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KENMOTSU MANIFOLDS ADMIT A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract This paper deals with the study of Kenmotsu manifolds endowed with a semi-symmetric metric connection. We also study the properties of symmetric and skew-symmetric parallel tensors within the framework of Kenmotsu manifolds. In this series, the properties of Ricci symmetric, weakly symmetric, weakly Ricci symmetric Kenmotsu manifolds and Ricci solitons are derived. Finally, we construct a non-trivial example of Kenmotsu manifold and verify some of our results.

1 Introduction

Boothby and Wang [4], in 1958, initiated the study of odd dimensional manifolds with contact and almost contact structures from topological point of view. Sasaki and Hatakeyama [29] re-investigated the same structures using tensor calculus in 1961. In 1972, Kenmotsu [24] studied a class of almost contact metric manifolds and named as a Kenmotsu manifold. He proved that a Kenmotsu manifold satisfying the condition $R(X,Y)\cdot R=0$ is a manifold of negative curvature -1, where R is the Riemannian curvature tensor of type $(1,3)$ and $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the tangent space $T(M)$. The properties of Kenmotsu manifolds have been studied by many researchers (for instance, see [1], [6]-[10], [15], [20], [22], [33], [36]-[38]).

Eisenhart [16] has initiated to study the properties of second order parallel symmetric tensor in 1923. He proved that if a positive definite Riemannian manifold confesses a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. Levy [25], in 1925, demonstrated that a second order parallel symmetric non-degenerated tensor of type $(0,2)$ in a space form is proportional to the metric tensor. Eisenhart and Levy have studied the properties of second order parallel tensor locally while Sharma [30] studied the properties of same tensor globally based on Ricci identities on complex space forms. Since then, many authors examined the Eisenhart problems of finding the properties of symmetric and skew-symmetric parallel tensors on various spaces and obtained many geometrical results. As illustration, the Eisenhart problems on almost contact metric manifolds were considered by R. Sharma ([31], [32]), on trans-Sasakian manifolds in [11], on LCS-manifolds in [5] and Lorentzian manifolds in [12], and also by others.

The notion of a semi-symmetric linear connection on a differentiable manifold has been introduced by Friedmann and Schouten [17] in 1924. Hayden [21] in 1932, introduced and studied the idea of semi-symmetric linear connection with torsion on a Riemannian manifold. After a long interval, Yano [41] started the systematic study of a semi-symmetric metric connection on a Riemannian manifold in 1970. Since then, the properties of semi-symmetric metric connection on different spaces have studied in ([2], [12]-[14], [19], [23], [26], [28]) and by others.

The above studies motivated us to study the Eisenhart problems, and to investigate the properties of second order parallel symmetric and skew symmetric tensors on a Kenmotsu manifold equipped with a semi-symmetric metric connection. To find our goal, we organize the present work as: After introductory section, we brief the known results of Kenmotsu manifold and Ricci soliton in Section 2. Next section deals with the study of semi-symmetric metric connection on Kenmotsu manifolds and it is proven that the Ricci soliton with respect to the semi-symmetric metric connection on Kenmotsu manifold is expanding. The properties of second order parallel symmetric tensors are studied in Section 4, and consequently we derive some interesting results. Section 5 concerns with the study of second order parallel skew-symmetric tensor. In Section 6, we give an example of Kenmotsu manifold equipped with a semi-symmetric metric connection and validate some of our results.

2 Preliminaries

A $(2n+1)$-dimensional differentiable manifold M of class C^∞ is said to have a (ϕ,ξ,η)-structure or almost contact structure if it admits a tensor field ϕ of endomorphisms of the tangent spaces, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1 \text{ and } \phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

where I denotes the identity transformation [3]. From (2.1), it can be easily shown that $\phi\xi = 0$, $\eta o\phi = 0$ and rank $\phi = 2n$. A Riemannian metric g of type $(0,2)$ is said to be compatible with the almost contact structure (ϕ,ξ,η) if the relation

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y) \tag{2.2}$$

holds for arbitrary vector fields X and Y on M. An almost contact structure (ϕ,ξ,η) equipped with a compatible Riemannian metric g is known as an almost contact metric structure (ϕ,ξ,η,g) and the manifold M endowed with almost contact metric structure is called an almost contact metric manifold. If moreover,

$$\nabla_X\xi = X - \eta(X)\xi, \tag{2.3}$$

holds for all $X \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the collection of all the differentiable vector fields of M, then the manifold M is said to be a Kenmotsu manifold [24]. Here ∇ denotes the Levi-Civita connection of the metric g. For proving our results in next sections, we are going to recall some basic results of Kenmotsu manifolds as:

$$(\nabla_X\eta)(Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$\eta(R(X,Y)Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z), \tag{2.5}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.7}$$

$$S(X,\xi) = -2n\eta(X). \tag{2.8}$$

A Ricci soliton on a Riemannian manifold is a natural generalization of Einstein metric and it has many applications in physics and other branches of science. A triplet (g,V,λ) on a Riemannian manifold (M,g) is said to be a Ricci soliton if it satisfies the condition

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0, \tag{2.9}$$

where g is the Riemannian metric associated with the vector field V, S is a Ricci tensor and λ is a real constant [18]. A Ricci soliton (g,V,λ) is said to be shrinking, expanding or steady if λ is $<,>$ or $=0$, respectively. In [39], authors have studied the properties of Ricci solitons and proved some interesting results.

