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ON THE MONOTONICITY OF THE OPERATOR NORM AND THE SPECTRAL RADIUS

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Abstract In this paper, we establish the monotonicity of the operator norm and the spectral radius in some ordered Banach spaces equipped with an absolute value function.

1 Main result

In this paper, we let $(E, \|\cdot\|)$ be a real Banach space. As usual, $\mathcal{L}(E)$ and $r(L)$ refer respectively to the set of linear bounded self-mappings defined on E and to the spectral radius of a mapping $L \in \mathcal{L}(E)$. The Gelfand formula states that

$$r(L) = \lim_{n\to\infty} \|L^n\|^{1/n}.$$

Since it has been used to prove several powerful results, such as the inverse map theorem and Cauchy-Picard theorem, the Banach contraction principle is considered as one of the most important results in modern Analysis. This theorem states that if a mapping $T : \Omega \to \Omega$ satisfies the following contraction condition

$$\|Tx - Ty\| \le k \|x - y\| \text{ for all } x, y \in \Omega, \tag{1.1}$$

holds for some $k \in [0, 1)$, where Ω is a closed subset of E, then T has a unique fixed point $\overline{x} \in \Omega$. The power of the Banach contraction principle consists in the fact that under Hypothesis (1.1), for any $x_0 \in \Omega$, the corresponding iterative sequence, $x_{n+1} = Tx_n$, converges to $\overline{x}$ with the estimate $\|x_n - \overline{x}\| \le \frac{k^n}{1-k} \|x_1 - x_0\|$. This estimate provides the ability to compute the number of iterations needed to produce approximations of any required accuracy. This is why this theorem has been extensively used to solve differential equations, PDE's, integral and integrodifferential equations, see for instance [8], [11], [12] and [13].

Naturally, many generalizations in various directions of this principle can be found in the literature, see for example, [2], [3], [4], [9] and [10]. In particular authors of [2], [3] and [4] introduced a new version of a contraction mapping adapted to the case of ordered Banach spaces equipped with an absolute value function. Roughly speaking, they proved that if a mapping $T : E \to E$ satisfies $|Tx - Ty| \preceq cL |x - y|$, where L is a positive operator with $cr(L) < 1$, then T has a unique fixed point in E.

The above condition $cr(L) < 1$ makes feel the need to have calculation formulas for the spectral radius of a linear operator. Several authors have already been interested in this question and estimates of the spectral radius of the sum of two operators have been established, see for instance, [1], [7], [14] and references therein. The main goal of this paper is to establish the monotonicity of the spectral radius as well as an inequality for the spectral radius of the sum of two operators.

Let us first introduce some concepts related to cones and positivity needed for the statement of the main result of this work. For detailed presentation, we refer the reader to [5] and [6].

Definition 1.1. A nonempty closed convex subset K of E is said to be a cone in E if $(tK) \subset K$ for all $t \geq 0$ and $K \cap (-K) = \{0_E\}$.

It is well known that if K is a cone in E, then K induces a partial order in the Banach space E. We write for all $x, y \in E$: $x \preceq_K y$ (or $y \succeq_K x$) if $y - x \in K$ and $x \prec_K y$ (or $y \succ_K x$) if $y - x \in K \setminus \{0_E\}$. Consequently, the vectors lying in $K \setminus \{0_E\}$ are said to be positive.

Definition 1.2. Let K be a cone in E. The norm $\|\cdot\|$ is said to be monotonic if for all $x, y \in E$, $0_E \preceq_K x \preceq_K y \Rightarrow \|x\| \leq \|y\|$.

Definition 1.3. Let K be a cone in E. A mapping $|\cdot| : E \to K$ is said to be an absolute value function on E if

i) $|x| = 0_E \Rightarrow x = 0_E$,

ii) $|tx| = |t|\,|x|$ for all $x \in E$ and $t \in \mathbb{R}$,

iii) $|x + y| \preceq_K |x| + |y|$ for all $x, y \in E$.

Definition 1.4. Let K be a cone in E and let $|\cdot| : E \to K$ be an absolute value function on E. The supremum of $x, y \in E$ is the vector denoted by $x \curlyvee y$ and defined by

$$x \curlyvee y = \frac{1}{2}\left(x + y + |x - y|\right).$$

In particular for any $x \in E$, the positive part and the negative part of x are respectively the vectors denoted by x^+ and x^- and defined by

$$x^+ = x \curlyvee 0_E = \frac{1}{2}\left(x + |x|\right) \text{ and } x^- = (-x) \curlyvee 0_E = \frac{1}{2}\left(-x + |x|\right).$$

