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THE IDEAL OF WEAKLY p-NUCLEAR POLYNOMIALS AND ITS DUAL

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Abstract In this paper, we introduce the concepts of weakly p-nuclear m-homogeneous polynomials and quasi Cohen p-nuclear linear operators and m-homogeneous polynomials. The main finding of this study shows that, under usual conditions, linear functionals in the space of weakly p-nuclear polynomials are represented, by quasi Cohen p-nuclear polynomials.

1 Introduction

The theory of operator ideals has been extended to ideals of nonlinear operators, for example, multilinear operators, homogeneous polynomials, Lipschitz operators..., and the different ways of generating such non-linear operator ideals as well as the study of their properties, we refer to the monographs (see e.g. ([2], [1], [4])). In 1983, Pietsch [6] made leading contributions through his work titled "Ideals of Multilinear Functionals," extending the concept of linear nuclear operators to nonlinear ones, particularly in multilinear and polynomial cases. In this paper, we further this extension by introducing the concept of weakly p-nuclear operators,$1 \leq p < \infty$, to the case of the polynomial recently introduced by Kim J. M [5].

We fix some notations and notions used throughout this paper. For X and Y Banach spaces, we denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X into Y.

Let m be a natural number. A map $P : X \to Y$ is called an $m-$homogeneous polynomial if there exists a unique symmetric $m-$linear operator $\widehat{P} : X \times \cdots \times X \to Y$ such that

$$P(x) = \widehat{P}(x, \overset{(m)}{\dots}, x) \ for\ every\ x \in X.$$

We denote by $\mathcal{P}(^m X; Y)$ the space of all continuous $m-$homogeneous polynomials of degree m from X into Y,This space is a Banach space with the norm

$$\begin{aligned} \|P\| &= \sup\{\|P(x)\| : \quad \|x\| \leq 1\} \\ &= \inf\{C : \|P(x)\| \leq C\|x\|^m, \quad x \in X\} \end{aligned}$$

We denote $\otimes^m X := X \otimes \overset{(m)}{\cdots} \otimes X$ for the m-fold tensor product of X is defined as the vector space, by the set

$$\left\{ u = \sum_{i=1}^{n} \lambda_i x_i \otimes \overset{(m)}{\cdots} \otimes x_i, \quad (\lambda_i)_{i=1}^n \subset \mathbb{K} \text{ and } x_i \in X \right\}.$$

We denote by $\ell_p^w(X)$ the Banach space of all weakly p-summable sequences $(x_n)_n$ in X with the norm

$$\begin{cases} \|(x_n)_n\|_p^w = \sup_{x^* \in B_{X^*}} \left(\sum_{n=1}^{\infty} |\langle x^*, x_n \rangle|^p \right)^{1/p}, & \text{if} \quad 1 \leq p < +\infty, \\ \|(x_n)_n\|_\infty^w = \sup_{x^* \in B_{X^*}} \sup_n |\langle x^*, x_n \rangle|, & \text{if} \quad p = +\infty \end{cases}.$$

A polynomials ideal $\mathcal{Q}$ is a subclass of the class $\mathcal{P}$ of all continuous homogeneous polynomials between Banach spaces such that for all $m \in \mathbb{N}$ and Banach spaces X and Y its components $\mathcal{Q}(^m X; Y) := \mathcal{P}(^m X; Y) \cap \mathcal{Q}$ satisfy the following conditions:

(i) $\mathcal{Q}(^m X; Y)$ is a linear subspace of $\mathcal{P}(X, Y)$ which contains the $m-$homogeneous polynomials of finite type.

(ii) The ideal property: If $P \in \mathcal{Q}(^m X; Y)$, $u \in \mathcal{L}(G, X)$ and $v \in \mathcal{L}(Y, F)$ then $v \circ P \circ u$ is in $\mathcal{Q}(^m G, F)$. If $\|.\|_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{R}^+$ satisfies

(i') $(\mathcal{Q}(^m X; Y), \|.\|_{\mathcal{Q}})$ is a normed space for all Banach spaces X and Y and all m,

(ii') $\|I_m : \mathbb{K} \to \mathbb{K} : I_m(x) = x^m\|_{\mathcal{Q}} = 1$ for all m,

(iii') if $P \in \mathcal{Q}(^m X; Y)$, $u \in \mathcal{L}(G, X)$ and $v \in \mathcal{L}(Y, F)$ then $\|v \circ P \circ u\|_{\mathcal{Q}} \leq \|v\| \|P\|_{\mathcal{Q}} \|u\|^m$,

then $(\mathcal{Q}, \|.\|_{\mathcal{Q}})$ is called a normed polynomial ideal.

