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All Relationship Between Some Regular Submonoid of $Relhyp_G((n),(m))$

P. Kunama and S. Leeratanavalee

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Corresponding Author: P. Kunama

Abstract An algebraic system is a structure which consists of a nonempty set together with a sequence of operations and a sequence of relations on this set. The properties of the structure are expressed by terms and relational terms. The set of all generalized relational hypersubstitutions for algebraic systems of type (τ,τ') together with a binary operation defined on the set and its identity forms a monoid. In this paper, we determine the set of all completely regular elements and intra-regular elements of the monoid of all generallized relational hypersubstitution for algebraic systems of type $((n),(m))$ for arbitrary natural number $n,m \geqslant 2$. Furthermore, the relationship between some regular submonoids of the monoid of all generalized relational hypersubstitutions for algebraic systems of type $((n),(m))$ is presented.

1 Introduction

Varieties are collections of algebras that are classified by identities. Hypervarities are collections of varieties that are classified by hyperidentities. The main tool which is used to study hyperidentities and hypervarieties is the concept of hypersubstitutions introduced by W. Taylor [9]. The notation of hypersubstitutions of a given type (τ) in universal algebras was originated by K. Denecke et al. [2] in 1991. On the other hand, to classify algebraic systems into subclasses by logical sentences we can use the concept of hypersubstitutions for algebraic systems. The concept of hypersubstitutions for algebraic systems of a given type (τ,τ') was first introduced by K. Denecke and D. Phusanga [3] in 2008. In this paper, we use the notation of algebraic systems in the sense of Mal'cev [4]. An algebraic system of type $(\tau,\tau^{'})$ is a triple $(A,(f_i^A)_{i\in I},(\gamma_j^A)_{j\in J})$ consisting of a nonempty set A, a sequence $(f_i^A)_{i\in I}$ of n_i-ary operations defined on A and a sequence $(\gamma_j^A)_{j\in J}$ of m_j-ary relations on A, where $\tau=(n_i)_{i\in I}$ is a sequence of the arities of each operation f_i^A and $\tau^{'}=(m_j)_{j\in J}$ is a sequence of the arities of each relation γ_j^A. The pair $(\tau,\tau^{'})$ is called the *type* of an algebraic system, see more details in [5, 6].

A hypersubstitution for algebraic systems is a mapping that assigns any operation symbol to a term and assigns any relation symbol to a formula which preserves the arity. In 2016 [7] D. Phusanga et al. extended this concept to generalized hypersubstitutions for algebraic systems of type (τ,τ'). Later, D. Phusanga and J. Koppitz introduced the concept of relational hypersubstitutions for algebraic systems of type (τ,τ') and proved that the set of all relational hypersubstitutions for algebraic systems of type (τ,τ') together with an associative binary operation and the identity element forms a monoid [8]. To study algebraic systems, first main approach is to produce new algebraic systems of the same type from given one, the second main approach is to study the semigroup properties. In semigroup theory, the principle special study of a regular element are inverse of an element and a completely regular element with a great diversity of their various generalization. The present paper will determine the set of all completely regular elements and intra-regular elements of generalized relational hypersubstitutions for algebraic systems of type $((n),(m))$. Finally, we show that the set of all completely regular elements and the set of all intra-regular elements of generalized relational hypersubstitutions for algebraic systems of type $((n),(m))$ are the same.

Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of symbols called variables. For each $n \geq 1$, let $X_n := \{x_1, \ldots, x_n\}$ be an n-element set which is called an n-element alphabet. Let $\{f_i : i \in I\}$ be the set of n_i-ary operation symbols indexed by the indexed set I, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the natural number n_i as its arity. The function $\tau = (n_i)_{i \in I}$ is called a *type*. An *n-ary term of type* τ is defined inductively as follows.

(i) Every variable $x_k \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

We denote the set of all n-ary terms of type τ which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii), by $W_\tau(X_n)$ and let $W_\tau(X) := \bigcup_{n \in \mathbb{N}^+} W_\tau(X_n)$ be the set of all terms of type τ.

2 The Monoid of Generalized Relational Hypersubstitutions for Algebraic Systems

Any generalized relational hypersubstitution for algebraic systems is a mapping that assigns any operation symbol to a term and assigns any relation symbol to a relational term which does not necessarily preserve the arity.

Definition 2.1 ([5]). Let I, J be indexed sets. If $i \in I, j \in J$ and $t_1, t_2, ..., t_{m_j}$ are $n-$ary terms of type $\tau = (n_i)_{i \in I}$ and γ_j is an m_j-ary relation symbol, then $\gamma_j(t_1, t_2, ..., t_{m_j})$ is an $n-$ary relational term of type $((n_i)_{i \in I}, (m_j)_{j \in J})$.

We denote the set of all $n-$ary relational terms of type $(\tau, \tau^{'})$ by $F^*_{\tau'}(W_\tau(X_n))$ and let $F^*_{\tau'}(W_\tau(X)) := \cup_{n \in \mathbb{N}} F^*_{\tau'}(W_\tau(X_n))$ be the set of all relational terms of type $(\tau, \tau^{'})$.

A generalized relational hypersubstitution for algebraic systems of type $(\tau, \tau^{'})$ is a mapping

$$\sigma : \{f_i |\ i \in I\} \cup \{\gamma_j |\ j \in J\} \rightarrow W_\tau(X) \cup F^*_{\tau'}(W_\tau(X)),$$

where $\sigma(f_i) \in W_\tau(X)$ and $\sigma(\gamma_j) \in F^*_{\tau'}(W_\tau(X))$.

The set of all generalized relational hypersubstitutions for algebraic systems of type $(\tau, \tau^{'})$ is denoted by $Relhyp_G(\tau, \tau^{'})$. To define a binary operation on this set, we define inductively the concept of a superposition of terms $S^n : W_\tau(X) \times (W_\tau(X))^n \rightarrow W_\tau(X)$ by the following steps. For any $t, t_1, \ldots, t_{n_i}, s_1, \ldots, s_n \in W_\tau(X)$,

(i) if $t = x_j$ for $1 \leq j \leq n$, then $S^n(t, s_1, \ldots, s_n) := s_j$;

(i) if $t = x_j$ for $n < j$, then $S^n(t, s_1, \ldots, s_n) := x_j$;

(ii) if $t = f_i(t_1, \ldots, t_{n_i})$, then
$S^n(t, s_1, \ldots, s_n) := f_i(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_{n_i}, s_1, \ldots, s_n))$.

For any $t \in W_\tau(X)$ and $F = \gamma_j(s_1, \ldots, s_{m_j}) \in F^*_{\tau'}(W_\tau(X))$, we define the superposition of relational terms $R^n : (W_\tau(X) \cup F^*_{\tau'}(W_\tau(X))) \times (W_\tau(X))^n \rightarrow W_\tau(X) \cup F^*_{\tau'}(W_\tau(X))$ by

(i) $R^n(t, t_1, \ldots, t_n) := S^n(t, t_1, \ldots, t_n)$,

(ii) $R^n(F, t_1, \ldots, t_n) := \gamma_j(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{m_j}, t_1, \ldots, t_n))$.