Lemma 1.5. *Let K be a cone in E and let $|\cdot| : E \to K$ be an absolute value function on E. The following properties hold true.*

a) *The mapping $\|\cdot\|_* : x \to \|x\|_* = \||x|\|$ define a norm on E.*

b) *For all $x \in E$,*

$$x = x \curlyvee x = x^+ - x^-,$$
$$|x| = x \curlyvee (-x) = x^+ + x^-.$$

c) *For all $x, y, z \in E$, $(x \preceq_K z$ and $y \preceq_K z) \Rightarrow x \curlyvee y \preceq_K z$.*

Proof. Properties a) and b) are obvious. In view of c), let $x, y, z \in E$ be such that $x \preceq_K z$ and $y \preceq_K z$. We have

$$\begin{aligned} z - x \curlyvee y &= z - \frac{1}{2}\left(x + y + |x - y|\right) \\ &= \frac{1}{2}\left((z - x) + (z - y) + |(z - y) - (z - x)|\right) \succeq_K 0_E. \end{aligned}$$

□

Lemma 1.6. *Let K be a cone in E and let $|\cdot| : E \to K$ be an absolute value function on E. The following assertions are equivalent.*

d) *For all $x \in E$, we have $-x, x \preceq_K |x|$.*

e) *For all $x \in E$, we have $x^+, x^- \succeq_K 0_E$.*

f) *For all $x, y \in E$, we have $x \preceq_K x \curlyvee y$.*

Proof. At first, notice that the equivalence between Assertions d) and e) is due to the fact that

$$x^{+}=x\curlyvee 0_E=\frac{1}{2}\left(|x|+x\right) \text{ and } x^{-}=(-x)\curlyvee 0_E=\frac{1}{2}\left(|x|-x\right)\succeq_K 0_E.$$

e) ⇒ f). Indeed, for all $x,y\in E$ we have

$$\begin{aligned} x\curlyvee y-x &= \frac{1}{2}\left(x+y+|x-y|\right)-x \\ &= \frac{1}{2}\left((y-x)+|y-x|\right) \\ &= (y-x)^{+}\succeq_K 0_E. \end{aligned}$$

f) ⇒ e). Applying f) with $x=0_E$, we obtain for any $y\in E$

$$0_E\preceq_K 0_E\curlyvee y=y^{+} \text{ and } 0_E\preceq_K 0_K\curlyvee(-y)=y^{-}.$$

□

Remark 1.7. Notice that Assertion d) implies that for all $x\in K$, we have $x^{-}=0$ and $x=|x|=x^{+}$. Indeed, for any $x\in K$ we have $-x\preceq_K 0_E$ and $0_E\preceq_K 0_E$. Hence, we obtain by Assertion c), $x^{-}=0_E\curlyvee(-x)\preceq_K 0_E$. This together with Assertions e) and b) leads to $x^{-}=0_E$ and $x=|x|=x^{+}\succeq_K 0_E$.

Definition 1.8. Let K be a cone in E. A mapping $L\in\mathcal{L}(E)$ is said to be positive, if $L(K)\subset K$.

Hereafter for any cone K in E, we denote below by $\mathcal{L}_K(E)$ the set of all positive operators in $\mathcal{L}(E)$. Clearly, $\mathcal{L}_K(E)$ is a closed convex set of $\mathcal{L}(E)$ satisfying $t\mathcal{L}_K(E)\subset\mathcal{L}_K(E)$ for all $t\geq 0$. However, in general $\mathcal{L}_K(E)\cap(-\mathcal{L}_K(E))\neq\left\{0_{\mathcal{L}(E)}\right\}$ and $\mathcal{L}_K(E)$ becomes a cone in $\mathcal{L}(E)$ whenever K is generating, i.e. $E=K-K$. In the remainder of this paper even if the cone K is not generator, we still say that $\mathcal{L}_K(E)$ is a cone in $\mathcal{L}(E)$.

Lemma 1.9. *Let K be a cone in E and let $|\cdot|:E\to K$ be an absolute value function on E. Assume that*

$$-x,x\preceq_K |x| \text{ for all } x\in E, \tag{1.2}$$

then for all positive mapping L in $\mathcal{L}(E)$ we have

$$|Lx|\preceq_K L|x| \text{ for all } x\in E.$$

Proof. At first, from Lemma 1.6 we have $x^{+},x^{-}\in K$ for all $x\in E$. Hence, for any $x\in E$, we obtain from Assertion b) of Lemma 1.5 and Property iii) of the absolute value function,

$$|Lx|=\left|Lx^{+}-Lx^{-}\right|\preceq_K\left|Lx^{+}\right|+\left|Lx^{-}\right|\ \preceq_K Lx^{+}+Lx^{-}=L|x|.$$

□

The main result of this paper consists of the following theorem.

