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Generalized Derivation on Hyperrings

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Abstract In this paper, we explore the notion of derivation and introduce generalized derivation in Krasner hyperring, and present several examples of it. Then, we derive various properties of derivation and generalized derivation in Krasner hyperring. Later on, we prove results regarding the commutativity of a Krasner hyperring. Moreover, if f is a generalized derivation and Z is a center, then $f(Z)$ is contained within Z.

1 Introduction

An operation is a relation that combines two elements from a set to generate an element in another set. A hyperoperation is a generalization of an operation in which it yields a set of elements instead of a single element. Structures that involve a set and at least one hyperoperation are referred to as algebraic hyperstructures.

In 1934, the study of hyperstructures was started by the French mathematician Frederic Marty [15] during the 8^{th} Congress of Scandinavian Mathematicians with the presentation of the paper "Sur une generalization de la notion de groupe". In this paper, Marty explained the concept of hypergroups and analyzed their properties. Later, hyperstructures have been studied by numerous researchers over the following decades and continue to be studied worldwide today.

In 1937, Krasner [14] and Wall [19] also presented their definitions of hypergroups in different senses. The theory of hyperstructure has been studied in various contexts and applied to diverse branches of Computer Science and Mathematics, as documented in [9].

The study of hyperrings and hyperfields was initiated by Krasner [13]. Krasner defined a special type of hyperring known as a Krasner hyperring, which resembles a ring but with modified axioms where addition serves as the hyperoperation and multiplication remains a usual binary operation. The basic definitions and results of hyperstructures and hyperrings are found in [10]. In 1957, Posner introduced the concept of derivation in rings and provided several interesting results regarding derivations in prime rings. Furthermore, many researchers have expanded on the concept of derivation in various directions, such as generalized derivation, Jordan derivation, generalized n-derivation in near rings and rings [5, 6]. In [4], Asokkumar gave some examples and important results on derivations in Krasner hyperrings.

Derivation on hyperring is studied along with several examples, and we introduce generalized derivation on hyperrings, presenting corresponding examples. Furthermore, we prove that for a generalized derivation f on a Krasner hyperring R, if $f^2 = 0$, then $f = 0$. It is also proved that a generalized derivation $f : R \to R$ is non-zero on a non-zero hyperideal I of R, along with additional results on the commutativity of Krasner hyperring.

Throughout this paper, the term "hyperring" refers specifically to a Krasner hyperring.

2 Preliminaries

In this section, we recall some basic definitions which can be found in [4, 10].

Definition 2.1. A hyperoperation $*$ on a non-empty set H is a mapping of $H \times H$ into the family of non-empty subsets of H i.e., $\mathcal{P}^*(H)$ [15]. A hypergroup $(H, *)$ is a non-empty set H equipped with a hyperoperation $*$ which satisfies the following axioms:

(i) $x * (y * z) = (x * y) * z$, for every $x, y, z \in H$;

(ii) $x * H = H * x = H$, for every $x \in H$.

Definition 2.2. A non-empty set R with a hyperaddition '+' and a multiplication '$\cdot$' is called an additive hyperring or Krasner hyperring if it satisfies the following:

(i) $(R, +)$ is a canonical hypergroup, i.e.,

a. $x + (y + z) = (x + y) + z$, for all $x, y, z \in R$;

b. $x + y = y + x$, for all $x, y \in R$;

c. there exists $0 \in R$ such that $0 + x = x$, for all $x \in R$;

d. for all $x \in R$ there exists an unique element denoted by $-x \in R$ such that $0 \in x + (-x)$;

e. $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in R$.

(ii) $(R, \cdot)$ is a semigroup having 0 as a bilaterally absorbing element i.e.,

a. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in R$;

b. $x \cdot 0 = 0 \cdot x = 0$, for all $x \in R$.

(iii) The multiplication '$\cdot$' is distributive with respect to the hyperoperation '+', i.e. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$.