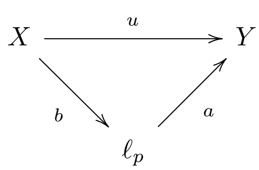
Recall that a bounded linear operator $u : X \to Y$ is weakly p-nuclear $(1 \leq p \leq \infty)$ if can be written in the form $u = \sum_{n=1}^{\infty} x_n^* \otimes y_n$, where $(x_n^*)_n \in \ell_p^w(X^*)$ and $(y_n)_n \in \ell_{p^*}^w(Y)$.

We denote by $\mathcal{N}_{wp}(X, Y)$ the space of all weakly p-nuclear operators from X into Y endowed with the weakly p-nuclear norm

$$\|u\|_{\mathcal{N}_{wp}} := \inf \|(x_n^*)_n\|_p^w \|(y_n)_n\|_{p^*}^w,$$

where the infimum is taken over all such weakly p-nuclear representations of u.

We known that in [5], a bounded linear operator $u : X \to Y$ is weakly p-nuclear $(1 \leq p < \infty)$ if and only if u has a factorization $u = a \circ b$ such that the following diagram commutes:



with $b \in \mathcal{L}(X, \ell_p)$, $a \in \mathcal{L}(\ell_p, Y)$. In this case,

$$\|u\|_{\mathcal{N}_{wp}} = \inf \|a\|.\|b\|,$$

the infimum is taken over all factorization as above.

2 Weakly p-nuclear polynomials

Now, we give the following concept of weakly p-nuclear polynomials, extending the notion of weakly p-nuclear operators introduced in [5].

Definition 2.1. Given $1 \leq p < \infty$. A polynomial $P \in \mathcal{P}(^m X; Y)$ is weakly p-nuclear if it can be written in the form

$$P(x) = \sum_{n=1}^{\infty} a_n(x)^m y_n, \qquad (x \in X)$$

where $(a_n) \subset X^*$ and $(y_n) \in \ell_{p^*}^w(Y)$ such that

$$\sup_{x\in B_X}\left(\sum_{n=1}^{\infty}|\langle a_n,x\rangle|^{mp}\right)^{1/p}<\infty$$

We use $\mathcal{P}_{\mathcal{N}_{wp}}({}^mX;Y)$ to denote the set of all weakly p-nuclear polynomials from X into Y and define a norm on $\mathcal{P}_{\mathcal{N}_{wp}}({}^mX;Y)$ by

$$\|P\|_{\mathcal{N}_{wp}}:=\inf\sup_{x\in B_X}\left(\sum_{n=1}^{\infty}|\langle a_n,x\rangle|^{mp}\right)^{1/p}\sup_{y^*\in B_{Y^*}}\left(\sum_{n=1}^{\infty}|\langle y^*,y_n\rangle|^{p^*}\right)^{1/p^*},$$

where the infimum is taken over all such representations as above.

Theorem 2.2. *For* $1\le p<\infty$, $\left[\mathcal{P}_{\mathcal{N}_{wp}},\|.\|_{\mathcal{N}_{wp}}\right]$ *is a Banach polynomial ideal.*

Proof. Let $P_1,P_2,\ldots\in\mathcal{P}_{\mathcal{N}_{wp}}({}^mX;Y)$ such that $\sum_{k=1}^{\infty}\|P_k\|_{\mathcal{N}_{wp}}<\infty$, and consider such representations that for each k, $P_k=\sum_{n=1}^{\infty}a_{k,n}^m\otimes y_{k,n}$ such that

$$\sup_{y^*\in B_{Y^*}}\left(\sum_{n=1}^{\infty}|\langle y^*,y_{k,n}\rangle|^{p^*}\right)^{1/p^*}\le\left[(1+\epsilon)\|P_k\|_{\mathcal{N}_{wp}}\right]^{1/p^*}$$

$$\sup_{x\in B_X}\left(\sum_{n=1}^{\infty}|a_{k,n}(x)^m|^{p}\right)^{1/p}\le\left[(1+\epsilon)\|P_k\|_{\mathcal{N}_{wp}}\right]^{1/p}.$$