Theorem 1.10. *Let K be a cone in E and let $|\cdot|:E\to K$ be an absolute value function on E. If the norm $\|\cdot\|$ is monotonic and the absolute value function $|\cdot|$ satisfies (1.2) and*

$$\|x\|=\|x\|_* \text{ for all } x\in E, \tag{1.3}$$

then for any $A,B\in\mathcal{L}_K(E)$ we have

i) $0_{\mathcal{L}(E)}\preceq_{\mathcal{L}_K(F)} A\preceq_{\mathcal{L}_K(F)} B$ implies $\|A\|\leq\|B\|$ and $r(A)\leq r(B)$,

ii) $BA\preceq_{\mathcal{L}_K(E)} AB$ implies and $r(AB)\leq r(A)r(B)$ and $r(A+B)\leq r(A)+r(B)$.

Proof. At first, for any $L \in \mathcal{L}_K(E)$ we obtain from the monotonicity of the norm $\|\cdot\|$, Hypothesis (1.3) and Lemma 1.9,

$$\|L\| = \sup_{\|x\|=1} \|Lx\| = \sup_{\|x\|_*=1} \|Lx\|_* = \sup_{\||x|\|=1} \||Lx|\| \leq \sup_{\||x|\|=1} \|L|x|\| \leq \sup_{x\in K, \|x\|=1} \|Lx\| \leq \|L\|.$$

This leads to

$$\|L\| = \sup_{x\in K, \|x\|=1} \|Lx\|.$$

Now let A, B be two mappings in $\mathcal{L}_K(E)$ such that $A \preceq_{\mathcal{L}_K(E)} B$. We have then

$$\|A\| = \sup_{x\in K, \|x\|=1} \|Ax\| \leq \sup_{x\in K, \|x\|=1} \|Bx\| = \|B\|.$$

Hence, the monotonicity of the operator norm is proved.

Also, $A \preceq_{\mathcal{L}_K(E)} B$ implies $A^n \preceq_{\mathcal{L}_K(E)} B^n$ for any integer n. Hence the monotonicity of the operator norm, leads to $\|A^n\|^{1/n} \leq \|B^n\|^{1/n}$ for any integer n. Passing to the limits as $n \to \infty$, we obtain $r(A) \leq r(B)$.

At this stage, if $BA \preceq_{\mathcal{L}_K(E)} AB$, then by induction we prove that for all integer n, $(AB)^n \preceq_{\mathcal{L}_K(E)} A^n B^n$. This together with the normality of the operator norm leads to

$$\left\|(AB)^n\right\|^{\frac{1}{n}} \leq \|A^n B^n\|^{\frac{1}{n}} \leq \|A^n\|^{\frac{1}{n}} \|B^n\|^{\frac{1}{n}}. \tag{1.4}$$

Passing to the limits in (1.4) as $n \to +\infty$, we get $r(AB) \leq r(A)r(B)$.

Furthermore, by induction one proves that for any integer n, we have

$$(A+B)^n \preceq_{\mathcal{L}_K(E)} \sum_{i=0}^{i=n} \begin{pmatrix} n \\ i \end{pmatrix} A^i B^{n-i}.$$

This together with the monotonicity of the operator norm leads to

$$\left\|(A+B)^n\right\| \leq \sum_{i=0}^{i=n} \begin{pmatrix} n \\ i \end{pmatrix} \|A^i\| \|B^{n-i}\|.$$

Let $\epsilon > 0$ small enough, there exists n_0 such that

$$\|A^n\| \leq (r(A)+\epsilon)^n \text{ and } \|B^n\| \leq (r(B)+\epsilon)^n \text{ for all } n \geq n_0.$$

Let $\delta > 0$, such that

$$\|A^n\| \leq \delta^n \text{ and } \|B^n\| \leq \delta^n \text{ for all } n \geq 1.$$