Analogous to the definition of Ricci soliton corresponding to the Levi-Civita connection ∇ on (M,g), we define the following:

Definition 2.1. Let (M, g) be an n-dimensional Riemannian manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. A triplet (g, V, λ) on (M, g) is said to be a Ricci soliton with respect to the connection $\tilde{\nabla}$ if it satisfies the expression

$$\tilde{\mathcal{L}}_V g + 2\tilde{S} + 2\lambda g = 0. \tag{2.10}$$

Here $\tilde{\mathcal{L}}$ and $\tilde{S}$ denote the Lie derivative operator and the Ricci tensor with respect to the connection $\tilde{\nabla}$, respectively. Here we suppose that $\tilde{S}$ is symmetric on M.

3 Semi-symmetric metric connection

Let M be a $(2n+1)$-dimensional Kenmotsu manifold and ∇ denotes the Levi-Civita connection. A linear connection $\tilde{\nabla}$ on M is said to be semi-symmetric if the torsion tensor $\tilde{T}$ of type $(1, 2)$ defined as

$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$$

satisfies

$$\tilde{T}(X, Y) = \eta(Y)X - \eta(X)Y \tag{3.1}$$

for $X, Y \in \mathfrak{X}(M)$. A semi-symmetric connection $\tilde{\nabla}$ satisfying the relation

$$\tilde{\nabla} g = 0 \tag{3.2}$$

is called a semi-symmetric metric connection. A semi-symmetric connection $\tilde{\nabla}$ is said to be non-metric if $\tilde{\nabla} g \neq 0$. A relation between the semi-symmetric metric and Levi-Civita connections is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi \tag{3.3}$$

for $X, Y \in \mathfrak{X}(M)$ ([41], p. 15). With the help of equations (2.1), (2.3) and (3.3), we can easily observe that

$$(\tilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - \eta(X)\eta(Y) + g(X, Y). \tag{3.4}$$

If R and $\tilde{R}$ denote the curvature tensors with respect to the Levi-Civita connection and semi-symmetric metric connection of M, respectively, then we have

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + \theta(X, Z)Y - \theta(Y, Z)X \\ &\quad + g(X, Z)LY - g(Y, Z)LX \end{aligned} \tag{3.5}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where

$$\theta(X, Y) = g(LX, Y) = (\nabla_X \eta)(Y) - \eta(X)\eta(Y) + \frac{1}{2}g(X, Y) \tag{3.6}$$

is a symmetric tensor of type (0, 2) on M. In consequence of (2.1), (2.4) and (3.6), equation (3.5) assumes the form

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z - 3g(Y, Z)X + 3g(X, Z)Y + 2\eta(Y)\eta(Z)X \\ &\quad - 2\eta(X)\eta(Z)Y + 2\eta(X)g(Y, Z)\xi - 2\eta(Y)g(X, Z)\xi. \end{aligned} \tag{3.7}$$

The contraction of equation (3.7) along the vector field X gives

$$\tilde{S}(Y, Z) = S(Y, Z) - 2(3n - 1)g(Y, Z) + 2(2n - 1)\eta(Y)\eta(Z), \tag{3.8}$$

which gives

$$\tilde{r} = r - 2n(6n - 1), \tag{3.9}$$

where $\tilde{S}$ denotes the Ricci tensor with respect to $\tilde{\nabla}$ and $\tilde{r}$, r represent the scalar curvature with respect to the semi-symmetric metric and Levi-Civita connections, respectively. In view of equations (2.1), (2.8) and (3.8), we can find

$$\tilde{S}(Y, \xi) = -4n\eta(Y). \tag{3.10}$$

Taking covariant derivative of (3.8) and then using (2.1), (2.4), (3.3), (3.4) and (3.8), we obtain

$$(\tilde{\nabla}_X \tilde{S})(Y,Z) = (\nabla_X S)(Y,Z) - \eta(Y)S(X,Z) - \eta(Z)S(X,Y)$$
$$+2(2n-1)\{2\eta(Z)g(X,Y) + \eta(Y)g(X,Z)$$
$$-4\eta(X)\eta(Y)\eta(Z)\} - 2n\{\eta(Y)g(X,Z) + \eta(Z)g(X,Y)\}.$$

In view of (2.1), (2.6), (2.7) and (3.7), we can easily calculate the following:

$$\tilde{R}(\xi,Y)Z = 2\{\eta(Z)Y - g(Y,Z)\xi\} \tag{3.11}$$

and

$$\tilde{R}(X,Y)\xi = 2\{\eta(X)Y - \eta(Y)X\}. \tag{3.12}$$

Equation (3.12) shows that the manifold M equipped with $\tilde{\nabla}$ is regular (i.e., $\tilde{R}(X,Y)\xi \neq 0$).