Every generalized relational hypersubstitution for algebraic systems σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \cup F^*_{\tau'}(W_\tau(X)) \rightarrow W_\tau(X) \cup F^*_{\tau'}(W_\tau(X))$ as follows:

(i) $\widehat{\sigma}[x_i] := x_i$ for $i \in \mathbb{N}$;

(ii) $\widehat{\sigma}[f_i(t_1 \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_{n_i}])$,
where $i \in I$ and $t_1, \dots, t_{n_i} \in W_\tau(X)$, i.e., any occurrence of the variable x_k in $\sigma(f_i)$ is replaced by the term $\widehat{\sigma}[t_k], 1 \leq k \leq n_i$;

(iii) $\widehat{\sigma}[\gamma_j(s_1 \dots, s_{m_j})] := R^{m_j}(\sigma(\gamma_j), \widehat{\sigma}[s_1], \dots, \widehat{\sigma}[s_{m_j}])$, where $j \in J$, $s_1, \dots, s_{m_j} \in W_\tau(X)$, i.e., any occurrence of the variable x_k in $\sigma(\gamma_j)$ is replaced by the term $\widehat{\sigma}[s_k], 1 \leq k \leq m_j$.

We define a binary operation $\circ_g$ on $Relhyp_G(\tau, \tau^{'})$ by $\sigma \circ_g \alpha := \hat{\sigma} \circ \alpha$ where $\circ$ is the usual composition of mappings and $\sigma, \alpha \in Relhyp_G(\tau, \tau^{'})$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \dots, x_{n_i})$ and maps each m_j-ary relation symbol γ_j to the relational term $\gamma_j(x_1, \dots, x_{m_j})$. Then the structure $\underline{Relhyp_G(\tau, \tau')} = (Relhyp_G(\tau, \tau'), \circ_g, \sigma_{id})$ forms a monoid.

Throughout this paper, we focus on the algebraic systems of type $((n), (m))$. Let f be an $n-$ary operation symbol and γ be an $m-$ary relation symbol. We denote the generalized relational hypersubstitution for algebraic systems of type $((n), (m))$ which maps f to a term $t \in W_{(n)}(X)$ and maps γ to a relational term $F \in F^*_{(m)}(W_{(n)}(X))$ by $\sigma_{t,F}$.

For $t = f(t_1, \dots, t_n) \in W_{(n)}(X)$ and $F = \gamma(s_1, \dots, s_m) \in F^*_{(m)}(W_{(n)}(X))$, we introduce the following notation:
$var(t)$:= the set of all variables occurring in the term t.
$var(F)$:= the set of all variables occurring in the relational term F.
$I(t)$:= the set of all indices of variables occur in $var(t) \cap X_n$.
$I(F)$:= the set of all indices of variables occur in $var(F) \cap X_m$.
$var(t)^{X_n} := \{t_i \mid t_i \in X \text{ for some } i = 1, \dots, n\}$.
$var(F)^{X_m} := \{s_j \mid s_j \in X \text{ for some } j = 1, \dots, m\}$.

To determine the set of all completely regular elements and intra-regular elements of $Relhyp_G((n), (m))$ in Section 3 and Section 4, we need the concept of the $i - most$ of terms and the concept of subterms which were defined as the following definition.

Definition 2.2 ([12]). Let $\tau = (n)$ be a type with an n-ary operation symbol f, $t \in W_{(n)}(X)$ and $1 \leq i \leq n$. An $i - most(t)$ is defined inductively as follows:
(i) if t is a variable, then $i - most(t) = t$;
(ii) if $t = f(t_1, \dots, t_n)$ where $t_1, \dots, t_n \in W_{(n)}(X)$, then $i - most(t) := i - most(t_i)$.

Example 2.3. Let $\tau = (3)$ with a ternary operation symbol f. Let $t = f(f(x_5, x_1, x_2), x_3, f(x_8, x_1, x_9))$. Then $1 - most(t) = 1 - most(f(x_5, x_1, x_2)) = x_5, 2 - most(t) = 2 - most(x_3) = x_3$ and $3 - most(t) = 3 - most(f(x_8, x_1, x_9)) = x_9$.

Lemma 2.4 ([12]). *Let $s, t \in W_{(n)}(X)$. If $j - most(t) = x_k \in X_n$ and $k - most(s) = x_i$, then $j - most(\hat{\sigma}_t[s]) = x_i$.*

The above lemma can be applied to any generalized relational hypersubstitution for algebraic systems of type $((n), (m))$, such as the following. Let $s, t \in W_{(n)}(X)$ and $F \in F^*_{(m)}(W_{(n)}(X))$. If $i - most(t) = x_j$, then $i - most(\widehat{\sigma}_{t,F}[s]) = j - most(s)$.

Lemma 2.5 ([11]). *Let $t, u \in W_{(n)}(X)$ and $F, H \in F^*_{(m)}(W_{(n)}(X))$ such that $t = \widehat{\sigma}_{t,F}[u]$ and $F = \widehat{\sigma}_{t,F}[H]$ with $x_i \in var(t)$ and $x_j \in var(F)$. Then we have*
(i) if $t = x_k \in X_n$, then $k - most(u_i) = x_i$ and $k - most(h_j) = x_j$;
(ii) if $t \in W_{(n)}(X) \backslash X$, then $u_i = x_i$ and $h_j = x_j$.

Definition 2.6 ([1]). Let $t \in W_{(n)}(X)$, a subterm of t is defined inductively by the following.
(i) Every variable $x \in var(t)$ is a subterm of t;
(ii) if $t = f(t_1, \dots, t_n)$ then t itself, $t_1, \dots, t_n$ are subterms of t.
The set of all subterms of t is denoted by $sub(t)$.

Example 2.7. Let $\tau = (3)$ with a ternary operation symbol f. Let $t = f(t_1, t_2, t_3)$ such that $t_1 = f(x_5, f(x_1, x_7, x_3), x_2)$, $t_2 = x_3$ and $t_3 = f(x_8, x_1, x_9)$. Then
$sub(t_1) = \{t_1, x_5, f(x_1, x_7, x_3), x_1, x_7, x_3, x_2\}$,
$sub(t_2) = \{x_3\}$,
$sub(t_3) = \{t_3, x_8, x_1, x_9\}$,
$sub(t) = \{t, t_1, t_2, t_3, x_5, f(x_1, x_7, x_3), x_1, x_7, x_2, x_8, x_9\}$.

3 All Completely Regular Elements in $Relhyp_G((n),(m))$

To determine the set of all completely regular elements of the monoid $Relhyp_G((n),(m))$, we first consider the structure of regular elements of the monoid. An element $\sigma_{t,F} \in Relhyp_G((n),(m))$ is called regular if and only if there exists $\sigma_{u,H} \in Relhyp_G((n),(m))$ such that $\sigma_{t,F} = \sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F}$. In this section, we use the concept of regular elements as a tool to determine the set of all completely regular elements of the monoid of all generalized relational hypersubstitutions for algebraic systems of type $((n),(m))$ and we have that a completely regular element is both a left regular and a right regular element of the monoid of all generalized relational hypersubstitutions for algebraic systems of type $((n),(m))$. For any $\sigma_{t,F} \in Relhyp_G((n),(m))$, where $t \in W_{(n)}(X)$ and $F \in F^*_{(m)}(W_{(n)}(X))$, $\sigma_{t,F}$ is called completely regular if and only if there exists $\sigma_{u,H} \in Relhyp_G((n),(m))$ such that

$$\sigma_{t,F} = \sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F} \text{ and}$$
$$\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}.$$

Theorem 3.1. [11] Let $t = x_i \in X_n$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is regular if and only if one of the following conditions is satisfied:
(i) $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $k - most(s_{b'_l}) = x_{b_l}$ where $\{b'_1, \ldots, b'_j\} \subseteq \{1, \ldots, m\}$;
(ii) $var(F) \cap X_m = \emptyset$.

Theorem 3.2. [11] Let $t \in W_{(n)}(X \backslash X_n)$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is regular if and only if one of the following conditions is satisfied:
(i) $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b'_l} = x_{b_l}$ where $\{b'_1, \ldots, b'_j\} \subseteq \{1, \ldots, m\}$;
(ii) $var(F) \cap X_m = \emptyset$.

