in [a_p^r]$. Then $\{a_{mk}\}_{k\in\mathbb{N}} \in \{[a_p^r]\}^\beta$ for every fixed $m \in \mathbb{N}$, and this shows the existence of $\mathcal{A}u$.
Now, consider the following equality obtained from the n^{th} partial sum of the series $\sum_k a_{mk}u_k$.

$$\sum_{k=0}^{n} a_{mk}u_k = \sum_{k=0}^{n-1} \tilde{a}_{mk}v_k + \frac{[n+1]_q}{[1+r^n]_q} a_{mn}v_n, \ \forall\, m, n \in \mathbb{N}. \tag{5.4}$$

Therefore as $n \to \infty$, we derive from (5.4) with (5.2) that

$$\sum_k a_{mk}u_k = \sum_k \tilde{a}_{mk}v_k, \ \forall\, m \in \mathbb{N}. \tag{5.5}$$

Thus by taking supremum over $m \in \mathbb{N}$ in (5.5), we get by using the Hölder's inequality with (5.3) that

$$\sup_{m\in\mathbb{N}} \|\mathcal{A}u\| \le \|v\|_p \left\{ \sup_{m\in\mathbb{N}} \left(\sum_k |\tilde{a}_{mk}|^{p'} \right)^{1/p'} \right\} < \infty,$$

which provides the sufficiency of (5.2) and (5.3).
Conversely, suppose that $\mathcal{A} \in ([a_p^r] : \ell_\infty)$. Then by hypothesis $(a_{mk}) \in \{[a_p^r]\}^\beta$ for all $m \in \mathbb{N}$ and necessity of (5.2) is trivial and (5.5) holds. Let us consider the continuous function g_m defined on $[a_p^r]$ by the sequence (a_m) as $g_m(u) = \sum_k a_{mk}u_k \ \forall\, m \in \mathbb{N}$. Since $[a_p^r] \cong \ell_p$, it follows that $\|g_n\| = \|\tilde{a}_m\|_{p'}$, with (5.5). This implies that the functional defined by rows of $\mathcal{A}$ in $[a_p^r]$ are pointwise bounded. Hence by uniform boundedness principle, they are uniformly bounded which implies that there exists a constant K such that $\|g_m\| \le K$ for every $m \in \mathbb{N}$. Then, it follows that $\left(\sum_k |\tilde{a}_{mk}|^{p'}\right)^{1/p'} = \|g_m\| \le K$ hold for all $m \in \mathbb{N}$, which gives the necessity of (5.3). □

Theorem 5.4. *Let $1 < p < \infty$. Then $\mathcal{A} \in ([a_p^r] : c)$ if and only if (5.2) and (5.3) hold and*

$$\lim_{m\to\infty} \tilde{a}_{mk} = \beta_k \text{ for each } k \in \mathbb{N}. \tag{5.6}$$

Proof. Suppose the condition (5.2), (5.3) and (5.6) hold, and let us take any $u \in [a_p^r]$. Then $\mathcal{A}u$ exists and by condition (5.6), we have that $\lim_{m\to\infty} |\tilde{a}_{mk}|^{p'} = |\beta_k|^{p'}$ for each $k \in \mathbb{N}$, which gives us the fact that with (5.3)

$$\sum_{i=0}^{k} |\beta_k|^{p'} \le \sup_{m\in\mathbb{N}} \sum_i |\tilde{a}_{mi}|^{p'} = K < \infty$$

holds for every $k \in \mathbb{N}$. This implies that $(\beta_k) \in \ell_p$. Since by hypothesis $(u_k) \in [a_p^r]$ and $[a_p^r] \cong \ell_p$ leads us the fact that $(v_k) \in \ell_p$. Thus we obtain by using Hölder inequality that $(\beta_k v_k) \in \ell_1$ for each $(v_k) \in \ell_p$. Given any $\epsilon > 0$, choose a fixed $k_0 \in \mathbb{N}$ with

$$\left(\sum_{k=k_0+1}^{\infty} |v_k|^p \right)^{1/p} < \frac{\epsilon}{4K^{1/p'}}$$