It follows that

$$\begin{aligned}\left\|(\langle y^*,y_{k,n}\rangle)_{n,k=1}^{\infty}\right\|_{p^*}^{w}&:=\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|\langle y^*,y_{k,n}\rangle|^{p^*}\right)^{1/p^*}=\left[\sum_{k=1}^{\infty}\left(\left[\sum_{n=1}^{\infty}|\langle y^*,y_{k,n}\rangle|^{p^*}\right]^{1/p^*}\right)^{p^*}\right]^{1/p^*}\\&\le\left[\sum_{k=1}^{\infty}\left(\left[(1+\epsilon)\|P_k\|_{\mathcal{N}_{wp}}\right]^{1/p^*}\right)^{p^*}\right]^{1/p^*}=(1+\epsilon)^{1/p^*}\left[\sum_{k=1}^{\infty}\|P_k\|_{\mathcal{N}_{wp}}\right]^{1/p^*}\end{aligned}$$

and

$$\begin{aligned}\sup_{x\in B_X}\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|a_{k,n}(x)^m|^p\right)^{1/p}&=\left[\sup_{x\in B_X}\sum_{k=1}^{\infty}\left(\left[\sum_{n=1}^{\infty}|a_{k,n}(x)^m|^p\right]^{1/p}\right)^p\right]^{1/p}\\&\le\left[\sum_{k=1}^{\infty}\left(\sup_{x\in B_X}\left[\sum_{n=1}^{\infty}|a_{k,n}(x)^m|^p\right]^{1/p}\right)^p\right]^{1/p}\\&\le\left[\sum_{k=1}^{\infty}\left(\left[(1+\epsilon)\|P_k\|_{\mathcal{N}_{wp}}\right]^{1/p}\right)^p\right]^{1/p}\\&=(1+\epsilon)^{1/p}\left[\sum_{k=1}^{\infty}\|P_k\|_{\mathcal{N}_{wp}}\right]^{1/p}.\end{aligned}$$

Then, $P = \sum\limits_{k=1}^{\infty} \sum\limits_{n=1}^{\infty} a_{k,n}^{m} \otimes y_{k,n}$ and

$$\begin{aligned} &\sup_{x \in B_X} \left(\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle a_{k,n}, x \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle y^*, y_{k,n} \rangle|^{p^*} \right)^{1/p^*} \\ &\leq (1+\epsilon)^{1/p^*} \left[\sum_{k=1}^{\infty} \|P_k\|_{\mathcal{N}_{wp}} \right]^{1/p^*} (1+\epsilon)^{1/p} \left[\sum_{k=1}^{\infty} \|P_k\|_{\mathcal{N}_{wp}} \right]^{1/p} \\ &= (1+\epsilon) \left[\sum_{k=1}^{\infty} \|P_k\|_{\mathcal{N}_{wp}} \right]. \end{aligned}$$

For every $\epsilon > 0$, follows that

$$\|P\|_{\mathcal{N}_{wp}} \leq \sum_{k=1}^{\infty} \|P_k\|_{\mathcal{N}_{wp}} < \infty$$

Let $Q : X_1 \longrightarrow X$, $O : Y \longrightarrow Y_1$ be a bounded linear operators and a polynomial $P \in \mathcal{P}_{\mathcal{N}_{wp}}\left({}^m X; Y\right)$. We want to show that: $OPQ \in \left[\mathcal{P}_{\mathcal{N}_{wp}}\left({}^m X_1, Y_1\right), \|\cdot\|_{\mathcal{N}_{wp}}\right]$.

$$\begin{aligned} OPQ(x) &= O\left(\sum_{n=1}^{\infty} a_n (Q(x))^m y_j \right) \\ &= \sum_{n=1}^{\infty} a_n (Q(x))^m \cdot O\left(y_n\right) \\ &= \sum_{n=1}^{\infty} \left(Q^* a_n\right)(x)^m \cdot O\left(y_n\right). \end{aligned}$$

Hence,

$$\begin{aligned} &\sup_{x \in B_{X_1}} \left(\sum_{n=1}^{\infty} |\langle x, Q^* a_n \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y_1^*}} \left(\sum_{n=1}^{\infty} |\langle O(y_n), y^* \rangle|^{p^*} \right)^{1/p^*} \\ &= \sup_{x \in B_{X_1}} \left(\sum_{n=1}^{\infty} |\langle Qx, a_n \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y_1^*}} \left(\sum_{n=1}^{\infty} |\langle y_n, O^* y^* \rangle|^{p^*} \right)^{1/p^*} \\ &= \|Q\|^m \|O^*\| \sup_{x \in B_{X_1}} \left(\sum_{n=1}^{\infty} \left| \left\langle \frac{Qx}{\|Q\|}, a_n \right\rangle \right|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y_1^*}} \left(\sum_{n=1}^{\infty} \left| \left\langle y_n, \frac{O^* y^*}{\|O^*\|} \right\rangle \right|^{p^*} \right)^{1/p^*} \\ &= \|Q\|^m \|O\| \sup_{x \in B_{X_1}} \left(\sum_{n=1}^{\infty} \left| \left\langle \frac{Qx}{\|Q\|}, a_n \right\rangle \right|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y_1^*}} \left(\sum_{n=1}^{\infty} \left| \left\langle y_n, \frac{O^* y^*}{\|O^*\|} \right\rangle \right|^{p^*} \right)^{1/p^*} \\ &\leq \|Q\|^m \|O\| \sup_{u \in B_X} \left(\sum_{n=1}^{\infty} |\langle u, a_n \rangle|^{mp} \right)^{1/p} \sup_{\varphi \in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle y_n, \varphi \rangle|^{p^*} \right)^{1/p^*}. \end{aligned}$$