A non-empty subset I of a canonical hypergroup R is called a canonical subhypergroup of R if I itself is a canonical hypergroup under the same hyperoperation as that of R. Equivalently, a non-empty subset I of a canonical hypergroup R is a canonical subhypergroup of R if $x - y \subseteq I$, for all $x, y \in I$. Here after we denote xy instead of $x \cdot y$. Moreover, for $A, B \subseteq R$ and $x \in R$, by $A + B$ we mean the set $\cup_{a \in A, b \in B}(a + b)$ and $AB = \cup_{a \in A, b \in B}(ab)$, $A + x = A + \{x\}$, $x + B = \{x\} + B$ and also $-A = \{-a : a \in A\}$.

The following elementary facts in a hyperring easily follow from the axioms: (i) $-(-a) = a$, for all $a \in R$; (ii) 0 is the unique element such that for every $a \in R$, there is an element $-a \in R$ with the property $0 \in a + (-a)$ and $-0 = 0$; (iii) $-(a + b) = -a - b$, for all $a, b \in R$; (iv) $-(ab) = (-a)b = a(-b)$, for all $a, b \in R$.

In a hyperring R, if there exists an element $1 \in R$ such that $1a = a1 = a$ for all $a \in R$, then the element 1 is called an identity element of a hyperring R. In fact, the element 1 is unique. Further, if $ab = ba$ for all $a, b \in R$, then the hyperring R is called a commutative hyperring.

Note 2.3. If $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ and $(x + y) \cdot z \subseteq x \cdot z + y \cdot z$, for all $x, y, z \in R$ in (iii), then R is called Weak Distributive Hyperring.

Example 2.4. [4] Consider the set $R = \{0, a, b\}$ with the hyperaddition and multiplication defined as follows:

+	0	a	b
0	{0}	{a}	{b}
a	{a}	{a,b}	R
b	{b}	R	{a,b}

·	0	a	b
0	0	0	0
a	0	b	a
b	0	a	b

$(R, +, \cdot)$ is a Hyperring.

Example 2.5. Consider the set $R = \{0, a, b, c\}$ with the hyperaddition and multiplication defined as follows:

+	0	a	b	c
0	{0}	{a}	{b}	{c}
a	{a}	R	{a,b}	{a,c}
b	{b}	{a,b}	R	{b,c}
c	{c}	{a,c}	{b,c}	R

·	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

$(R, +, \cdot)$ is a (Weak Distributive)Hyperring.

Definition 2.6. Let R be a hyperring. A non-empty subset S of R is called a subhyperring of R if $x - y \subseteq S$ and $xy \in S$, for all $x, y \in S$.

Definition 2.7. Let R be a hyperring and I be a non-empty subset of R. I is called a left (resp. right) hyperideal of R if

(i) $(I, +)$ is a canonical subhypergroup of R, i.e., $x - y \subseteq I$, for all $x, y \in I$;

(ii) $ra \subseteq I$ (resp. $ar \subseteq I$), for all $a \in I, r \in R$.

A hyperideal of R is one which is a left as well as a right hyperideal of R.

Definition 2.8. A hyperring R is said to be a prime hyperring if $xRy = 0$, for $x, y \in R$ implies either $x = 0$ or $y = 0$.

Definition 2.9. [3] A hyperring R is said to be a semiprime hyperring if $xRx = 0$, for $x \in R$ implies that $x = 0$.

Definition 2.10. A hyperring R is said to be 2-torsion free if $0 \in x + x$, for $x \in R$ implies $x = 0$.

Definition 2.11. [3] Let $(R, +, \cdot)$ be a hyperring. A center of R denoted by $Z(R)$, is

$$Z(R) = \{x \in R | x \cdot y = y \cdot x, \text{ for all } y \in R\}.$$

Lemma 2.12. *[3] Let R be a hyperring and $[x, y]$ denotes the set $xy - yx$, for all $x, y \in R$. Then, for all $x, y, z \in R$, we have,*

(i) $[x + y, z] = [x, z] + [y, z]$*;*

(ii) $[xy, z] \subseteq x[y, z] + [x, z]y$*;*

(iii) If $x \in Z(R)$, then $[xy, z] = x[y, z]$;

(iv) If d is a derivation of R, then $d[x, y] \subseteq [d(x), y] + [x, d(y)]$.

3 Derivation on Hyperrings

In this section, we recall the definition of a derivation and provide some examples. We then prove some of its properties following the approach outlined in [7, 16, 18].