Theorem 3.3. [11] Let $t \in W_{(n)}(X) \backslash X$ such that $var(t) \cap X_n = \{x_{a_1}, \ldots, x_{a_i}\}$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is regular if and only if $t_{a'_k} = x_{a_k}$ where $\{a'_1, \ldots, a'_i\} \subseteq \{1, \ldots, n\}$ and $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b'_l} = x_{b_l}$ where $\{b'_1, \ldots, b'_j\} \subseteq \{1, \ldots, m\}$ or $var(F) \cap X_m = \emptyset$.

Theorem 3.4. *Let $t = x_i \in X_n$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is completely regular if and only if one of the following conditions is satisfied:*
(i) $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $i - most(s_{b_l}) = x_{\phi(b_l)}$ where ϕ is a bijective map on $\{b_1, \ldots, b_j\}$;
(ii) $var(F) \cap X_m = \emptyset$.

Proof. Let $\sigma_{t,F}$ be completely regular. Then there exists $\sigma_{u,H} \in Relhyp_G((n),(m))$ such that

$$\sigma_{t,F} = \sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F}, \tag{3.1}$$

and

$$\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}. \tag{3.2}$$

Assume that $var(F) \cap X_m \neq \emptyset$, and let $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$. We will show that $i - most(s_{b_l}) = x_{\phi(b_l)}$ for all $l = 1, \ldots, j$ where ϕ is a bijective map on $\{b_1, \ldots, b_j\}$. Consider

$$\begin{aligned}(\sigma_{u,H} \circ_g \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{u,H}[F] \\ &= R^m(H, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]) \\ &= \gamma(S^m(h_1, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]), \ldots, \\ &\quad S^m(h_m, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])) \\ &= \gamma(a_1, \ldots, a_m)\end{aligned}$$

where $a_i = S^m(h_i, \sigma_{u,H}[s_1], \ldots, \sigma_{u,H}[s_m])$ for all $i = 1, \ldots, m$.
Then $F = (\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F})(\gamma) = \widehat{\sigma}_{t,F}[\gamma(a_1, \ldots, a_m)]$ and $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$. By Lemma 2.5(i), we have $i - most(a_{b_l}) = x_{b_l}$ for all $l = 1, \ldots, j$. So $x_{b_l} = i - most(S^m(h_{b_l}, \sigma_{u,H}[s_1], \ldots, \sigma_{u,H}[s_m]))$. Without loss of generality, we may assume that $i - most(h_{b_l}) = x_{b_l} \in var(F)$ for all $l = 1, \ldots, j$. Then $x_{b_l} = \widehat{\sigma}_{u,H}[s_{b_l}]$, so there exist elements $x_{b_1}, \ldots, x_{b_j}$ which all are distinct, since if $x_{b_p} = x_{b_q}$ for some $p \neq q \in \{1, \ldots, j\}$ then $x_{b_p} = \widehat{\sigma}_{u,H}[s_{b_p}] = \widehat{\sigma}_{u,H}[s_{b_q}] = x_{b_q}$ but $x_{b_p} \neq x_{b_q}$. Hence all $b_1, \ldots, b_j$ are distinct. We can define a bijective map $\phi : \{b_1, \ldots, b_j\} \to \{b_1, \ldots, b_j\}$ by $\phi(b_l) = b_l$. Hence $i - most(s_{b_l}) = x_{b_l} = x_{\phi(b_l)}$.

Conversely, choose $\sigma_{u,H} \in F^*_{(m)}(W_{(n)}(X))$ where $u = x_i, H = \gamma(h_1, \ldots, h_m)$ with $var(H) = var(F)$. If $var(F) \cap X_m = \emptyset$ then we choose $h_k = s_k$ for all $k = 1, \ldots, m$. And if $i - mots(s_{b_l}) = x_{\phi(b_l)}$, we choose $i - most(h_{b_l}) = x_{\phi-1(b_l)}$. By Theorem 3.1, we have $\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F} = \sigma_{t,F}$. Next we will show that $\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H})(\gamma) &= R^n(F, i - most(h_1), \ldots, i - most(h_m)) \\ &= \gamma(w_1, \ldots, w_m),\end{aligned}$$

where $w_i = S^m(s_i, i - most(h_1), \ldots, i - most(h_m))$ for all $i = 1, \ldots, m$.

$$\begin{aligned}\sigma_{u,H} \circ_g \sigma_{t,F})(\gamma) &= R^n(H, i - most(s_1), \ldots, i - most(s_m)) \\ &= \gamma(v_1, \ldots, v_m),\end{aligned}$$

where $v_i = S^m(h_i, i - most(s_1), \ldots, i - most(s_m))$ for all $i = 1, \ldots, m$.
Consider

$$\begin{aligned}i - most(w_i) &= i - most(S^m(s_i, i - most(h_1), \ldots, i - most(h_m))) \\ &= S^m(i - most(s_i), i - most(h_1), \ldots, i - most(h_m)) \\ &= S^m(x_{\phi(b_l)}, i - most(h_1), \ldots, i - most(h_m)) \\ &= i - most(h_{\phi(b_l)})(i - most(h_{\phi(b_l)}) = x_{\phi(\phi-1(b_l))}),\end{aligned}$$

and

$$\begin{aligned}i - most(v_i) &= i - most(S^m(h_i, i - most(s_1), \ldots, i - most(s_m))) \\ &= S^m(i - most(h_i), i - most(s_1), \ldots, i - most(s_m)) \\ &= S^m(x_{\phi-1(b_l)}, i - most(s_1), \ldots, i - most(s_m)) \\ &= i - most(s_{\phi-1(b_l)})(i - most(s_{\phi-1(b_l)}) = x_{\phi-1(\phi(b_l))}).\end{aligned}$$

Thus $w_i = v_i$ for all $i = 1, \ldots, m$. So $\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}$. Therefore $\sigma_{t,F}$ is completely regular. □

Theorem 3.5. *Let $t \in W_{(n)}(X \backslash X_n)$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is completely regular if and only if one of the following conditions is satisfied:*

(i) $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b_l} = x_{\phi(b_l)}$ for all $l = 1, \ldots, j$ where ϕ is a bijective map on $\{b_1, \ldots, b_j\}$;

(ii) $var(F) \cap X_m = \emptyset$.

Proof. Let $\sigma_{t,F}$ be completely regular. Then there exists $\sigma_{u,H} \in Relhyp_G((n), (m))$ such that

$$\sigma_{t,F} = \sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F}, \tag{3.3}$$

and

$$\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}. \tag{3.4}$$

Assume that $var(F) \cap X_m \neq \emptyset$, and let $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b_l} \neq x_{\phi(b_l)}$

where ϕ is a bijective map on $\{b_1, \ldots, b_j\}$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[F]] \\ &= \widehat{\sigma}_{t,F}[R^m(H, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])] \\ &= \widehat{\sigma}_{t,F}[\gamma(a_1, \ldots, a_m)] \\ &= \gamma(S^m(s_1, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[a_m]), \ldots, \\ &\quad S^m(s_m, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[a_m])).\end{aligned}$$

By (3.3), we have $\widehat{\sigma}_{t,F}[a_{b_l}] = x_{b_l}$. Since $x_{b_l} \in var(F)$, we have to replace a variable x_{b_l} in the relational term F by $\widehat{\sigma}_{t,F}[a_{b_l}]$. Since $\widehat{\sigma}_{t,F}[a_{b_l}] = \widehat{\sigma}_{t,F}[S^m(h_{b_l}, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])]$, then $h_{b_l} \in X_m$ for all $l = 1, \ldots, j$. Without loss of generality, we may assume that $h_{b_l} = x_{b_l}$. So

$$\begin{aligned}\widehat{\sigma}_{t,F}[a_{b_l}] &= \widehat{\sigma}_{t,F}[S^m(x_{b_l}, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])] \\ &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[s_{b_l}]] \\ &\neq x_{b_l}.\end{aligned}$$

So $s_{b_l} = x_{b_l}$. By using this process, we obtain elements $x_{b_1}, \ldots, x_{b_j}$ which all are distinct. We can define a bijective map $\phi : \{b_1, \ldots, b_j\} \to \{b_1, \ldots, b_j\}$ by $\phi(b_l) = b_l$. Hence $s_{b_l} = x_{\phi(b_l)}$.