Then, OPQ is weakly p-nuclear and

$$\|OPQ\|_{\mathcal{N}_{wp}} \leq \|Q\|^m \cdot \|P\|_{\mathcal{N}_{wp}} \cdot \|O\|.$$

Then, $\mathcal{P}_{\mathcal{N}_{wp}}$ with the norm $\|\cdot\|_{\mathcal{N}_{wp}}$ is Banach ideal of polynomials. □

3 Connection with tensor product

We consider a tensor norm and associate it with an operator ideal. Let us define a cross norm $w_p(.)$ ($1 \leq p < \infty$) on the tensor product $\otimes^m X \otimes Y$ as follows: if $u \in \otimes^m X \otimes Y$, then

$$w_p(u) = \inf \left\{ \|(\lambda_i)_{i=1}^n\|_\infty \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^n |\langle y^*, y_i \rangle|^{p^*} \right)^{1/p^*} \right\}.$$

where the infimum is taken over all representations of u of the form

$$u = \sum_{i=1}^n \lambda_i x_i \otimes \cdots x_i \otimes y_i$$

with $(x_i)_{i=1}^n \subset X$ and $(y_i)_{i=1}^n \subset Y$.

Proposition 3.1. *ω_p is a reasonable crossnorm on $\otimes^m X \otimes Y$ and $\epsilon \leq \omega_p$, where ϵ denotes the injective tensor norm on $\otimes^m X \otimes Y$.*

Lemma 3.2. *If the norms $\|.\|_{\mathcal{N}_{wp}}$ and $\omega_p(.)$ are equivalent on $\mathcal{P}_f\left({}^m X; Y\right)$, then they coincide on our space.*

Proof. assume that there is a constant $c > 0$ such that $\omega_p(.) \leq c\| \cdot \|_{\mathcal{N}_{wp}}$ on $\mathcal{P}_f\left({}^m X; Y\right)$. Given $P \in \mathcal{P}_f\left({}^m X; Y\right)$ and $\varepsilon > 0$, take an infinite weakly p-nuclear representation

$$P = \sum_{i=1}^\infty a_i^m \otimes y_i$$

such that

$$\sup_{x \in B_X} \left(\sum_{i=1}^\infty |a_i^m(x)|^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^\infty |y^*(y_i)|^{p^*} \right)^{1/p^*} \leq \left(1 + \frac{\varepsilon}{2}\right) \|P\|_{\mathcal{N}_{wp}}.$$

In particular, for each $n \in \mathbb{N}$,

$$\begin{aligned} \omega_p\left(\sum_{i=1}^{n-1} a_i^m \otimes y_i\right) &\leq \sup_{x \in B_X} \left(\sum_{i=1}^{n-1} |a_i^m(x)|^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n-1} |y^*(y_i)|^{p^*} \right)^{1/p^*} \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \|P\|_{\mathcal{N}_{wp}} \end{aligned}.$$

For a sufficiently large $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \left\| \sum_{i=n}^\infty a_i^m \otimes y_i \right\|_{\mathcal{N}_{wp}} &\leq \sup_{x \in B_X} \left(\sum_{i=n}^\infty |a_i^m(x)|^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=n}^\infty |y^*(y_i)|^{p^*} \right)^{1/p^*} \\ &\leq \sup_{x \in B_X} \left(\sum_{i=1}^\infty |a_i^m(x)|^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^\infty |y^*(y_i)|^{p^*} \right)^{1/p^*} \\ &\leq \frac{\varepsilon}{2c} \|P\|_{\mathcal{N}_{wp}} \end{aligned}$$

It follows that

$$\begin{aligned} \omega_p(P) &\leq \omega_p\left(\sum_{i=1}^{n-1} a_i^m \otimes y_i\right) + \omega_p\left(\sum_{i=n}^\infty a_i^m \otimes y_i\right) \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \|P\|_{\mathcal{N}_{wp}} + c \left\| \sum_{i=n}^\infty a_i^m \otimes y_i \right\|_{\mathcal{N}_{wp}} \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \|P\|_{\mathcal{N}_{wp}} + \frac{\varepsilon}{2} \|P\|_{\mathcal{N}_{wp}} = (1 + \varepsilon) \|P\|_{\mathcal{N}_{wp}}. \end{aligned}$$