Therefore, for $n \geq 2n_0$ we have

$$\begin{aligned}
\left\|(A+B)^n\right\| &\leq \sum_{i=0}^{i=n-n_0}\begin{pmatrix} n \\ i \end{pmatrix}\left\|A^i\right\|\left\|B^{n-i}\right\| + \sum_{i=n-n_0+1}^{i=n-2n_0}\begin{pmatrix} n \\ i \end{pmatrix}\left\|A^i\right\|\left\|B^{n-i}\right\| \\
&\quad + \sum_{i=n-2n_0+1}^{i=n}\begin{pmatrix} n \\ i \end{pmatrix}\left\|A^i\right\|\left\|B^{n-i}\right\| \\
&\leq \sum_{i=0}^{i=n-n_0}\begin{pmatrix} n \\ i \end{pmatrix}\delta^i\left(r(B)+\epsilon\right)^{n-i} + \sum_{i=n-n_0+1}^{i=n-2n_0}\begin{pmatrix} n \\ i \end{pmatrix}\left(r(A)+\epsilon\right)^i\left(r(B)+\epsilon\right)^{n-i} \\
&\quad + \sum_{i=n-2n_0+1}^{i=n}\begin{pmatrix} n \\ i \end{pmatrix}\left(r(A)+\epsilon\right)^i\delta^{n-i} \\
&\leq \sum_{i=0}^{i=n-n_0}\begin{pmatrix} n \\ i \end{pmatrix}\left(r(A)+\epsilon\right)^i\left(r(B)+\epsilon\right)^{n-i}\left(\frac{\delta}{(r(A)+\epsilon)}\right)^i \\
&\quad + \sum_{i=n-n_0+1}^{i=n-2n_0}\begin{pmatrix} n \\ i \end{pmatrix}\left(r(A)+\epsilon\right)^i\left(r(B)+\epsilon\right)^{n-i} \\
&\quad + \sum_{i=n-2n_0+1}^{i=n}\begin{pmatrix} n \\ i \end{pmatrix}\left(r(A)+\epsilon\right)^i\left(r(B)+\epsilon\right)^{n-i}\left(\frac{\delta}{(r(B)+\epsilon)}\right)^{n-i} \\
&\leq \eta\sum_{i=0}^{i=n}\begin{pmatrix} n \\ i \end{pmatrix}\left(r(A)+\epsilon\right)^i\left(r(B)+\epsilon\right)^{n-i} \\
&= \eta\left(r(A)+r(A)+2\epsilon\right)^n,
\end{aligned}$$

where

$$\eta = \max\left(1, \max_{1\leq i\leq n_0}\left(\frac{\delta}{\min\left(r(A), r(B)\right)+\epsilon}\right)^i\right).$$

At the end, passing to the limits in

$$\left\|(A+B)^n\right\|^{\frac{1}{n}} \leq \eta^{\frac{1}{n}}\left(r(A)+r(B)+2\epsilon\right),$$

as $n \to +\infty$, we obtain $r(A+B) \leq r(A)+r(A)+2\epsilon$. Since ϵ is arbitrary, we conclude that $r(A+B) \leq r(A)+r(B)$. □

Remark 1.11. Since for any equivalent norm $\|\cdot\|_2$ we have

$$r(L) = \lim_{n\to+\infty}\left(\sup_{\|x\|=1}\|L^n x\|\right)^{\frac{1}{n}} = \lim_{n\to+\infty}\left(\sup_{\|x\|_2=1}\|L^n x\|_2\right)^{\frac{1}{n}},$$

the monotonicity of the spectral radius as well as inequalities in ii) of Theorem 1.10 hold true if we replace Hypothesis (1.3) by that $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent norm.

2 Examples

(i) For any open bounded subset Ω of $\mathbb{R}^n$, $C\left(\overline{\Omega}, \mathbb{R}\right)$ and $L^p(\Omega)$ $(p \geq 1)$ are typical spaces where Theorem 1.10 is applicable.

(ii) Let $E = C\left([0,1], \mathbb{R}\right)$ equipped with its sup-norm $\|\cdot\|_\infty$, and let K be the cone of nonnegative functions in E. Consider $A, B : E \to E$ the operators defined by

$$\begin{aligned}
Au(t) &= \int_0^t\left(e^{-t}+\frac{1}{\sqrt{s}}\right)u(s)ds, \\
Bu(t) &= \int_0^t e^s u(s)ds.
\end{aligned}$$

Clearly $A, B \in \mathcal{L}_K(E)$.

For all $u \in K$ we have

$$\begin{aligned} ABu(t) &= \int_0^t \left(e^{-t} + \frac{1}{\sqrt{s}}\right) \int_0^s e^{\xi} u(\xi) d\xi ds \\ &\leq \int_0^t \left(e^{-t} + \frac{1}{\sqrt{s}}\right) e^s \int_0^s u(\xi) d\xi ds \\ &\leq \int_0^t e^s \int_0^s \left(e^{-s} + \frac{1}{\sqrt{\xi}}\right) u(\xi) d\xi ds \\ &= BAu(t). \end{aligned}$$

By induction we show that for all $n \in \mathbb{N}$,

$$\|B^n\| = \|B^n(1)\|_\infty = e - \sum_{k=0}^{k=n-1} \frac{1}{k!} \to 0 \text{ as } n \to \infty.$$

Since $r(B) = 0$ and by Theorem 1.10 we have $r(AB) \leq r(A)r(B)$ and $r(B) \leq r(A+B) \leq r(A) + r(B)$, we conclude that

$$r(AB) = 0 \text{ and } r(A+B) = r(A).$$

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