Definition 3.1. [4] Let R be a hyperring. A map $d : R \to R$ is said to be a derivation of R if d satisfies

(i) $d(x + y) \subseteq d(x) + d(y)$ and

(ii) $d(xy) \in d(x)y + xd(y)$ for all $x, y \in R$.

If the map d is such that $d(x + y) = d(x) + d(y)$, for all $x, y \in R$ and satisfies the second condition, then d is called a strong derivation of R.

Example 3.2. Consider the hyperring given as in Example 2.4, define a map $d : R \to R$ such that $d(x) = x$, for all $x \in R$. Here d is a derivation on R.

Example 3.3. Consider the hyperring given as in Example 2.5. Define a map $d : R \to R$ such that $d(0) = 0$, $d(a) = b$, $d(b) = c$, and $d(c) = a$. The function d is a derivation on R.

Example 3.4. Let d_1 and d_2 be derivations on hyperrings R_1 and R_2 respectively. Consider the set $R = R_1 \times R_2$. We define hyperaddition $(a,b)+(c,d) = (a+c, b+d) = \{(x,y)|x \in a+c, y \in b+d\}$ and multiplication $(a,b)(c,d) = (ac, bd)$, for all, $a, c \in R_1$ and $b, d \in R_2$. Then R forms a hyperring. Now, we define a map d on R by $d(a,b) = (d_1(a), d_2(b))$. Here, d is a derivation on R.

Lemma 3.5. *[4] Let d be a derivation of a prime hyperring R and $a \in R$ such that $ad(u) = 0$ (or $d(u)a = 0$), for all $u \in R$. Then either $a = 0$ or $d = 0$.*

Proposition 3.6. *Let R be a prime hyperring. If for some $a, b \in R$, the relation $0 \in ax + xb$ holds for all $x \in R$, then $a, b \in Z(R)$. Furthermore, $b = -a$.*

Proof. Suppose that $0 \in ax + xb$, for all $x \in R$, replacing x by xy, we obtain,

$$0 \in axy + xyb \implies xyb = -axy \tag{3.1}$$

Multiplying the given equation in the hypothesis by y from the right, we get,

$$0 \in axy + xby \implies xby = -axy \tag{3.2}$$

Subtracting Equation 3.2 from Equation 3.1, we get, $0 \in xyb - xby = x(yb - by)$, for $x, y \in R$. Replacing x by xz for $z \in R$, we have $0 \in xR(yb - by) \implies x = 0$ or $0 \in yb - by$, for all $y \in R$. Since $R \neq 0$, $b \in Z(R)$. Thus, by the hypothesis, we have $0 \in (a+b)x$. Replacing x by zx, $0 \in (a+b)zx = (a+b)Rx \implies 0 \in a+b \implies a = -b \in Z(R)$. □

Lemma 3.7. *[3] Let I be a non-zero hyperideal on a prime hyperring R. Then, for all $x, y \in R$,*

(i) If $Ix = 0$ or $xI = 0$, then $x = 0$;

(ii) If $xIy = 0$, then $x = 0$ or $y = 0$.

Theorem 3.8. *Let R be a prime hyperring and d be a non-zero derivation on R such that $0 = d([x,y])$, for all $x, y \in R$. Then R is a commutative hyperring.*

Proof. Let $0 = d([x,y])$, for all $x, y \in R$. Replacing y by yx, $0 = d([x,yx]) = d([x,y]x) \implies 0 \in [x,y]d(x) \implies xyd(x) = yxd(x)$, for all $x, y \in R$. Now, by replacing y by zy, we get $0 \in xzyd(x) - zyxd(x) = xzyd(x) - zxyd(x) = [x,z]Rd(x)$, for $x, z \in R$. Since d is non-zero, $0 \in [x,z]$, for all $x, z \in R$. That is, R is a commutative hyperring. □

Theorem 3.9. *Let R be a 2-torsion free prime hyperring and $a \in R$. Suppose d be a non-zero derivation on R such that $0 = [a, d(R)]$. Then $a \in Z(R)$.*