Then there is some m_0 with (5.6) such that

$$\left| \sum_{k=0}^{m_0} (\tilde{a}_{mk} - \beta_k)v_k \right| < \epsilon/2$$

for every $m \geq m_0$. Thus

$$\begin{aligned}\left|\sum_k \tilde{a}_{mk}u_k - \sum_k \beta_k v_k\right| &= \left|\sum_k (\tilde{a}_{mk} - \beta_k)v_k\right| \\ &\leq \left|\sum_{k=0}^{k_0} (\tilde{a}_{mk} - \beta_k)v_k\right| + \left|\sum_{k=k_0+1}^{\infty} (\tilde{a}_{mk} - \beta_k)v_k\right| \\ &< \epsilon/2 + \left[\sum_{k=k_0+1}^{\infty} (|\tilde{a}_{mk}| + |\beta_k|)^{p'}\right]^{1/p'} \left(\sum_{k=k_0+1}^{\infty} |v_k|^p\right)^{1/p} \\ &< \epsilon/2 + 2K^{1/p'} \frac{\epsilon}{4K^{1/p'}} = \epsilon\end{aligned}$$

for all sufficiently large m. Therefore, $Au \in c$.
Conversely, suppose that $A \in ([a_p^r] : c)$. Since $c \subset \ell_\infty$, the necessity of (5.2) and (5.3) are immediately yield from Theorem 5.3. To prove the necessity of (5.6), take the sequence $b^{(k)}(q)$ given in (3.2) which is in $[a_p^r]$, for every $k \in \mathbb{N}$. Since $\mathcal{A}u$ exists and is in c, for every $u \in [a_p^r]$, it is evident that $\mathcal{A}b^{(k)}(q) = (\tilde{a}_{mk}) \in c$, for all $k \in \mathbb{N}$ which shows the necessity of (5.6). □

If the space c is replaced by the sequence space c_0 in Theorem 5.4, then we have the following Corollary.

Corollary 5.5. *Let $1 < p < \infty$. Then $\mathcal{A} \in ([a_p^r] : c_0)$ if and only if (5.2) and (5.3) hold, and (5.6) also holds with $\beta_k = 0$, $\forall\, k \in \mathbb{N}$.*

Theorem 5.6. *$\mathcal{A} \in ([a_c^r] : \ell_p)$ if and only if*
(i) For $1 \leq p < \infty$,

$$\sup_{F \in \mathcal{G}} \sum_m \left|\sum_{k \in F} \tilde{a}_{mk}\right|^p < \infty \tag{5.7}$$

$$\sum_k |\tilde{a}_{mk}| < \infty \text{ for all } m \in \mathbb{N} \tag{5.8}$$

$$\left(\frac{a_{mk}}{[1 + r^k]_q}\right)_{k \in \mathbb{N}} \in cs \text{ for all } m \in \mathbb{N}. \tag{5.9}$$

(ii) For $p = \infty$, (5.9) holds and

$$\sup_{m \in \mathbb{N}} \sum_k |\tilde{a}_{mk}| < \infty. \tag{5.10}$$

Proof. First, suppose that (5.7), (5.8) and (5.9) hold, and let us take any sequence $(u_k) \in [a_c^r]$. Then $(a_{mk}) \in \{[a_c^r]\}^\beta$ for every $m \in \mathbb{N}$ and this implies the existence of Au. Let us now define the matrix $T = (t_{mk})$ with $t_{mk} = \tilde{a}_{mk}$, for every $m, k \in \mathbb{N}$. Then since (5.1) is satisfied for the matrix $T \in (c : \ell_p)$. Reconsider the equality (5.4), obtained from the n^{th}-partial sum of series $\sum_k a_{mk}u_k$. Following the same procedure as in the proof of Theorem 4.5, one can easily obtain by combining the (5.8) and (5.9) such that

$$\left\{\frac{[n+1]_q}{[1 + r^n]_q} a_{mn}\right\}_{n \in \mathbb{N}} \in c_0$$

for every $m \in \mathbb{N}$. Therefore, keeping in mind the fact that as $n \to \infty$ in (5.4), the second term on right side approaches to zero and we again get the condition (5.5), which obtain by taking p-norm such that $\|\mathcal{A}u\|_p = \|Tv\|_p < \infty$. This shows $\mathcal{A} \in ([a_c^r] : \ell_p)$.
Conversely, suppose that $\mathcal{A} \in ([a_c^r] : \ell_p)$. Then since $[a_c^r]$ and ℓ_p are BK-spaces, so by Lemma 5.2, $\exists$ some real constant $M > 0$ that

$$\|\mathcal{A}u\|_{\ell_p} \leq M \|u\|_{[a_c^r]} \text{ for all } u \in [a_c^r]. \tag{5.11}$$