From equations (2.1), (2.3), (3.3) and the definition of Lie derivative, we have

$$(\tilde{\mathcal{L}}_\xi g)(X,Y) = g(X,\tilde{\nabla}_Y\xi) + g(\tilde{\nabla}_X\xi,Y) = 4\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{3.13}$$

In consequence of equations (2.1), (3.8) and (3.13), equation (2.10) takes the form

$$2S(X,Y) - 4(3n-2)g(X,Y) + 8(n-1)\eta(X)\eta(Y) + 2\lambda g(X,Y) = 0.$$

Setting $X = Y = \xi$ in the above equation and then using equations (2.1) and (3.10), we obtain $\lambda = 4n$. Since $n > 0 \Rightarrow \lambda > 0$. Thus, the Ricci soliton with respect to the semi-symmetric metric connection $\tilde{\nabla}$ on (M,g) is expanding. Hence, we state the following theorem.

Theorem 3.1. *Let (M,g) be a $(2n+1)$-dimensional Kenmotsu manifold endowed with a semi-symmetric metric connection $\tilde{\nabla}$. If (M,g) admits a Ricci soliton (g,V,λ) corresponding to $\tilde{\nabla}$, then (g,V,λ) is expanding.*

4 Second order parallel symmetric tensor with respect to a semi-symmetric metric connection

A symmetric tensor α of type $(0,2)$ is called parallel with respect to the Levi-Civita connection ∇ if $\nabla\alpha = 0$. Analogous to this definition, we define

Definition 4.1. A $(0,2)$ type symmetric tensor α on a Riemannian manifold (M,g) of dimension n is said to be a second order parallel tensor with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if $\tilde{\nabla}\alpha = 0$.

If $\tilde{\nabla}\alpha = 0$ then it can be easily shown that $\tilde{R}(X,Y)\cdot\alpha = 0$. Thus

$$\alpha(\tilde{R}(X,Y)Z,W) + \alpha(Z,\tilde{R}(X,Y)W) = 0$$

for arbitrary vector fields X, Y, Z and W on (M,g). Setting $W = X = \xi$ in the above equation and using (2.1), (3.11) and (3.12), we find that

$$\eta(Z)\alpha(Y,\xi) - g(Y,Z)\alpha(\xi,\xi) + \alpha(Y,Z) - \eta(Y)\alpha(\xi,Z) = 0.$$

Replacing Z by ξ in the above equation and then using (2.1), we obtain

$$\alpha(Y,\xi) = g(Y,\xi)\alpha(\xi,\xi). \tag{4.1}$$

Covariant differentiation of (4.1) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along the vector field X reveals that

$$\alpha(Y,\tilde{\nabla}_X\xi) = g(Y,\tilde{\nabla}_X\xi)\alpha(\xi,\xi) + 2g(Y,\xi)\alpha(\tilde{\nabla}_X\xi,\xi). \tag{4.2}$$

Replacing the vector field Y with $\tilde{\nabla}_X Y$ in (4.1), we get

$$g(\tilde{\nabla}_X Y,\xi)\alpha(\xi,\xi) - \alpha(\tilde{\nabla}_X Y,\xi) = 0. \tag{4.3}$$

In consequence of equations (4.2) and (4.3), we find that

$$\alpha(Y, \tilde{\nabla}_X \xi) = \{g(Y, \tilde{\nabla}_X \xi) + 2g(Y, \xi)g(\tilde{\nabla}_X \xi, \xi)\}\alpha(\xi, \xi). \tag{4.4}$$

With the help of equations (2.1), (2.3), (3.3), (4.1) and (4.4), we conclude that

$$\alpha(X, Y) = g(X, Y)\alpha(\xi, \xi). \tag{4.5}$$

The covariant differentiation of (4.5) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along any arbitrary vector field on (M, g) together with (2.1), (2.3) and (3.3) reveals that $\alpha(\xi, \xi)$ is constant. Thus the equation (4.5) implies that the second order symmetric parallel tensor with respect to the connection $\tilde{\nabla}$ in a regular Kenmotsu manifold (M, g) is a constant multiple of the metric tensor g. Thus, we have the following:

Theorem 4.2. *A Kenmotsu metric on a $(2n + 1)$-dimensional regular Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is irreducible. In other words, the tangent bundle of M does not admit a decomposition $TM = E_1 \times E_2$ parallel with respect to the connection $\tilde{\nabla}$ of g.*

Before going to prove some geometrical results, we have to give the following definition.

Definition 4.3. A Riemannian manifold (M, g) of dimension n is said to be β-symmetric with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if the tensor field β of type $(0, 2)$ defined on M satisfies the condition $\tilde{\nabla}\beta = 0$.