Proof. Let $[a, d(x)] = 0$, for all $x \in R$. Replacing x by xy, we have $[a, d(xy)] = 0$, for all $x, y \in R$. We observe, $0 \in [a, d(x)y + xd(y)] = [a, d(x)y] + [a, xd(y)] \subseteq d(x)[a,y] + [a, d(x)]y + x[a, d(y)] + [a,x]d(y) = d(x)[a,y] + [a,x]d(y)$, for all $x, y \in R$. Now, by replacing y by $d(y)$, we get $0 \in [a,x]d^2(y) \implies axd^2(y) = xad^2(y)$, for all $x, y \in R$. By replacing x by zx, we get $0 \in azxd^2(y) - zxad^2(y) = azxd^2(y) - zaxd^2(y) = [a,z]Rd^2(y)$, for all $y, z \in R$. By Proposition 4.2 in [4], $a \in Z(R)$. □

Theorem 3.10. *Let R be a semiprime hyperring and d be a derivation on R. If $d(x)d(y) = 0$, for all $x, y \in R$, then d must be the zero derivation.*

4 Generalized Derivation on Hyperrings

In this section, we introduce the concept of generalized derivation on hyperrings and provide some examples along with certain properties. Here, some results are proven following the approach outlined in [1, 2, 8, 11, 12, 17].

Definition 4.1. Let R be a hyperring. A map $f : R \to R$ is called generalized derivation associated with a derivation $d : R \to R$ if, for all $x, y \in R$,

(i) $f(x+y) \subseteq f(x) + f(y)$;

(ii) $f(xy) \in f(x)y + xd(y)$;

(iii) $f(xy) \in d(x)y + xf(y)$.

Example 4.2. Let R be a hyperring and $M(R) = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in R \right\}$ be a collection of 2×2 matrices over R. A hyperaddition $\oplus$ is defined on $M(R)$ by $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x \in a+c, y \in b+d \right\}$, for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in M(R)$. A multiplication $\otimes$ is defined on $M(R)$ by $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix}$, for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in M(R)$. Clearly, hyperaddition and multiplication are well defined. $(M(R), \oplus, \otimes)$ forms a Krasner Hyperring. Now, define a function g on $M(R)$ by $g\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$. This map g is well-defined. We shall now show that g is a derivation. For $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in M(R)$, the set $g\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\}$ and the set $g\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \oplus g\left(\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right)$ are equal and equal to the set $\left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in -b-d \right\}$.

Also, $g\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} 0 & -ad \\ 0 & 0 \end{pmatrix}$
$= \left\{g\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes g\left(\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right)\right\}$. Thus g is a derivation on $M(R)$. Here g is a strong derivation on $M(R)$. Now, define a function f on $M(R)$ by $f\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Clearly this map is well-defined. We shall now show that f is a generalized derivation. For $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in M(R)$, the set
$f\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\}$ and the set $f\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \oplus f\left(\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right)$ are equal and equal to the set $\left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in a+c \right\}$.

Also, $f\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} ac & 0 \\ 0 & 0 \end{pmatrix}$ and
$\left\{f\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes g\left(\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right)\right\}$
$= \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -d \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -ad \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & ad+(-ad) \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} ac & x \\ 0 & 0 \end{pmatrix} : x \in ad-ad \right\} = S \implies \begin{pmatrix} ac & 0 \\ 0 & 0 \end{pmatrix} \in S$. Which proves, $f\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} \subseteq \left\{f\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes g\left(\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right)\right\}$. Similarly, we have $f\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} \subseteq \left\{g\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \otimes\right.$

$\left(\begin{array}{cc} c & d \\ 0 & 0 \end{array}\right)\Big\}\oplus\Big\{\left(\begin{array}{cc} a & b \\ 0 & 0 \end{array}\right)\otimes f\left(\left(\begin{array}{cc} c & d \\ 0 & 0 \end{array}\right)\right)\Big\}$. Thus f is a generalized derivation on $M(R)$.

Example 4.3. Consider the hyperring as in Example 2.4. Define a map $f : R \to R$ such that $f(0) = 0$, $f(a) = b$, and $f(b) = a$ and another map $g : R \to R$ defined as $g(x) = x$, for all $x \in R$. Then f is a generalized derivation on R associated with g.