And as this holds for every $\varepsilon > 0$, the result follows. □

Proposition 3.3. *If $P \in \mathcal{P}_{\mathcal{N}_{wp}}\left({}^{m}X;Y\right)$ and $Q \in \mathcal{L}_f\left(D,X\right)$, then $\omega_p(P\circ Q) \leq \|P\|_{\mathcal{N}_{wp}}\|Q\|^m$.*

Proof. Let $J : Q(D) \to X$ be the formal inclusion, and $\tilde{Q} : D \to Q(D)$ be defined by $\tilde{Q} : u \mapsto \tilde{Q}(u) := Q(u)$. We can write Q as the composition $Q = J \circ \tilde{Q}$. Since each, $Q \circ J \in \mathcal{L}\left(Q(D), Y\right)$, where $\dim Q(D) < \infty$, By Lemma 3.2, we obtain

$$\omega_p(P\circ J) = \|P\circ J\|_{\mathcal{N}_{wp}} \leq \|P\|_{\mathcal{N}_{wp}}\|J\|^m = \|P\|_{\mathcal{N}_{wp}}$$

from which it follows that

$$\omega_p(P\circ Q) = \omega_p(P\circ J\circ \tilde{Q}) \leq \omega_p(P\circ J)\|\tilde{Q}\|^m = \|P\|_{\mathcal{N}_{wp}}\|Q\|^m.$$

□

Proposition 3.4. *If X^* has the bounded approximation property, then $\omega_p(P) = \|P\|_{\mathcal{N}_{wp}}$ on $\mathcal{P}_f\left({}^{m}X;Y\right)$ regardless of the Banach space Y.*

Proof. We provide a proof for $m = 2$, as for other values of m it is similar. Let $P \in \mathcal{P}_f\left({}^{2}X;Y\right)$. We know that $\mathcal{L}_f(X,X;Y)$ is isometrically isomorphic to $\mathcal{L}_f\left(X;\mathcal{L}_f(X;Y)\right)$, thanks the application that associates $\bar{S} \in \mathcal{L}_f\left(X;\mathcal{L}_f(X;Y)\right)$ through $S \in \mathcal{L}_f(X,X;Y)$ for $\bar{S}\left(x_1\right)\left(x_2\right) := S\left(x_1,x_2\right)$. So, $S = \check{P}$ is the symmetric bilinear application associate a P. Note that

$$\bar{S}\left(x_1\right)\left(x_2\right) = S\left(x_1,x_2\right) = S\left(x_2,x_1\right) = \bar{S}\left(x_2\right)\left(x_1\right).$$

Since X^* has the property approximation, by [6, Lemma 10.2.6], given $\epsilon > 0$, there is $\bar{T} \in \mathcal{N}_{w1}(X;X)$ such that $\|\bar{T}\| \leq (1+\epsilon)\gamma$, let $\gamma \geq 1$ and $\overline{ST} = \bar{S}$. So, we have

$$S\left(\bar{T}x_1,x_2\right) = \bar{S}\left(\bar{T}x_1\right)\left(x_2\right) = \overline{ST}\left(x_1\right)\left(x_2\right) = \bar{S}\left(x_1\right)\left(x_2\right) = S\left(x_1,x_2\right).$$

And by the symmetry of S,

$$S\left(x_1,\bar{T}x_2\right) = S\left(\bar{T}x_2,x_1\right) = S\left(x_2,x_1\right) = S\left(x_1,x_2\right).$$

So, $S\circ(\bar{T},\bar{T}) = S$, for all $x \in X$, we have

$$P(x) = S(x,x) = S\circ(\bar{T}x,\bar{T}x) = P(\bar{T}x) = P\circ\bar{T}(x).$$

for all $x \in X$, proving that $S = S\circ(\bar{T},\bar{T})$. Calling Proposition 3.3, we have

$$\begin{aligned}\omega_p(P) &= \omega_p(P\circ\bar{T})\\ &\leq \|P\|_{\mathcal{N}_{wp}}\|\bar{T}\|^2\\ &\leq (1+\epsilon)^2\gamma^2\|P\|_{\mathcal{N}_{wp}}\end{aligned}$$

For each $\epsilon > 0$, follows that $\omega_p(P) \leq \gamma^2\|P\|_{\mathcal{N}_{wp}}$. The result follows from Lemma 3.2. □

Proposition 3.5. *If X^* has the bounded approximation property, then $\mathcal{P}_f\left({}^{m}X;Y\right)$ dense in $\mathcal{P}_{\mathcal{N}_{wp}}\left({}^{m}X;Y\right)$ by the norm $\|\cdot\|_{\mathcal{N}_{wp}}$.*