If we replace the tensor field β with the Ricci tensor S, then it becomes the Ricci symmetric. Let us suppose that the Kenmotsu manifold M of dimension $(2n+1)$ equipped with the connection $\tilde{\nabla}$ be θ-symmetric, where θ is a symmetric tensor of type $(0, 2)$ defined by (3.6). With the help of (2.1), (2.4) and Theorem 4.2, we can find that

$$\theta(X, Y) = \theta(\xi, \xi)g(X, Y),$$

where $\theta(\xi, \xi) = -\frac{1}{2}$. Thus we can state:

Corollary 4.4. *If a $(2n+1)$-dimensional Kenmotsu manifold M equipped with $\tilde{\nabla}$ is θ-symmetric, then the symmetric tensor θ is a constant multiple of the metric tensor g.*

In view of (3.5), (3.6) and the above result, we conclude that

$$\tilde{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y, \tag{4.6}$$

which gives

$$\tilde{R}(X, Y)\xi = 0,$$

where equations (2.1) and (2.6) are used. Thus the manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is irregular.

Corollary 4.5. *Every $(2n+1)$-dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is irregular, provided it is θ-symmetric.*

Apart from the conformal curvature tensor, concircular curvature tensor plays an important role in differential geometry and mathematical physics (specially in the theory of relativity and cosmology). A tensor field C of type $(1, 3)$ defined as

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)}\{g(Y, Z)X - g(X, Z)Y\} \tag{4.7}$$

for all X, Y, $Z \in \mathfrak{X}(M)$, is called a concircular curvature tensor [40] of (M, g). Now the contraction of (4.6) along the vector X gives

$$\tilde{S}(Y, Z) = S(Y, Z) + 2ng(Y, Z),$$

which implies

$$\tilde{r} = r + 2n(2n+1). \tag{4.8}$$

If the scalar curvature with respect to the semi-symmetric metric connection $\tilde{\nabla}$ vanishes on (M, g), then the equations (4.6), (4.7) and (4.8) imply that

$$\tilde{R} = C$$

and therefore we state the following corollary:

Corollary 4.6. *If a $(2n+1)$-dimensional Kenmotsu manifold (M, g) endowed with a semi-symmetric metric connection $\tilde{\nabla}$ is θ-symmetric and the scalar curvature with respect to $\tilde{\nabla}$ vanishes, then the curvature tensor with respect to the connection $\tilde{\nabla}$ coincide with the concircular curvature tensor of the manifold.*

If we suppose that the curvature tensor with respect to the semi-symmetric metric connection $\tilde{\nabla}$ vanishes on (M, g), then equation (4.6) reduces to

$$R(X,Y)Z = -\{g(Y,Z)X - g(X,Z)Y\}, \tag{4.9}$$

which shows that the manifold is of constant curvature. Thus we can state:

Corollary 4.7. *If a $(2n+1)$-dimensional Kenmotsu manifold (M, g) equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is θ-symmetric and the curvature tensor with respect to $\tilde{\nabla}$ vanishes, then the manifold is of constant curvature.*

Let us suppose that the Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ be Ricci-symmetric corresponding to the connection $\tilde{\nabla}$, then equations (2.1), (2.8), (3.10) and Theorem 4.2 give

$$\tilde{S}(X,Y) = \tilde{S}(\xi,\xi)g(X,Y), \tag{4.10}$$

where

$$\tilde{S}(\xi,\xi) = -4n.$$

Let $\{e_1, e_2, e_3, \ldots, e_{2n}, e_{2n+1} = \xi\}$ be an orthonormal basis of the tangent space at each point of the manifold M. Setting $X = Y = e_i$ in (4.10) and then summing over i, $1 \le i \le 2n+1$, we find that

$$\tilde{r} = -4n(2n+1), \tag{4.11}$$

which is constant, where $\tilde{r} = \sum_{i=1}^{2n+1} \tilde{S}(e_i, e_i)$. From equation (4.10), it is clear that the Ricci tensor with respect to the connection $\tilde{\nabla}$ is a constant multiple of the metric tensor g and hence the manifold is Einstein with respect to $\tilde{\nabla}$. By considering (4.10), equation (3.8) leads to

$$S(X,Y) = 2(n-1)g(X,Y) - 2(2n-1)\eta(X)\eta(Y), \tag{4.12}$$

which shows that the manifold under consideration is an η-Einstein manifold with scalars $a = 2(n-1)$ and $b = -2(2n-1)$. It is obvious that the scalars a and b in a Kenmotsu manifold (M, g) satisfies the relation $a + b = -2n$ ([24], p - 97). It is well-known that an η-Einstein manifold with either a or b being constant is an Einstein manifold [24]. Thus we can state:

Corollary 4.8. *If a $(2n+1)$-dimensional regular Kenmotsu manifold M endowed with a semi-symmetric metric connection $\tilde{\nabla}$ is Ricci symmetric with respect to $\tilde{\nabla}$, then the manifold is Einstein.*

Corollary 4.9. *Let M be a $(2n+1)$-dimensional Ricci symmetric Kenmotsu manifold endowed with a semi-symmetric metric connection $\tilde{\nabla}$, then the scalar curvature with respect to the connection $\tilde{\nabla}$ is constant.*