Example 4.4. Let $R = \{0, 1\}$. Define the hyperoperations on R as following,

+	0	1
0	{0}	{1}
1	{1}	{0,1}

·	0	1
0	0	0
1	0	1

So, $(R, +, \cdot)$ forms a hyperring. Now define g on R as $g(x) = 0$, for all $x \in R$ and f on R as $f(x) = x$, for all $x \in R$. It is easy to verify that f is a generalized derivation associated with the derivation g on R.

Example 4.5. Consider the hyperring as in Example 2.5. Define a map $f : R \to R$ such that $f(0) = 0$, $f(a) = b$, $f(b) = c$, and $f(c) = a$ and another map $g : R \to R$ such that $g(0) = 0$, $g(a) = c$, $g(b) = a$, and $g(c) = b$. Then f is a generalized derivation on R associated with g.

Theorem 4.6. *Let R be a prime hyperring and I be a non-zero hyperideal of R. If f is a non-zero generalized derivation of R with an associated derivation d, then $f(I) \neq \{0\}$.*

Proof. Suppose $f(I) = \{0\}$. Then, $0 = f(ux) \in f(u)x + ud(x)$, for all $u \in I, x \in R \implies 0 = Id(x)$, for all $x \in R$. Using Lemma 3.7, we have $d = 0$. Now, $0 = f(xu) \in f(x)u + xd(u)$, for all $u \in I, x \in R \implies 0 = f(x)I$, for all $x \in R$. By Lemma 3.7, we have $f = 0$. A contradiction. Hence $f(I) \neq \{0\}$. □

Lemma 4.7. *[3] Let d be a derivation on a prime hyperring R and I be a non-zero hyperideal on R. If $d(I) = 0$, then $d = 0$.*

Theorem 4.8. *Let R be a prime hyperring and I be a non-zero hyperideal of R. If $f : R \to R$ is a generalized derivation associated with the derivation d such that for $a \in R$, $af(I) = 0$ (or $f(I)a = 0$), then $a = 0$.*

Proof. Suppose $af(I) = 0$, then $0 = af(uv) \in a(f(u)v + ud(v)) = aud(v)$, for all $u, v \in I$. This implies $0 = aId(v)$, for all $v \in I$. By Lemmas 3.7 and 4.7, $a = 0$ or $d = 0$. Using $af(I) = \{0\}$, we have, $0 = af(uv) \in a(d(u)v + uf(v)) = auf(v)$, for all $u, v \in I$ which implies $aIf(v) = 0$, for all $v \in I$. By Lemma 3.7, we have either $a = 0$ or $f(v) = 0$, for all $v \in I$. But $f(I) \neq \{0\}$ by Theorem 4.6. Thus, we conclude $a = 0$. Similarly, it can be proved in case of $f(I)a = 0$. □

Theorem 4.9. *Let R be a prime hyperring such that $Z(R)$ is a ring, and f be a generalized derivation associated with the derivation d. Then $f(Z) \subseteq Z$.*

Proof. Let $z \in Z$ and $x \in R$. Then $f(xz) = f(zx)$. That is $0 \in f(xz) - f(zx)$,

$$\begin{aligned}
&\implies 0 \in d(x)z + xf(z) - [f(z)x + zd(x)] = d(x)z + xf(z) - f(z)x - zd(x)\\
&\implies 0 \in d(x)z - zd(x) + xf(z) - f(z)x\\
&\implies 0 \in zd(x) - zd(x) + xf(z) - f(z)x\\
&\implies 0 \in (z - z)d(x) + xf(z) - f(z)x = xf(z) - f(z)x\\
&\implies xf(z) = f(z)x, \text{ for all } x \in R\\
&\implies f(Z) \subseteq Z.
\end{aligned}$$

□

Theorem 4.10. *Let R be a prime hyperring and f be a non-zero generalized derivation on R associated with non-zero derivation d such that $0 = f([x, y])$, for all $x, y \in R$. Then R is commutative.*