Conversely, let $t_k \in sub(t)$. Choose $\sigma_{u,H} \in Relhyp_G((n), (m))$ where $var(t) = var(u)$,$u_k = t_k$, and $h_{\phi(b_l)} = x_{b_l}$ for all $l = 1, \ldots, j$, $var(H) = var(F)$. Let $s_l \in sub(s_q)$, $h_l \in sub(h_q)$ for all $q \in \{1, \ldots, m\} \backslash \{b_1, \ldots, b_j\}$. If $var(s_l) \cap X_m = \emptyset$ then we choose $h_l = s_l$. And if $s_l = x_{\phi(b_q)}$ and $\phi(b_q) = b_l$ then we choose $h_{\phi(b_l)} = x_{b_l}$. If $var(F) \cap X_m = \emptyset$ then we choose $h_i = s_i$ for all $i = 1, \ldots, m$. By Theorem 3.2, we have $\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F} = \sigma_{t,F}$. Next, we will show that $\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H})(f) &= S^n(t, \widehat{\sigma}_{t,F}[u_1], \ldots, \widehat{\sigma}_{t,F}[u_n]) \\ &= f(w_1, \ldots, w_n)\end{aligned}$$

where $w_i = S^n(t_i, \widehat{\sigma}_{t,F}[u_1], \ldots, \widehat{\sigma}_{t,F}[u_n])$ for all $i = 1, \ldots, n$.

$$\begin{aligned}(\sigma_{u,H} \circ_g \sigma_{t,F})(f) &= S^n(u, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n]) \\ &= f(\tilde{w_1}, \ldots, \tilde{w_n})\end{aligned}$$

where $\tilde{w_i} = S^n(u_i, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n])$ for all $i = 1, \ldots, n$.

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H})(\gamma) &= R^m(F, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_m]) \\ &= \gamma(v_1, \ldots, v_m)\end{aligned}$$

where $v_j = R^m(s_j, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_m])$ for all $j = 1, \ldots, m$.

$$\begin{aligned}(\sigma_{u,H} \circ_g \sigma_{t,F})(\gamma) &= R^m(H, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]) \\ &= f(\tilde{v_1}, \ldots, \tilde{v_m})\end{aligned}$$

where $\tilde{v_j} = R^m(h_j, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])$ for all $j = 1, \ldots, m$.
We will show that $f(w_1, \ldots, w_n) = f(\tilde{w_1}, \ldots, \tilde{w_n})$ and $\gamma(v_1, \ldots, v_m) = \gamma(\tilde{v_1}, \ldots, \tilde{v_m})$.
Case 1 $var(F) = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b_l} = x_{\phi(b_l)}$:

Case 1.1 $l \in \{1, \ldots, j\}$:

$$\begin{aligned}v_{b_l} &= S^m(s_{b_l}, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_m]) \\ &= S^m(x_{\phi(b_l)}, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_m]) \\ &= \widehat{\sigma}_{t,F}[h_{\phi(b_l)}] \\ &= x_{b_l}\end{aligned}$$

and

$$\begin{aligned}\tilde{v}_{b_l} &= S^m(h_{b_l}, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]) \\ &= S^m(x_{\phi^{-1}(b_l)}, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]) \\ &= \widehat{\sigma}_{u,H}[s_{\phi^{-1}(b_l)}] \\ &= x_{b_l}.\end{aligned}$$

Case 1.2 $l \in \{1, \ldots, m\} \backslash \{b_1, \ldots, b_j\}$:
Let $s_l \in sub(s_q)$ and $h_l \in sub(h_q)$ for all $l = \{1, \ldots, m\}$. Then $v_q = S^m(s_q, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_m])$ and $\tilde{v_q} = S^m(h_q, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])$. If $var(s_l) \cap X_m = \emptyset$, then $v_l = s_l$ and $\tilde{v_l} = h_l = s_l$. If $s_l = x_{\phi(b_q)}$ and $\phi(b_q) = b_l$, then

$$v_l = S^m(s_l, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_m]) = \widehat{\sigma}_{t,F}[h_{\phi(b_l)}] = x_{b_l} \text{ and}$$

$$\tilde{v_l} = S^m(h_l, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]) = \widehat{\sigma}_{u,H}[s_l] = x_{\phi(b_q)} = x_{b_l}.$$

Hence $v_{b_l} = \tilde{v}_{b_l}$.
Case 2 $var(F) \cap X_m = \emptyset$: It is easily to calculate that $\gamma(v_1, \ldots, v_m) = \gamma(\tilde{v_1}, \ldots, \tilde{v_m})$. By straightforward calculation, we obtain $f(w_1, \ldots, w_n) = f(\tilde{w_1}, \ldots, \tilde{w_n})$. Thus $\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}$. Therefore $\sigma_{t,F}$ is completely regular. □

Theorem 3.6. *Let $t \in W_{(n)}(X) \backslash X$ such that $var(t) \cap X_n = \{x_{a_1}, \ldots, x_{a_i}\}$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is completely regular if and only if $t_{a_k} = x_{\pi(a_k)}$ for all $k = 1, \ldots, i$ where π is a bijective map on $\{a_1, \ldots, a_i\}$ and $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b_l} = x_{\phi(b_l)}$ for all $l = 1, \ldots, j$ where ϕ is a bijective map on $\{b_1, \ldots, b_j\}$ or $var(F) \cap X_m = \emptyset$.*

Proof. Let $\sigma_{t,F}$ be completely regular. Then there exists $\sigma_{u,H} \in Relhyp_G((n), (m))$ such that

$$\sigma_{t,F} = \sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F}, \tag{3.5}$$

and

$$\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}. \tag{3.6}$$

If $u \in X$, then $t = (\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F})(f) = u \in X$. This is a contradiction with $t \in W_{(n)}(X) \backslash X$. So $u = f(u_1, \ldots, u_n) \in W_{(n)}(X) \backslash X$. Consider

$$\begin{aligned}(\sigma_{u,H} \circ_g \sigma_{t,F})(f) &= S^n(u, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n]) \\ &= S^n(f(u_1, \ldots, u_n), \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n]) \\ &= f(S^n(u_1, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n]), \ldots, \\ &\quad S^n(u_n, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n])) \\ &= f(w_1, \ldots, w_n)\end{aligned}$$

where $w_i = S^n(u_i, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n])$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned}t &= \widehat{\sigma}_{t,F}[f(w_1, \ldots, w_n)] \\ &= S^n(f(t_1, \ldots, t_n), \widehat{\sigma}_{t,F}[w_1], \ldots, \widehat{\sigma}_{t,F}[w_n])\end{aligned}$$