In view of equations (3.11) and (4.12), we find that

$$(\nabla_X S)(Y,Z) = 2(2n-1)\eta(Z)\{2\eta(Y)\eta(X) - g(X,Y)\}. \tag{4.13}$$

Interchanging X and Y in (4.13), we get

$$(\nabla_Y S)(X,Z) = 2(2n-1)\eta(Z)\{2\eta(Y)\eta(X) - g(X,Y)\}. \tag{4.14}$$

From equations (4.13) and (4.14), we observe that

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{4.15}$$

Thus the Ricci tensor is of Codazzi type. Hence we state:

Corollary 4.10. *If a $(2n+1)$-dimensional Kenmotsu manifold (M,g) equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is Ricci symmetric with respect to $\tilde{\nabla}$, then the Ricci tensor of M is of Codazzi type.*

Now we define the following definitions as:

Definition 4.11. A non-flat differentiable manifold of dimension $m > 3$ is said to be a weakly symmetric manifold ([34], [35]) if its non-vanishing curvature tensor R satisfies

$$\begin{aligned}(\nabla_X R)(Y,Z)U &= A(X)R(Y,Z)U + B(Y)R(X,Z)U \\ &+C(Z)R(Y,X)U + D(U)R(Y,Z)X + g(R(Y,Z)U,X)P\end{aligned} \tag{4.16}$$

for all the vector fields $X, Y, Z, U \in \mathfrak{X}(M)$, where A, B, C and D are 1-forms (not simultaneously zero) on M and ∇ denotes the Levi-Civita connection of the manifold.

Definition 4.12. A non-flat differentiable manifold of dimension $m > 3$ is said to be a weakly Ricci symmetric manifold ([34], [35]) if its non-vanishing Ricci tensor S satisfies

$$(\nabla_X S)(Y,Z) = \rho(X)S(Y,Z) + \mu(Y)S(X,Z) + \nu(Z)S(X,Y) \tag{4.17}$$

for all the vector fields $X, Y, Z, U \in \mathfrak{X}(M)$, where ρ, μ and ν are 1-forms (not simultaneously zero) on M and ∇ denotes the Levi-Civita connection of the manifold.

We consider a weakly symmetric and weakly Ricci symmetric Kenmotsu manifolds and prove the following results as:

Theorem 4.13. *Let (g,ξ,λ) be a Ricci soliton on a $(2n+1)$-dimensional weakly symmetric Kenmotsu manifold (M,g) admitting a semi-symmetric metric connection $\tilde{\nabla}$. If $\tilde{\mathcal{L}}_\xi g + 2\tilde{S}$ is parallel with respect to the connection $\tilde{\nabla}$, then (g,ξ,λ) on (M,g) is expanding.*

Proof. Let (M,g) be a weakly symmetric Kenmotsu manifold of dimension $(2n+1)$. It is noticed in [27] that

$$\begin{aligned}S(X,Y) &= \frac{1}{1+D(\xi)}\{-[2n+p(\xi)]g(X,Y) + [2nA(X) + B(X) \\ &+p(X)]\eta(Y) + [2nC(Y) - B(Y)]\eta(X)\}\end{aligned} \tag{4.18}$$

and

$$A(\xi) + C(\xi) + D(\xi) = 0, \tag{4.19}$$

provided $1 + D(\xi) \neq 0$ and $p(X) = g(X,P)$. With the help of equation (3.8), equation (4.18) assumes the form

$$\begin{aligned}\tilde{S}(X,Y) &= \frac{1}{1+D(\xi)}\{-[2n+p(\xi)]g(X,Y) + [2nA(X) + B(X) \\ &+p(X)]\eta(Y) + [2nC(Y) - B(Y)]\eta(X)\} \\ &-2(3n-1)g(X,Y) + 2(2n-1)\eta(X)\eta(Y).\end{aligned} \tag{4.20}$$

Let us define $\alpha(X,Y) = (\tilde{\mathcal{L}}_\xi g + 2\tilde{S})(X,Y)$. Then equations (4.20) and (3.13) give

$$\begin{aligned}\alpha(X,Y) &= \frac{2}{1+D(\xi)}\{-[2n+p(\xi)]g(X,Y)+[2nA(X)+B(X) \\ &\quad +p(X)]\eta(Y)+[2nC(Y)-B(Y)]\eta(X\} \\ &\quad -6(n-1)g(X,Y)+2(2n-3)\eta(X)\eta(Y).\end{aligned} \tag{4.21}$$

Let us suppose that $\tilde{\mathcal{L}}_\xi g + 2\tilde{S}$ be parallel with respect to the semi-symmetric metric connection $\tilde{\nabla}$ and therefore from Theorem 4.2, we conclude that

$$\tilde{\mathcal{L}}_\xi g + 2\tilde{S} = \alpha(\xi,\xi)g. \tag{4.22}$$

In view of equations (2.9) and (4.22), we observe that

$$\alpha(\xi,\xi) = -2\lambda. \tag{4.23}$$

Setting $X = Y = \xi$ in (4.21) and then using equation (2.1), we find

$$\alpha(\xi,\xi) = \frac{4n}{1+D(\xi)}\{-1+A(\xi)+C(\xi)\} - 2n.$$

In consequence of (4.19), the above equation takes the form

$$\lambda = 3n.$$

Since $n > 0$, and therefore the above equation gives $\lambda > 0$. Thus, the Ricci soliton (g,ξ,λ) on (M,g) is expanding. Hence the proof is completed. □