Proof. Let $P \in \mathcal{P}_{\mathcal{N}_{wp}}\left({}^{m}X;Y\right)$ and given $\epsilon > 0$ consider a representation of P such that

$$\sup_{x\in B_X}\left(\sum_{n=1}^{\infty}|\langle a_n,x\rangle|^{mp}\right)^{1/p}\sup_{y^*\in B_{Y^*}}\left(\sum_{n=1}^{\infty}|\langle y^*,y_n\rangle|^{p^*}\right)^{1/p^*} \leq (1+\epsilon)\|P\|_{\mathcal{N}_{wp}}.$$

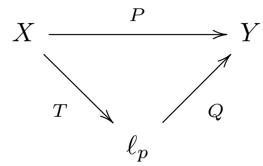
Consider $P_n = \sum\limits_{i=1}^{n} a_i^m \otimes y_i \in \mathcal{P}_f\left({}^{m}X;Y\right)$, then, P_n converges to P in the norm $\|\cdot\|_{\mathcal{N}_{wp}}$. □

The following theorem shows that weakly p-nuclear polynomial has factorization through ℓ_p.

Theorem 3.6. *Let X and Y be Banach spaces, and let $P : X \to Y$ be an m-homogeneous polynomial. Then the following are equivalent:*

(a) *P is weakly p-nuclear.*

(b) *There exists $T \in \mathcal{L}(X; \ell_p)$ and $Q \in \mathcal{P}(^m\ell_p; Y)$ such that its associated m-linear symmetric application $\widehat{Q} \in \mathcal{L}(^m\ell_p; Y)$ is diagonal. the following diagram commutes:*



In this case,

$$\|P\|_{\mathcal{N}_{wp}} = \inf \|Q\| \cdot \|T\|^m$$

where the infimum is taken over all such factorizations of P.

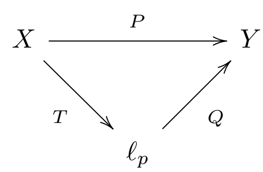
Proof. $(\Rightarrow)$ Assume that P is weakly p-nuclear. Let $(a_n)_n \in \ell_{mp,w}(X^*)$ and $(y_n)_n \in \ell_{p^*,w}(Y)$ such that

$$P = \sum_{n=1}^{\infty} a_n^m \otimes y_n.$$

Consider

$$T: \quad X \to \ell_p, \quad x \mapsto (a_n(x))_n$$
$$Q: \quad \ell_p \to Y, \quad (s_n)_n \mapsto \sum_{n=1}^{\infty} s_n^m y_n.$$

Then, we observe that $\|T\|^m \leq \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^{mp} \right)^{1/p}$ and $\|Q\| \leq \|(y_n)_n\|_{\ell_{p^*,w}(Y)}$. Additionally, the following diagram is commutative



Thus

$$\|P\|_{\mathcal{N}_{wp}} \geq \inf \|Q\| \|T\|^m.$$

$(\Leftarrow) P = Q \circ T$. Let $(f_n)_{n=1}^{\infty}$ be the sequence of coordinate functional a Schauder basis $\{e_n\}_{n=1}^{\infty}$ for ℓ_p, we have

$$T(x) = \sum_{n=1}^{\infty} f_n(T(x)) \cdot e_n = \sum_{n=1}^{\infty} (T^* f_n)(x) \cdot e_n,$$

and

$$\begin{aligned}
P(x) &= Q(T(x)) \\
&= Q\left(\sum_{n=1}^{\infty} (T^* f_n)(x) \cdot e_n\right) \\
&= \widehat{Q}\left(\sum_{n=1}^{\infty} (T^* f_n)(x) \cdot e_n, \overset{(m)}{\ldots}, \sum_{n=1}^{\infty} (T^* f_n)(x) \cdot e_n\right) \\
&= \sum_{n=1}^{\infty} (T^* f_n)(x)^m \widehat{Q}\left(e_n, \overset{(m)}{\ldots}, e_n\right) \\
&= \sum_{n=1}^{\infty} (T^* f_n)(x)^m Q(e_n)
\end{aligned}$$

We set $a_n = (T^* f_n) \in X^*$ and $y_n = Q(e_n) \in Y$. We have the representation $P = \sum\limits_{n=1}^{\infty} a_n^m \otimes y_n$, where $(y_n)_{n=1}^{\infty} \in \ell_{p^*,w}(Y)$, and

$$\sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^{mp} \right)^{1/p} \leq \infty.$$

Showing that P is weakly p-nuclear. Furthermore, we have:

$$\begin{aligned}
\|P\|_{\mathcal{N}_{wp}} &\leq \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle y_n, y^* \rangle|^{p^*} \right)^{1/p^*} \\
&\leq \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, T^* f_n \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle Q(e_n), y^* \rangle|^{p^*} \right)^{1/p^*} \\
&= \|Q\| \, \|T\|^m \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} \left| \left\langle \frac{Tx}{\|T\|}, f_n \right\rangle \right|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} \left| \left\langle e_n, \frac{Q^* y^*}{\|Q^*\|} \right\rangle \right|^{p^*} \right)^{1/p^*} \\
&\leq \|Q\| \, \|T\|^m .
\end{aligned}$$