In addition, condition (5.11) is satisfied for $u = (u_k) = \sum_{k\in F} b^{(k)}(q)$ belonging to $[a_c^r]$, where $b^{(k)}(q) = \{b_m^{(k)}(q)$ is given in (3.2). Thus we have for any $F \in \mathcal{G}$

$$\|\mathcal{A}u\|_p = \left(\sum_m \left|\sum_{k\in F} \tilde{a}_{mk}\right|^p\right)^{1/p} \leq M\,\|u\|_{[a_c^r]}$$

which gives the necessity of (5.7). Furthermore, $\mathcal{A}$ is applicable to space $[a_c^r]$ by hypothesis, so the necessity of (5.8) and (5.9) is obvious. This proves Part (i).
Part (ii) can be obtained along the lines similar to that of part (i). □

Theorem 5.7. $\mathcal{A} = (a_{mk}) \in ([a_c^r] : c)$ *if and only if (5.6), (5.9) and (5.10) hold and*

$$\lim_{m\to\infty} \sum_k \tilde{a}_{mk} = \beta. \tag{5.12}$$

Proof. Suppose that the matrix $\mathcal{A}$ satisfies the conditions (5.6), (5.9), (5.10) and (5.12). Let us take any $u \in [a_c^r]$. Then $\mathcal{A}u$ exists and, it is clear that $v = (v_k)$ is connected with $u = (u_k)$ by the condition (3.1) is in c with $v_k \to t$ as $k \to \infty$. We immediately see from (5.6) and (5.10) that

$$\sum_{i=0}^{k} |\beta_i| \leq \sup_{m\in\mathbb{N}} \sum_i |\tilde{a}_{mi}| < \infty$$

holds, for all $k \in \mathbb{N}$. This implies to the fact that $(\beta_k) \in \ell_1$. Now considering (5.5), we write

$$\sum_k a_{mk}u_k = \sum_k \tilde{a}_{mk}(v_k - t) + t\sum_k \tilde{a}_{mk}. \tag{5.13}$$

Thus, as $m \to \infty$ in (5.13) we observe the first term on right side approaches to $\sum_k \beta_k(v_k - t)$ by (5.10) and (5.4) and second term approaches to $t\beta$ by (5.12). Therefore, we obtain $\lim_{m\to\infty}(\mathcal{A}u)_m = \sum_k \beta_k(v_k - t) + t\beta$ which gives that $\mathcal{A} \in ([a_c^r] : c)$.
Conversely, suppose that $A \in ([a_c^r] : c)$. Also $c \subset \ell_\infty$ and so the necessity of (5.9) and (5.10) are immediately yield from the Theorem 5.6. To show the necessity of (5.5), let us take the sequence $u = u^{(k)} = \{u_m^{(k)}(q)\}_{m\in\mathbb{N}} \in [a_c^r]$ given as

$$u_m^{(k)}(q) = \begin{cases} (-1)^{m-k}\frac{[k+1]_q}{[1+r^k]_q}, & k \leq m \leq k+1 \\ 0, & 0 \leq m < k-1 \text{ or } m > k+1 \end{cases}$$

for every $k \in \mathbb{N}$. As $\mathcal{A}u$ exists and is in c for all $u \in [a_c^r]$, we can observe that $\mathcal{A}u^{(k)} = (\tilde{a}_{mk})_{m\in\mathbb{N}} \in c$, for all $k \in \mathbb{N}$ gives the necessity of (5.6). Similarly, putting $u = e$ in (5.5), we have $\mathcal{A}u = (\sum_k \tilde{a}_{mk})_{m\in\mathbb{N}}$ which belongs to c, and this gives the necessity of (5.12). □

6 Conclusion

This paper aims to study new transformations of the q-analogue of A^r-matrix of order r. In addition, some topological properties and inclusion relations have been discussed. In addition to this, the Toeplitz duals and matrix transformations have been examined. Therefore, the results of this work are variant, significant, and so future research may focus on generalizing them with other special number sequences and thus studying their topological properties and examining their duals, matrix transformation and compact operators.

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