Theorem 4.14. *The Ricci soliton (g,ξ,λ) on a $(2n+1)$-dimensional weakly Ricci symmetric Kenmotsu manifold (M,g) endowed with a semi-symmetric metric connection $\tilde{\nabla}$ is expanding.*

Proof. Let (M,g) be a $(2n+1)$-dimensional weakly Ricci symmetric Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. In [27], author proved that a weakly Ricci symmetric Kenmotsu manifold satisfies

$$S(X,Y) = \frac{2n}{1+\nu(\xi)}\{\rho(X)\eta(Y)+\mu(Y)\eta(X)-g(X,Y)\} \tag{4.24}$$

and

$$\rho(\xi)+\mu(\xi)+\nu(\xi) = 0. \tag{4.25}$$

With the help of (3.8), equation (4.24) becomes

$$\begin{aligned}\tilde{S}(X,Y) &= \frac{2n}{1+\nu(\xi)}\{\rho(X)\eta(Y)+\mu(Y)\eta(X)-g(X,Y)\} \\ &\quad -2(3n-1)g(X,Y)+2(2n-1)\eta(X)\eta(Y).\end{aligned} \tag{4.26}$$

Equations (4.22) and (4.26) give

$$\begin{aligned}\alpha(X,Y) &= (\tilde{\mathcal{L}}_\xi g)(X,Y)+2\tilde{S}(X,Y) \\ &= \frac{4n}{1+\nu(\xi)}\{\rho(X)\eta(Y)+\mu(Y)\eta(X)-g(X,Y)\} \\ &\quad -4(3n-2)g(X,Y)+8(n-1)\eta(X)\eta(Y).\end{aligned} \tag{4.27}$$

We suppose that α is symmetric with respect to the semi-symmetric metric connection $\tilde{\nabla}$. Therefore by Theorem 4.2 and equation (2.9), we obtain

$$\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y) \Longleftrightarrow -2\lambda = \alpha(\xi,\xi).$$

Replacing X and Y with ξ in (4.27) and using equations (2.1) and (4.25), we conclude that $\lambda = 4n$. Since $n > 0$ on (M,g) and therefore $\lambda > 0$. Thus the Ricci soliton (g,ξ,λ) on (M,g) is expanding. □

Let us consider that $\tilde{\mathcal{L}}_V g$ be parallel and a regular Kenmotsu manifold is Ricci symmetric for the connection $\tilde{\nabla}$, $(i.e., \ \tilde{\nabla}\tilde{S} = 0)$, where $\tilde{\mathcal{L}}_V g$ denotes the Lie derivative of g along the vector field V with respect to $\tilde{\nabla}$. Here we have two situations regarding the vector field V: the first is that $V \in Span\{\xi\}$ and second $V \perp \xi$. From the analysis point of view, second situation becomes complex and therefore we are going to consider the first case, $i.e.$, $V = \xi$. In a 3-dimensional Kenmotsu manifold, the Ricci tensor S assumes the form

$$S(X,Y) = (\frac{r}{2} + 1)g(X,Y) - (\frac{r}{2} + 3)\eta(X)\eta(Y) \tag{4.28}$$

for all X, $Y \in \mathfrak{X}(M)$. We consider $\alpha(X,Y) = (\tilde{\mathcal{L}}_\xi g + 2\tilde{S})(X,Y)$, then from equations (2.1), (3.8), (3.9), (3.13) and (4.28), we find

$$\alpha(X,Y) = (\tilde{r} + 8)g(X,Y) - (\tilde{r} + 16)\eta(X)\eta(Y). \tag{4.29}$$

The covariant derivative of (4.29) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along the vector field Z gives

$$\begin{aligned}(\tilde{\nabla}_Z\alpha)(X,Y) &= d\tilde{r}(Z)g(\phi X,\phi Y) \\ &\quad -2(\tilde{r} + 16)\{g(\phi X,\phi Z)\eta(Y) + g(\phi Y,\phi Z)\eta(X)\}.\end{aligned} \tag{4.30}$$

Let us suppose that α be a symmetric parallel tensor and $Z = \xi$, $X = Y \in (Span\{\xi\})^{\perp}$, then (4.30) becomes

$$d\tilde{r}(\xi) = 0,$$

which gives $\tilde{r} = constant$. Thus from equations (2.9), (4.28), (4.29) and Theorem 4.2, we can conclude that

$$\alpha(X,Y) = -2\lambda g(X,Y),$$

where $\lambda = 4 > 0$. Hence we can state the following:

Corollary 4.15. *Let a* 3*-dimensional Kenmotsu manifold endowed with a semi-symmetric metric connection* $\tilde{\nabla}$ *be Ricci symmetric and* $\tilde{\mathcal{L}}_\xi g$ *is parallel. Then the Ricci soliton* (g,ξ,λ) *on* M^3 *is expanding.*

Definition 4.16. A vector field $X \in \mathfrak{X}(M)$ on a Riemannian manifold is said to be affine Killing vector field if $\nabla\mathcal{L}_X g = 0$.