Proof. Let $0 = f([x,y])$, for all $x, y \in R$. Replacing y by xy, we have $0 = f([x,xy]) = f(x[x,y]) \in d(x)[x,y] \implies d(x)xy = d(x)yx$. Now, by replacing y by yz, we get $0 \in d(x)xyz - d(x)yzx = d(x)yxz - d(x)yzx \implies 0 \in d(x)R[x,z] \implies 0 \in [x,z]$, for all $x, z \in R$ or $d = 0$. Since d is non-zero, R is commutative. □

Theorem 4.11. *Let R be a prime hyperring and f be a generalized derivation on R associated with a non-zero surjective derivation d. If $f^2 = 0$, then $f = 0$.*

Proof. We have $0 = f^2(xy) \in f(f(x)y + xd(y)) \subseteq f(x)d(y) + f(x)d(y) + xd^2(y)$, for all $x, y \in R$. By substituting $f(x)$ for x, we obtain $0 = f(x)d^2(y)$, for all $x, y \in R$. Since d is surjective, we have $0 = f(x)d(z)$, for all $x, z \in R$. Using Lemma 3.5, we conclude that either $d = 0$ or $f = 0$. Since d is a non-zero derivation on R, it follows that $f = 0$. □

Definition 4.12. Let S be a non-empty subset of a hyperring R. A mapping $f : R \to R$ is said to commute on S if $0 \in [f(x), x]$, for all $x \in S$.

Theorem 4.13. *Let R be a non-zero prime hyperring, I a non-zero hyperideal of R, and $f : R \to R$ a non-zero generalized derivation associated with the derivation d such that $f(R)$ is a ring. If either $f(x)x = 0$ or $xf(x) = 0$, for all $x \in I$, then f commutes on I.*

Proof. Let

$$f(x)x = 0, \text{ for all } x \in I \tag{4.1}$$

For $y \in I$, replacing x by $x + y$, we obtain,

$$0 \in f(x+y)(x+y) \subseteq f(x)y + f(y)x, \text{ for all } x, y \in I \tag{4.2}$$

Replacing y by y^2, we get,

$$0 \in f(x)y^2 + f(y^2)x, \text{ for all } x, y \in I \tag{4.3}$$

Right multiplying Equation 4.2 by y and subtracting from Equation 4.3, we obtain,

$$0 \in f(y^2)x + f(y)xy, \text{ for all } x, y \in I \tag{4.4}$$

Replacing x by $xf(y)$ and using Equation 4.1, we get

$$0 = f(y^2)xf(y), \text{ for all } x, y \in I \tag{4.5}$$

Right multiplying Equation 4.4 by $f(y)$ and using Equation 4.5, we get, $f(y)xyf(y) = 0$, for all $x, y \in I$. Replacing x by rx, we obtain, $f(y)rxyf(y) = 0$, for all $x, y \in I$, $r \in R$.. Left multiplying by xy, we have, $xyf(y)rxyf(y) = 0$, for all $x, y \in I$. The Semiprimeness of R implies that, $xyf(y) = 0$, for all $x, y \in I$. Replacing x by xr, we get, $xryf(y) = 0$, for all $x, y \in I$. Replacing r by $f(y)r$, $xf(y)ryf(y) = 0$, for all $x, y \in I$. Substituting x for y, we get $xf(x)Rxf(x) = 0$, for all $x \in R$. By primeness of R, we have,

$$xf(x) = 0, \text{ for all } x \in I \tag{4.6}$$

Subtracting Equation 4.6 from Equation 4.1, we have $f(x)x - xf(x) = 0$, for all $x \in I$. That is $[f(x), x] = 0$, for all $x \in I$ and hence f is commuting on I. Similarly, we can prove the theorem in case $xf(x) = 0$, for all $x \in I$. □

5 Conclusion remarks

In this paper, we introduced the generalized derivation on Krasner hyperring and obtained some results which establish various important relationships between generalized derivations, derivations, hyperideals, and the structure of hyperrings. Moreover, the theorems demonstrate the significance of generalized derivations in understanding the algebraic structure of hyperrings, particularly in the context of commutativity and interaction with hyperideals. Overall, these results contribute to a deeper understanding of the role and impact of generalized derivations in the study of hyperrings, paving the way for further exploration in hyperring theory.

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