where $t_i = S^n(t_i, \widehat{\sigma}_{t,F}[w_1], \ldots, \widehat{\sigma}_{t,F}[w_n])$ for all $i = 1, \ldots, n$. Since $var(t) \cap X_n = \{x_{a_1}, \ldots, x_{a_i}\}$ by Lemma 2.5(ii), we have $x_{a_k} = \widehat{\sigma}_{t,F}[w_{a_k}]$ for all $k = 1, \ldots, i$. Since $t \in W_{(n)}(X) \backslash X$ and $w_{a_k} = x_{a_k}$. Hence $x_{a_k} = w_{a_k} = S^n(u_i, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[u_n])$. If $u_{a_k} = f(u_{a_{k_1}}, \ldots, u_{a_{k_n}}) \in W_{(n)}(X)$, then $x_{a_k} = S^n(u_{a_k}, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n]) \in W_{(n)}(X) \backslash X$, a contradition. So $u_{a_k} = x_{a_k} \in var(t)$ for some $a_k \in \{a_1, \ldots, a_i\}$, then $x_{a_k} = w_{a_k} = S^n(u_{a_k}, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n])$. Thus $t_{a_k} = x_{a_k}$ for all $a_k \in \{a_1, \ldots, a_i\}$, by using this prosess we obtain elements $x_{a_1}, \ldots, x_{a_i}$ which all are distinct. We can define a bijective map $\pi : \{a_1, \ldots, a_i\} \to \{a_1, \ldots, a_i\}$ by $\pi(a_k) = a_k$. Hence $t_{a_k} = x_{a_k} = x_{\pi(a_k)}$. The proof of $s_{b_l} = x_{\phi(b_l)}$ is similary of Theorem 3.2.

Conversely, choose $\sigma_{u,H} \in Relhyp_G((n),(m))$ where $u_{\pi(a_k)} = x_{a_k}$ for all $k = 1,\ldots,i$, $var(t) = var(u)$ and $h_{\phi(b_l)} = x_{b_l}$ for all $l = 1,\ldots,j$, $var(H) = var(F)$. Let $t_k \in sub(t_p)$, $u_k \in sub(u_p)$ for all $p \in \{1,\ldots,n\}\backslash\{a_1,\ldots,a_i\}$. If $var(t_k) \cap X_n = \emptyset$ then we choose $u_k = t_k$. And, if $t_k = x_{\pi(a_p)}$ and $\pi(a_p) = a_k$ then we choose $u_{\pi(a_k)} = x_{a_k}$. Let $s_l \in sub(s_q)$, $h_l \in sub(h_q)$ for all $q \in \{1,\ldots,m\}\backslash\{b_1,\ldots,b_j\}$. If $var(s_l) \cap X_m = \emptyset$ then we choose $h_l = s_l$. And if $s_l = x_{\phi(b_q)}$ and $\phi(b_q) = b_l$ then we choose $h_{\phi(b_l)} = x_{b_l}$. If $var(F) \cap X_m = \emptyset$ then we choose $h_i = s_i$ for all $i = 1,\ldots,m$. By Theorem 3.3, we have $\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F} = \sigma_{t,F}$. Next we will show that $\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H})(f) &= S^n(t, \widehat{\sigma}_{t,F}[u_1],\ldots,\widehat{\sigma}_{t,F}[u_n]) \\ &= f(w_1,\ldots,w_n)\end{aligned}$$

where $w_i = S^n(t_i, \widehat{\sigma}_{t,F}[u_1],\ldots,\widehat{\sigma}_{t,F}[u_n])$ for all $i = 1,\ldots,n$.

$$\begin{aligned}(\sigma_{u,H} \circ_g \sigma_{t,F})(f) &= S^n(u, \widehat{\sigma}_{u,H}[t_1],\ldots,\widehat{\sigma}_{u,H}[t_n]) \\ &= f(\tilde{w_1},\ldots,\tilde{w_n})\end{aligned}$$

where $\tilde{w_i} = S^n(u_i, \widehat{\sigma}_{u,H}[t_1],\ldots,\widehat{\sigma}_{u,H}[t_n])$ for all $i = 1,\ldots,n$.

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H})(\gamma) &= R^m(F, \widehat{\sigma}_{t,F}[h_1],\ldots,\widehat{\sigma}_{t,F}[h_m]) \\ &= \gamma(v_1,\ldots,v_m)\end{aligned}$$

where $v_j = R^m(s_j, \widehat{\sigma}_{t,F}[h_1],\ldots,\widehat{\sigma}_{t,F}[h_m])$ for all $j = 1,\ldots,m$.

$$\begin{aligned}(\sigma_{u,H} \circ_g \sigma_{t,F})(\gamma) &= R^m(H, \widehat{\sigma}_{u,H}[s_1],\ldots,\widehat{\sigma}_{u,H}[s_m]) \\ &= f(\tilde{v_1},\ldots,\tilde{v_m})\end{aligned}$$

where $\tilde{v_j} = R^m(h_j, \widehat{\sigma}_{u,H}[s_1],\ldots,\widehat{\sigma}_{u,H}[s_m])$ for all $j = 1,\ldots,m$.
We will show that $f(w_1,\ldots,w_n) = f(\tilde{w_1},\ldots,\tilde{w_n})$ and $\gamma(v_1,\ldots,v_m) = \gamma(\tilde{v_1},\ldots,\tilde{v_m})$.
Consider $var(t) = \{x_{a_1},\ldots,x_{a_i}\}$ such that $t_{a_k} = x_{\phi(a_k)}$:
Case 1 $k \in \{1,\ldots,i\}$:

$$\begin{aligned}w_{a_k} &= S^n(t_{a_k}, \widehat{\sigma}_{t,F}[u_1],\ldots,\widehat{\sigma}_{t,F}[u_n]) \\ &= S^n(x_{\pi(a_k)}, \widehat{\sigma}_{t,F}[u_1],\ldots,\widehat{\sigma}_{t,F}[u_n]) \\ &= \widehat{\sigma}_{t,F}[u_{\pi(a_k)}] \\ &= x_{a_k}\end{aligned}$$

and

$$\begin{aligned}\tilde{w}_{a_k} &= S^n(u_{\pi(a_k)}, \widehat{\sigma}_{u,H}[t_1],\ldots,\widehat{\sigma}_{u,H}[t_n]) \\ &= S^n(x_{a_k}, \widehat{\sigma}_{u,H}[t_1],\ldots,\widehat{\sigma}_{u,H}[t_n]) \\ &= \widehat{\sigma}_{u,H}[t_{a_k}] \\ &= x_{\pi(a_k)}.\end{aligned}$$

Case 2 $k \in \{1,\ldots,n\}\backslash\{a_1,\ldots,a_i\}$:
Let $t_k \in sub(t_p)$. Then $w_k = S^n(t_k, \widehat{\sigma}_{t,F}[u_1],\ldots,\widehat{\sigma}_{t,F}[u_n])$ and $\tilde{w_k} = S^n(u_k, \widehat{\sigma}_{u,H}[t_1],\ldots,\widehat{\sigma}_{u,H}[t_n])$ for all $k = 1,\ldots,n$. If $var(t_k) \cap X_m = \emptyset$, then $w_k = t_k$ and $\tilde{w_k} = u_k = t_k$. If $t_k = x_{\pi(a_k)}$, then

$$w_k = S^n(t_k, \widehat{\sigma}_{t,F}[u_1],\ldots,\widehat{\sigma}_{t,F}[u_n]) = \widehat{\sigma}_{t,F}[u_{\pi(a_k)}] = x_{a_k} \text{ and}$$

$$\tilde{w_k} = S^n(u_k, \widehat{\sigma}_{u,H}[t_1],\ldots,\widehat{\sigma}_{u,H}[t_n]) = \widehat{\sigma}_{u,H}[t_{\pi^{-1}(a_k)}] = x_{a_k}.$$

Hence $w_{a_k} = \tilde{w}_{a_k}$, so $f(w_1,\ldots,w_n) = f(\tilde{w_1},\ldots,\tilde{w_n})$. Similary we have $f(v_1,\ldots,v_m)$ $= f(\tilde{v_1},\ldots,\tilde{v_m})$. Thus $\sigma_{t,F} \circ_g \sigma_{u,H} = \sigma_{u,H} \circ_g \sigma_{t,F}$. Therefore $\sigma_{t,F}$ is completely regular. □

4 All Intra-regular Elements in $Relhyp_G((n),(m))$

It is well-known in semigroup theory that every completely regular element is an intra-regular element. In general, every intra-regular element need not be a completely regular element. In this section, we use the concept in Section 3 to show that every intra-regular element of the monoid of all generalized relational hypersubstitutions for algebraic systems of type $((n),(m))$ is a completely regular element. Moreover, we show that completely regular, left regular, right regular and intra-regular elements of the monoid of all generalized relational hypersubstitutions for algebraic systems of type $((n),(m))$ are the same.