Analogous to this definition, we can define as:

Definition 4.17. A vector field $X \in \mathfrak{X}(M)$ on a $(2n + 1)$-dimensional Kenmotsu manifold endowed with a semi-symmetric metric connection $\tilde{\nabla}$ is said to be affine Killing vector field with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if $\tilde{\nabla}\mathcal{L}_X g = 0$.

With the help of Theorem 4.2 and above definition, we have

$$(\mathcal{L}_X g)(Y,Z) = cg(Y,Z),$$

where $c = -2g(\mathcal{L}_X\xi,\xi)$. With the help of equation (3.10), we can easily calculate that $(\mathcal{L}_X\tilde{Q})(\xi) = 0$ and hence $(\mathcal{L}_X\tilde{S})(\xi,\xi) = 0$. Also, $(\mathcal{L}_X\tilde{S})(\xi,\xi) = -2\tilde{S}(\mathcal{L}_X\xi,\xi) = 8ng(\mathcal{L}_X\xi,\xi) = 0$ and thus $g(\mathcal{L}_X\xi,\xi) = 0$. It is obvious that $(\mathcal{L}_X g)(\xi,\xi) = -2g(\mathcal{L}_X\xi,\xi) = 0$ and therefore $(\mathcal{L}_X g)(Y,Z) = 0$. This shows that the vector field X is a Killing vector field. Thus we have the following:

Corollary 4.18. *An affine Killing vector field with respect to the semi-symmetric metric connection* $\tilde{\nabla}$ *on a* $(2n + 1)$*-dimensional regular Kenmotsu manifold* (M,g) *equipped with* $\tilde{\nabla}$ *is Killing.*

5 Second order parallel skew-symmetric tensor with respect to semi-symmetric metric connection

In this section, we study the properties of second order parallel skew-symmetric tensor with respect to a semi-symmetric metric connection $\tilde{\nabla}$ in a regular Kenmotsu manifold. Let us suppose that α be a second order skew symmetric parallel tensor with respect to the semi-symmetric metric connection $\tilde{\nabla}$, $i.e.$, $\alpha(X,Y) = -\alpha(Y,X)$ and $\tilde{\nabla}\alpha = 0$. By considering $\tilde{\nabla}\alpha = 0$, we obtain $R(X,Y)\cdot\alpha = 0$ and hence

$$\alpha(\tilde{R}(W,X)Y,Z) + \alpha(Y,\tilde{R}(W,X)Z) = 0$$

for arbitrary vector fields X, Y, Z and W on (M,g). Setting $W = Y = \xi$ in the above equation and using (2.1), (3.11) and (3.12), we obtain

$$\alpha(X,Z) = \eta(X)\alpha(\xi,Z) - \eta(Z)\alpha(\xi,X). \tag{5.1}$$

Let A be a $(1,1)$-tensor field which is metrically equivalent to α, $i.e.$,

$$\alpha(X,Y) = g(AX,Y). \tag{5.2}$$

From equations (5.1) and (5.2), we conclude that

$$AX = \eta(X)A\xi - g(A\xi,X)\xi. \tag{5.3}$$

Since α is parallel and therefore A is parallel. Thus we have

$$\tilde{\nabla}_X(A\xi) = -2g(A\xi,X)\xi, \tag{5.4}$$

where equations (2.1), (2.3), (3.3) and (5.3) are used. In view of (2.1) and (5.3), we have

$$g(AX,\xi) = \eta(X)g(A\xi,\xi) - g(A\xi,X). \tag{5.5}$$

Putting $X = \xi$ in (5.5), we obtain

$$g(A\xi,\xi) = 0.$$

It is obvious from above discussion that

$$g(\tilde{\nabla}_X(A\xi),A\xi) = 0,$$

which reflects that $||A\xi|| =$ constant on M. The above equations reveal that

$$A^2\xi = -||A\xi||^2\xi. \tag{5.6}$$

Differentiating (5.6) covariantly with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along the vector field X and then using the equations (2.1), (2.3) and (3.3), we find that

$$A^2X = -||A\xi||^2X. \tag{5.7}$$

Now if $||A\xi||^2 \neq 0$, then $J = \frac{1}{||A\xi||}A$ is an almost complex structure on M. Indeed, (J,g) is a Kähler structure on M. Thus the fundamental 2-form is $g(JX,Y) = \lambda\alpha(X,Y)$ with $\lambda = \frac{1}{||A\xi||} =$ constant. But α satisfies the relation (5.1) and thus it is degenerate, which is a contradiction. Therefore $||A\xi|| = 0$ and hence $\alpha = 0$ on M. Thus we state:

Theorem 5.1. *Let (M,g) be a $(2n+1)$-dimensional regular Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. Then there does not exist a second order skew-symmetric parallel vector field α on M.*

6 Example

In this section, we construct an example of Kenmotsu manifold admitting a semi-symmetric metric connection and after that we validate some of our results.

Example 6.1.