Then,

$$\|P\|_{\mathcal{N}_{wp}} \leq \inf \|Q\| \|T\|^m.$$

□

4 Dual of $\mathcal{P}_{\mathcal{N}_{wp}}(^m X; Y)$

Definition 4.1. We say that an $m-$homogeneous polynomial $P : X \to Y^*$ is quasi Cohen p-nuclear, $1 \leq p \leq \infty$, if there is a constant $C > 0$ such that for any $(x_i)_{i=1}^n \subset X$ and any $(y_i)_{i=1}^n \subset Y$

$$\sum_{i=1}^{n} |\langle P(x_i), y_i \rangle| \leq C \sup_{a \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle a, x_i \rangle|^{mp} \right)^{\frac{1}{p}} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y^*, y_i \rangle|^{p^*} \right)^{1/p^*}. \tag{4.1}$$

The class of all quasi Cohen p-nuclear $m-$homogeneous polynomials from X into Y^* is denoted by $\mathcal{P}_{\mathcal{QN}_p}(^m X; Y^*)$. Our space is a Banach space with the norm $\|\cdot\|_{\mathcal{QN}_p}$, which is the smallest constant C such that the inequality (4.1) holds.

Denoting by $\mathcal{P}_{p,N}^c(^m X; Y^*)$ the space of Cohen p-nuclear polynomials operators from [1], it is straightforward that $\mathcal{P}_{p,N}^c(^m X; Y^*) \subset \mathcal{P}_{\mathcal{QN}_p}(^m X; Y^*)$ with $\|\cdot\|_{\mathcal{QN}_p} \leq \|\cdot\|_{p,N}$ for every Y, and further, $\mathcal{P}_{p,N}^c(^m X; Y^*) = \mathcal{P}_{\mathcal{QN}_p}(^m X; Y^*)$ isometrically for reflexive Y.

Theorem 4.2. *If X^* has the bounded approximation property, then, for every Banach space Y and $1 \leq p < \infty$, the space $\mathcal{P}_{\mathcal{QN}_p}(^m X^*; Y^*)$ is isometrically isomorphic to $\left[\mathcal{P}_{\mathcal{N}_{wp}}(^m X; Y)\right]^*$.*

Proof. Given $\varphi \in \left[\mathcal{P}_{\mathcal{N}_{wp}}(^m X; Y)\right]^*$.

$$\begin{aligned}
\varphi : \quad \mathcal{P}_{\mathcal{N}_{wp}}(^m X; Y) &\longrightarrow \quad \mathbb{K} \\
a = \sum_{n=1}^{\infty} a_n^m \otimes y_n &\mapsto \quad \varphi(a) = \varphi\left(\sum_{n=1}^{\infty} a_n^m \otimes y_n \right)
\end{aligned}$$

we define

$$\begin{aligned}
P_\varphi : X^* &\longrightarrow Y^* \\
a &\mapsto P_\varphi(a) : \quad Y \longrightarrow \mathbb{K} \\
&\qquad y \mapsto P_\varphi(a)(y) := \varphi(a^m \otimes y)
\end{aligned}$$

In order to prove that $P_\varphi \in \mathcal{P}_{Q\mathcal{N}_p}(^mX^*;Y^*)$, let $n \in \mathbb{N}, x_1^*, \ldots, x_n^* \in X^*$, $y_1, \ldots, y_n \in Y$. So,

$$\begin{aligned}
\left|\sum_{i=1}^{n} P_\varphi(a_i))(y_i)\right| &= \left|\sum_{i=1}^{n} \varphi(a_i^m \otimes y_i)\right| \\
&= \left|\varphi\left(\sum_{i=1}^{n} a_i^m \otimes y_i\right)\right| \\
&\le \|\varphi\| \cdot \left\|\sum_{i=1}^{n} a_i^m \otimes y_i\right\|_{\mathcal{N}_{wp}} \\
&\le \|\varphi\| \cdot \sup_{x \in B_X}\left(\sum_{i=1}^{n} |\langle a, x\rangle|^{mp}\right)^{1/p} \sup_{y^* \in B_{Y^*}}\left(\sum_{i=1}^{\infty} |\langle y^*, y_i\rangle|^{p^*}\right)^{1/p^*}
\end{aligned}$$

proving that P_φ is quasi Cohen p-nuclear, and furthermore, $\|P_\varphi\|_{Q\mathcal{N}_p} \le \|\varphi\|$.