Lemma 4.1. ([1]) Let $t, u \in W_{(n)}(X)\backslash(X)$. Then $\left|var(\hat{\sigma}_t[u])^{X_n}\right| \leq \left|var(t)^{X_n}\right|$.

Lemma 4.2. ([10]) Let $F, H \in F^*_{(m)}(W_{(n)}(X))$. Then $\left|var(\hat{\sigma}_{t,F}[H])^{X_m}\right| \leq \left|var(F)^{X_m}\right|$.

Theorem 4.3. *Let $t = x_i \in X_n$ and $F = \gamma(s_1,\ldots,s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is intra-regular if and only if one of the following conditions is satisfied:*
(i) $var(F) \cap X_m = \{x_{b_1},\ldots,x_{b_j}\}$ such that $i-most(s_{b_l}) = x_{\phi(b_l)}$ where ϕ is bijective map on $\{b_1,\ldots,b_j\}$;
(ii) $var(F) \cap X_m = \emptyset$.

Proof. Let $\sigma_{t,F}$ be intra-regular. Then there exist $\sigma_{u,H}, \sigma_{v,G} \in Relhyp_G((n),(m))$ such that

$$\sigma_{t,F} = \sigma_{u,H} \circ_g \sigma_{t,F} \circ_g \sigma_{t,F} \circ_g \sigma_{v,G}. \tag{4.1}$$

Assume that $var(F) \cap X_m \neq \emptyset$. Let $var(F) \cap X_m = \{x_{b_1},\ldots,x_{b_j}\}$ and $i-most(s_{b_l}) \neq x_{\phi(b_l)}$ where ϕ is bijective map on $\{b_1,\ldots,b_j\}$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{t,F}) &= \widehat{\sigma}_{x_i,F}[F]\\ &= R^m(F, \widehat{\sigma}_{x_i,F}[s_1],\ldots,\widehat{\sigma}_{x_i,F}[s_m])\\ &= \gamma(S^m(s_1, i-most(s_1),\ldots,i-most(s_m)),\ldots,\\ &\quad S^m(s_m, i-most(s_1),\ldots,i-most(s_m))).\end{aligned}$$

Since every variable $x_{b_k} \in var(F)$ is replaced by $i-most(s_{b_k})$ but $x_{\phi(b_l)} \notin \{i-most(s_{b_k}) : k = 1,\ldots,j\}$, $x_{b_l} \notin var((\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma))$. This implies that $|var((\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma))| < |var(F)|$. Consider

$$\begin{aligned}(\sigma^2_{t,F} \circ_g \sigma_{v,G})(\gamma) &= \widehat{\sigma}^2_{t,F}[G]\\ &= R^m(\widehat{\sigma}^2_{t,F}, i-most(g_1),\ldots,i-most(g_m)).\end{aligned}$$

Since every vaiable $x_{b_k} \in \sigma^2_{t,F}$ is replaced by $i-most(g_{b_k})$, so $|var((\sigma^2_{t,F} \circ_g \sigma_{v,G})(\gamma))| \leqslant |var(\sigma^2_{t,F}(\gamma))| < |var(F)|$. By (4.1) and $var(F) = var((\sigma_{u,H} \circ_g \sigma^2_{t,F} \circ_g \sigma_{v,G})(\gamma)) \subseteq var((\sigma^2_{t,F} \circ_g \sigma_{v,G})(\gamma))$, so $|var(F)| \leqslant |var((\sigma^2_{t,F} \circ_g \sigma_{v,G})(\gamma))| < |var(F)|$, this is contradiction. Hence $i-most(s_{b_l}) = x_{\phi(b_l)}$. Conversely , by Theorem 3.4, $\sigma_{t,F}$ is completely regular. It follows that $\sigma_{t,F}$ is intra-regular. □

Theorem 4.4. *Let $t \in W_{(n)}(X\backslash X_n)$ and $F = \gamma(s_1,\ldots,s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is regular if and only if one of the following conditions is satisfied:*
(i) $var(F) \cap X_m = \{x_{b_1},\ldots,x_{b_j}\}$ such that $s_{b_l} = x_{\phi(b_l)}$ for all $l = 1,\ldots,j$ where ϕ is bijective map on $\{b_1,\ldots,b_j\}$;
(ii) $var(F) \cap X_m = \emptyset$.

Proof. Let $\sigma_{t,F}$ be intra-regular. Then there exist $\sigma_{u,H}, \sigma_{v,G} \in Relhyp_G((n),(m))$ such that

$$\sigma_{t,F} = \sigma_{u,H} \circ_g \sigma_{t,F} \circ_g \sigma_{t,F} \circ_g \sigma_{v,G}. \tag{4.2}$$

Consider

$$(\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) = \gamma(w'_1,\ldots,w'_m) \text{ where } w'_j = R^m(s_j, \widehat{\sigma}_{t,F}[s_1],\ldots,\widehat{\sigma}_{t,F}[s_m])$$

and

$$(\sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma) = \gamma(z_1', \ldots, z_m') \text{ where } z_j' = R^m(w_j', \widehat{\sigma}_{t,F}^2[g_1], \ldots, \widehat{\sigma}_{t,F}^2[g_m])$$

for all $j = 1, \ldots, m$.

Assume $var(F) \cap X_m \neq \emptyset$. Let $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b_l} \neq x_{\phi(b_l)}$ for all $l = 1, \ldots, j$ where ϕ is bijective map on $\{b_1, \ldots, b_j\}$. We consider two cases:

(1) $x_{\phi(b_l)} \notin var(F)^{X_m}$.

(2) $s_{b_1}, \ldots, s_{b_j} \in var(F)$ such that $s_p = s_q$ for some $p, q \in \{b_1, \ldots, b_j\}$.

Case 1 Assume that the condition (1) holds. Then $var(\widehat{\sigma}_{t,F}[F])^{X_m} \subseteq var(F)^{X_m} = \{x_{b_1}, \ldots, x_{b_j}\}$ and $|var(\widehat{\sigma}_{t,F}[F])^{X_m}| \leqslant |var(F)^{X_m}|$. Fisrt, we will show that $x_{\phi(b_i)} \notin var(\widehat{\sigma}_{t,F}[F])^{X_m}$. Consider $(\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) = \gamma(w_1', \ldots, w_m')$ where $w_j' = R^m(s_j, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$ for all $j = 1, \ldots, m$. If $x_{\phi(b_i)} \in var(\widehat{\sigma}_{t,F}[F])^{X_m}$, then $w_k = x_{\phi(b_i)}$. So there exists $k \in \{b_1, \ldots, b_j\}$ such that $s_k = x_{\phi(b_l)}$ and $x_{\phi(b_l)} \in var(\widehat{\sigma}_{t,F}[F])^{X_m}$, which contradicts to $x_{\phi(b_l)} \notin var(F)^{X_m}$. Hence $x_{\phi(b_i)} \notin var(\widehat{\sigma}_{t,F}[F])^{X_m}$. Then $var(\widehat{\sigma}_{t,F}[F])^{X_m} \subset var(F)^{X_m}$. Therefore $|var(\widehat{\sigma}_{t,F}[F])^{X_m}| < |var(F)^{X_m}|$. Finally, we will show that $\sigma_{t,F} \neq \sigma_{u,H} \circ_g \sigma_{t,F}^2 \circ_g \sigma_{v,G}$ for all $\sigma_{u,H}, \sigma_{v,G} \in Relhyp_G((n),(m))$. By Lemma 4.2, we have $|var(\sigma_{t,F}^2 \circ_g \sigma_{v,G})^{X_m}| \leqslant |var(\sigma_{t,F}^2)^{X_m}|$ and $|var(\sigma_{u,H} \circ_g \sigma_{t,F}^2 \circ_g \sigma_{v,G})^{X_m}| \leqslant |var(\sigma_{t,F}^2 \circ_g \sigma_{v,G})^{X_m}|$. Hence $|var(\sigma_{u,H} \circ_g \sigma_{t,F}^2 \circ_g \sigma_{v,G})^{X_m}| < |var(F)^{X_m}|$. It follows that $\sigma_{t,F}$ is not intra-regular.

Case 2 $s_{b_1}, \ldots, s_{b_j} \in var(F)$ such that $s_p = s_q$ for all $p, q \in \{b_1, \ldots, b_j\}$. Then there exist at least one element of $var(F)$ which is not an element of the set $\{s_{b_1}, \ldots, s_{b_j}\}$, say x_{b_l}. So $x_{b_l} \notin \widehat{\sigma}_{t,F}[s_k]$ for all $k = b_1, \ldots, b_j$. Consider $(\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) = R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$. We will replace every x_k in the term F by $\widehat{\sigma}_{t,F}[s_k]$. But $x_{b_l} \notin \widehat{\sigma}_{t,F}[s_k]$ for all $k = b_1, \ldots, b_j$, so $x_{b_l} \notin var((\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma))$. Thus $|var((\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma))| \leqslant |var(F)| - 1$. Let $A = \{p \in I(F) | s_p \neq s_q$ for all $q = 1, \ldots, m\}$, $B = \{p \in I(F) | w_p' \in var(F)$ and $w_p \neq w_q$ for all $q = 1, \ldots, m\}$, $\tilde{B} = \{p' | w_p' = x_{p'}$ for all $p \in B\}$. Then $A, B, \tilde{B} \neq \emptyset$ and $B = \tilde{B}$. By the definition of the set B, we can consider 2 cases.

Case 2.1: $g_{p'} = x_j$ for all $p' \in \tilde{B}$. Then $|var((\sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma))| \leqslant |var((\sigma_{t,F}^2)(\gamma))| \leqslant |var(F)| - 1$. Since $var((\sigma_{u,H} \circ_g \sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma)) \subseteq var((\sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma))$, $|var((\sigma_{u,H} \circ_g \sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma))| \leqslant |var((\sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma))|$. But $F = \sigma_{t,F}(\gamma) = (\sigma_{u,H} \circ_g \sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma)$. So $var(F) = |var((\sigma_{u,H} \circ_g \sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma))| \leqslant |var(F)| - 1$, this is a contradiction.

Case 2.2: There is $p' \in \tilde{B}$ such that $g_{p'} \in W_{(m)}(X) \backslash X$. Let $C = \{p | z_p' \in X_m, z_p' \neq z_q'$ for all $q = 1, \ldots, m\}$. Then $|C| \leqslant |\tilde{B}| - 1 = |B| - 1$, because q_p is replaced by $\widehat{\sigma}_{t,F}[g_{p'}]$. Since $F = (\sigma_{u,H} \circ_g \sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma)$, $var(F) \subseteq var((\sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma))$. So $|A| \leqslant |C|$, but $|A| \geqslant |B| \geqslant |B| - 1 \geqslant |C| \geqslant |A|$, this is a contradiction.

Conversely, by Theorem 3.5 $\sigma_{t,F}$ is completely regular. It follows that $\sigma_{t,F}$ is intra-regular. □

Theorem 4.5. *Let $t \in W_{(n)}(X) \backslash X$ such that $var(t) \cap X_n = \{x_{a_1}, \ldots, x_{a_i}\}$ and $F = \gamma(s_1, \ldots, s_m) \in F_{(m)}^*(W_{(n)}(X))$. Then $\sigma_{t,F}$ is intra-regular if and only if $t_{a_k} = x_{\pi(a_k)}$ for all $k = 1, \ldots, i$ where π is a bijective map on $\{a_1, \ldots, a_i\}$ and $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b_l} = x_{\phi(b_l)}$ for all $l = 1, \ldots, j$ where ϕ is a bijective map on $\{b_1, \ldots, b_j\}$ or $var(F) \cap X_m = \emptyset$.*

Proof. Let $\sigma_{t,F}$ be intra-regular. Then there exist $\sigma_{u,H}, \sigma_{v,G} \in Relhyp_G((n),(m))$ such that

$$\sigma_{t,F} = \sigma_{u,H} \circ_g \sigma_{t,F} \circ_g \sigma_{t,F} \circ_g \sigma_{v,G} \tag{4.3}$$

Consider

$$(\sigma_{t,F} \circ_g \sigma_{t,F})(f) = f(w_1, \ldots, w_n) \text{ where } w_i = S^n(t_i, \widehat{\sigma}_{t,F}[t_1], \ldots, \widehat{\sigma}_{t,F}[t_n])$$

and

$$(\sigma_{t,F}^2 \circ_g \sigma_{v,G})(f) = f(z_1, \ldots, z_n) \text{ where } z_i = S^n(w_i, \widehat{\sigma}_{t,F}^2[v_1], \ldots, \widehat{\sigma}_{v,F}^2[v_n])$$

for all $i = 1, \ldots, n$.

$$(\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) = \gamma(w_1', \ldots, w_m') \text{ where } w_j' = R^m(s_j, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$$

and

$$(\sigma_{t,F}^2 \circ_g \sigma_{v,G})(\gamma) = \gamma(z_1', \ldots, z_m') \text{ where } z_j' = R^m(q_j, \widehat{\sigma}_{t,F}^2[g_1], \ldots, \widehat{\sigma}_{t,F}^2[g_m])$$

for all $j = 1, \dots, m$.
Assume $var(t) \cap X_n = \{x_{a_1}, \dots, x_{a_i}\}$ such that $t_{a_k} \neq x_{\pi(a_k)}$ for all $k = 1, \dots, i$ where π is bijective map on $\{a_1, \dots, a_i\}$ and $var(F) \cap X_m \neq \emptyset$, $var(F) \cap X_m = \{x_{b_1}, \dots, x_{b_j}\}$ such that $s_{b_l} \neq x_{\phi(b_l)}$ for all $l = 1, \dots, j$ where ϕ is bijective map on $\{b_1, \dots, b_j\}$. We consider four cases:

(1) $x_{\pi(a_k)} \notin var(t)^{X_n}$.

(2) $t_{a_1}, \dots, t_{a_i} \in var(t)$ such that $t_c = t_d$ for some $c, d \in \{a_1, \dots, a_i\}$.