Conversely, given $P \in \mathcal{P}_{Q\mathcal{N}_p}(^mX^*;Y^*)$, define

$$\begin{aligned}
P : X^* &\longrightarrow \quad Y^* \\
a &\mapsto P(a) : \quad Y \longrightarrow \mathbb{K} \\
&\qquad\qquad\quad y \mapsto P(a)(y)
\end{aligned}$$

having in mind that $\otimes^m X^* \otimes Y = \mathcal{P}_f(^mX;Y)$, by the universal property of the tensor product there exists a linear operator $\mathcal{T}_P : \mathcal{P}_f(^mX;Y) \longrightarrow \mathbb{K}$ such that

$$\mathcal{T}_P(a^m \otimes y) = P(a)(y)$$

for all $a \in X^*$ and $y \in Y$. Now we shall prove that $\mathcal{T}_P$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{N}_{wp}}$. Given $\varepsilon > 0$ and $T \in \mathcal{P}_f(^mX;Y)$, by definition of the norm $w_p(.)$ we can choose a representation $A = \sum_{i=1}^{n} a_i^m \otimes y_i$ such that

$$\sup_{x \in B_X}\left(\sum_{i=1}^{n} |\langle a_i, x\rangle|^{mp}\right)^{1/p} \sup_{y^* \in B_{Y^*}}\left(\sum_{i=1}^{n} |\langle y^*, y_i\rangle|^{p^*}\right)^{1/p^*} \le (1+\varepsilon)w_p(T).$$

Therefore,

$$\begin{aligned}
|\mathcal{T}_P(A)| &= \left|\mathcal{T}_P\left(\sum_{i=1}^{n} a_i^m \otimes y_i\right)\right| \\
&= \left|\sum_{i=1}^{n} P(a_i)(y_i)\right| \\
&\le \|P\|_{Q\mathcal{N}_p} \cdot \sup_{x \in B_X}\left(\sum_{i=1}^{n} |\langle a_i, x\rangle|^{mp}\right)^{1/p} \sup_{y^* \in B_{Y^*}}\left(\sum_{i=1}^{n} |\langle y^*, y_i\rangle|^{p^*}\right)^{1/p^*} \\
&\le \|P\|_{Q\mathcal{N}_p}(1+\varepsilon)w_p(A).
\end{aligned}$$

As this holds for arbitrary $\varepsilon > 0$, and the spaces X^* has the bounded approximation property, by Proposition 3.4, we conclude that

$$|\mathcal{T}_P(A)| \le \|S\|_{Q\mathcal{N}_p} \cdot w_p(A) = \|P\|_{Q\mathcal{N}_p} \cdot \|T\|_{\mathcal{N}_{wp}}.$$

So, $\mathcal{T}_P \in \left[\mathcal{P}_f(^mX;Y), \|\cdot\|_{\mathcal{N}_{wp}}\right]^*$ and $\|\mathcal{T}_P\| \le \|P\|_{Q\mathcal{N}_p}$. As $\mathcal{P}_f(^mX;Y)$ is $\|\cdot\|_{\mathcal{N}_{wp}}$-dense in $\mathcal{P}_{\mathcal{N}_{wp}}(^mX;Y)$, there is a unique norm-preserving continuous linear extension φ_S of $\mathcal{T}_P$ to the whole of $\mathcal{P}_{\mathcal{N}_{wp}}\left({}^{m}X;Y\right)$. In particular, $\|\varphi_P\| \leq \|P\|_{\mathcal{QN}_p}$ and for $A = \sum_{i=1}^{\infty} a_i^m \otimes y_i \in \mathcal{P}_{\mathcal{N}_{wp}}\left({}^{m}X;Y\right)$,

$$\varphi_P(A) = \varphi_P\left(\sum_{i=1}^{\infty} a_i^m \otimes y_i\right) = \sum_{i=1}^{\infty} \varphi_P\left(a_i^m \otimes y_i\right)$$
$$= \sum_{i=1}^{\infty} \mathcal{T}_p\left(a_i^m \otimes y_i\right) = \sum_{i=1}^{\infty} P\left(a_i\right)\left(y_i\right).$$

From the expression above, it follows easily that the correspondences $\varphi \mapsto P_\varphi$ and $S \mapsto \varphi_S$ are each other's inverse in the sense that $\varphi_{P_\varphi} = \varphi$ and $P_{\varphi s} = S$ for $\varphi \in \left[\mathcal{P}_{\mathcal{N}_{wp}}\left({}^{m}X;Y\right)\right]^*$ and $P \in \mathcal{P}_{\mathcal{N}_{wp}}\left({}^{m}X;Y\right)$. The equality $\|P_\varphi\|_{\mathcal{QN}_p} = \|\varphi\|$ completes the proof. □

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