(3) $x_{\phi(b_l)} \notin var(F)^{X_m}$.

(4) $s_{b_1}, \dots, s_{b_j} \in var(F)$ such that $s_p = s_q$ for some $p, q \in \{b_1, \dots, b_j\}$.

Case 1 Assume that the condition (1) holds. Then $var(\widehat{\sigma}_{t,F}[t])^{X_n} \subseteq var(t)^{X_n} = \{x_{a_1}, \dots, x_{a_i}\}$ and $|var(\widehat{\sigma}_{t,F}[t])^{X_n}| \leqslant |var(t)^{X_n}|$. Fisrt, we will show that $x_{\pi(a_j)} \notin var(\widehat{\sigma}_{t,F}[t])^{X_n}$. Consider $(\sigma_{t,F} \circ_g \sigma_{t,F})(f) = f(w_1, \dots, w_n)$ where $w_i = S^n(t_i, \widehat{\sigma}_{t,F}[t_1], \dots, \widehat{\sigma}_{t,F}[t_n])$ for all $i = 1, \dots, n$. If $x_{\pi(a_j)} \in var(\widehat{\sigma}_{t,F}[t])^{X_n}$, then $w_p = x_{\pi(a_j)}$. So there exists $p \in \{a_1, \dots, a_i\}$ such that $t_p = x_{\pi(a_k)}$ and $x_{\pi(a_k)} \in var(\widehat{\sigma}_{t,F}[t])^{X_n}$, which contradicts to $x_{\pi(a_k)} \notin var(t)^{X_n}$. Hence $x_{\pi(a_j)} \notin var(\widehat{\sigma}_{t,F}[t])^{X_n}$. Then $var(\widehat{\sigma}_{t,F}[t])^{X_n} \subset var(t)^{X_n}$. Therefore $|var(\widehat{\sigma}_{t,F}[t])^{X_n}| < |var(t)^{X_n}|$. Finally, we will show that $\sigma_{t,F} \neq \sigma_{u,H} \circ_g \sigma^2_{t,F} \circ_g \sigma_{v,G}$ for all $\sigma_{u,H}, \sigma_{v,G} \in Relhyp_G((n),(m))$. By Lemma 4.1, we have $|var(\sigma^2_{t,F} \circ_g \sigma_{v,G})^{X_n}| \leqslant |var(\sigma^2_{t,F})^{X_n}|$ and $|var(\sigma_{u,H} \circ_g \sigma^2_{t,F} \circ_g \sigma_{v,G})^{X_n}| \leqslant |var(\sigma^2_{t,F} \circ_g \sigma_{v,G})^{X_n}|$. Hence $|var(\sigma_{u,H} \circ_g \sigma^2_{t,F} \circ_g \sigma_{v,G})^{X_n}| < |var(F)^{X_n}|$. It folows that $\sigma_{t,F}$ is not intra-regular.

Case 2 $t_{a_1}, \dots, t_{a_i} \in var(t)$ such that $t_c = t_d$ for all $c, d \in \{a_1, \dots, a_i\}$. Then there exist at least one element of $var(t)$ which is not an element of the set $\{t_{a_1}, \dots, t_{a_i}\}$, say x_{a_k}. So $x_{a_k} \notin \widehat{\sigma}_{t,F}[t_j]$ for all $j = a_1, \dots, a_i$. Consider $(\sigma_{t,F} \circ_g \sigma_{t,F})(f) = S^n(t, \widehat{\sigma}_{t,F}[t_1], \dots, \widehat{\sigma}_{t,F}[t_n])$. We will replace every x_j in the term t by $\widehat{\sigma}_{t,F}[t_j]$. But $x_{a_k} \notin \widehat{\sigma}_{t,F}[t_j]$ for all $j = a_1, \dots, a_i$, so $x_{a_k} \notin var((\sigma_{t,F} \circ_g \sigma_{t,F})(f))$. Thus $|var((\sigma_{t,F} \circ_g \sigma_{t,F})(f))| \leqslant |var(t)| - 1$. Let $A = \{c \in I(t) | t_c \neq t_d$ for all $d = 1, \dots, n\}$, $B = \{c \in I(t) | w_c \in var(t)$ and $w_c \neq w_d$ for all $d = 1, \dots, n\}$, $\tilde{B} = \{c' | w_c = w_{c'}$ for all $c \in B\}$. Then $A, B, \tilde{B} \neq \emptyset$ and $B = \tilde{B}$. By definition of the set B, we can consider 2 cases.

Case 2.1: $v_{c'} = x_j$ for all $c' \in \tilde{B}$. Then $|var((\sigma^2_{t,F} \circ_g \sigma_{v,G})(f))| \leqslant |var((\sigma^2_{t,F})(f))| \leqslant |var(t)| - 1$. Since $var((\sigma_{u,H} \circ_g \sigma^2_{t,F} \circ_g \sigma_{v,G})(f)) \subseteq var((\sigma^2_{t,F} \circ_g \sigma_{v,G})(f))$, $|var((\sigma_{u,H} \circ_g \sigma^2_{t,F} \circ_g \sigma_{v,G})(f))| \leqslant |var((\sigma^2_{t,F} \circ_g \sigma_{v,G})(f))|$. But $t = \sigma_{t,F}(f) = (\sigma_{u,H} \circ_g \sigma^2_{t,F} \circ_g \sigma_{v,G})(f)$. So $var(t) = |var((\sigma_{u,H} \circ_g \sigma^2_{t,F} \circ_g \sigma_{v,G})(f))| \leqslant |var(t)| - 1$, this is a contradiction.

Case 2.2: There is $c' \in B$ such that $v_{c'} \in W_{(n)}(X) \backslash X$. Let $C = \{c | z_c \in X_n, z_c \neq z_d$ for all $d = 1, \dots, n\}$. Then $|C| \leqslant |\tilde{B}| - 1 = |B| - 1$, because w_c is replaced by $\widehat{\sigma}_{t,F}[v_{c'}]$. Since $t = (\sigma_{u,H} \circ_g \sigma^2_{t,F} \circ_g \sigma_{v,G})(f)$, $var(t) \subseteq var((\sigma^2_{t,F} \circ_g \sigma_{v,G})(f))$. So $|A| \leqslant |C|$, but $|A| \geqslant |B| \geqslant |B| - 1 \geqslant |C| \geqslant |A|$, this is a contradiction. The proof of Case 3 and 4 are similar to Case 1 and 2 of Theorem 4.4, respectively. Conversely, by Theorem 3.6 $\sigma_{t,F}$ is completely regular. It follows that $\sigma_{t,F}$ is intra-regular. □

Corollary 4.6. *Let $\sigma_{t,F} \in Relhyp((m),(n))$. Then the following statements are equivalent:*

(i) $\sigma_{t,F}$ is completely regular in $Relhyp_G((n),(m))$;

(ii) $\sigma_{t,F}$ is left regular in $Relhyp_G((n),(m))$;

(iii) $\sigma_{t,F}$ is right regular in $Relhyp_G((n),(m))$;

(iv) $\sigma_{t,F}$ is intra-regular in $Relhyp_G((n),(m))$.

5 Conclusion remarks

This paper aims is to determine the set of all completely regular elements and intra-regular elements of $Relhyp_G((n),(m))$. Furthermore, we show that the sets of all completely regular elements, left(right) regular elements and intra-regular elements of the monoid of all generallized relational hypersubstitution for algebraic systems of type $((n),(m))$ are the same.

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Author information

P. Kunama, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Chiang Rai 57120, Thailand.
E-mail: pornpimol@rmutl.ac.th

S. Leeratanavalee, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.
E-mail: sorasak.l@cmu.ac.th

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