Let

$$M^3 = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \in \mathbb{R}\},$$

be a three-dimensional differentiable manifold, where (x, y, z) denotes the standard coordinate of a point in $\mathbb{R}^3$. Let us suppose that

$$e_1 = e^z\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad e_2 = e^z\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right), \quad e_3 = -\frac{\partial}{\partial z}$$

be a set of linearly independent vector fields at each point of the manifold M^3 and therefore they form a basis for the tangent space $T(M^3)$. We also define the Riemannian metric g of the manifold M^3 as $g(e_i, e_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta and $i, j = 1, 2, 3$. Let us consider a 1-form η defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in T(M^3)$ and a tensor field ϕ of type $(1, 1)$ defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

By the linearity properties of ϕ and g, we can easily verify that the following relations

$$\phi^2 X = -X + \eta(X)e_3, \quad \eta(e_3) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

hold for arbitrary vector fields $X, Y \in T(M^3)$. These equations show that for $\xi = e_3$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M^3.

If ∇ represents the Levi-Civita connection with respect to the Riemannian metric g, then with the help of above equations, we can easily calculate that

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

We recall the Koszul's formula as

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \end{aligned}$$

for arbitrary vector fields $X, Y, Z \in T(M^3)$. It is obvious from Koszul's formula that

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3, \quad &&\nabla_{e_1} e_2 = 0, \quad &&\nabla_{e_1} e_3 = e_1, \\ &\nabla_{e_2} e_1 = 0, &&\nabla_{e_2} e_2 = -e_3, &&\nabla_{e_2} e_3 = e_2, \\ &\nabla_{e_3} e_1 = 0, &&\nabla_{e_3} e_2 = 0, &&\nabla_{e_3} e_3 = 0. \end{aligned}$$

From above calculations, we can observe that $\nabla_X \xi = X - \eta(X)\xi$ holds for $\xi = e_3$ and $X \in \mathfrak{X}(M^3)$. Thus the manifold (M^3, g) is a Kenmotsu manifold of dimension 3 and the structure (ϕ, η, ξ, g) denotes the Kenmotsu structure on M^3.
It is obvious from the above results that

$$\begin{aligned} &R(e_1, e_2)e_3 = 0, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_3, e_2)e_2 = -e_3, \quad R(e_3, e_1)e_1 = -e_3, \\ &R(e_2, e_1)e_1 = -e_2, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_2)e_2 = -e_1, \quad R(e_3, e_1)e_2 = 0, \\ &S(e_1, e_1) = -2, \qquad S(e_2, e_2) = -2, \qquad S(e_3, e_3) = -2. \end{aligned}$$

In consequence of (3.3) and above results, we can find that

$$\begin{aligned} &\tilde{\nabla}_{e_1} e_1 = -2e_3, \quad &&\tilde{\nabla}_{e_1} e_2 = 0, \quad &&\tilde{\nabla}_{e_1} e_3 = 2e_1, \\ &\tilde{\nabla}_{e_2} e_1 = 0, &&\tilde{\nabla}_{e_2} e_2 = -2e_3, &&\tilde{\nabla}_{e_2} e_3 = 2e_2, \\ &\tilde{\nabla}_{e_3} e_1 = 0, &&\tilde{\nabla}_{e_3} e_2 = 0, &&\tilde{\nabla}_{e_3} e_3 = 0 \end{aligned}$$

and also the components of torsion tensor $\tilde{T}$ are

$$\tilde{T}(e_i, e_i) = \tilde{\nabla}_{e_i} e_i - \tilde{\nabla}_{e_i} e_i - [e_i, e_i] = 0, \ \ for \ \ i = 1, 2, 3$$
$$\tilde{T}(e_1, e_2) = 0, \quad \tilde{T}(e_1, e_3) = e_1, \quad \tilde{T}(e_2, e_3) = e_2.$$

This shows that $\tilde{T} \neq 0$ and therefore by equation (3.1), we can say that the linear connection defined by (3.3) is a semi-symmetric connection on (M^3, g). By the straight forward calculations, we can also find

$$(\tilde{\nabla}_{e_1} g)(e_i, e_j) = 0, \quad (\tilde{\nabla}_{e_2} g)(e_i, e_j) = 0, \quad (\tilde{\nabla}_{e_3} g)(e_i, e_j) = 0$$

for all $i, j = 1, 2, 3$. This demonstrates that the equation (3.2) is satisfied and hence the linear connection $\tilde{\nabla}$ defined by (3.3) is a semi-symmetric metric connection on M^3. Thus, we can say that the manifold (M^3, g) be a three-dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ defined by (3.3).

With the help of above discussions, we can calculate the curvature and Ricci tensors with respect to the semi-symmetric metric connection $\tilde{\nabla}$ as

$$\tilde{R}(e_1, e_2)e_3 = 0, \ \ \tilde{R}(e_1, e_3)e_3 = -2e_1, \ \ \tilde{R}(e_3, e_2)e_2 = -2e_3,$$
$$\tilde{R}(e_3, e_1)e_1 = -2e_3, \ \ \tilde{R}(e_2, e_1)e_1 = -4e_2, \ \ \tilde{R}(e_2, e_3)e_3 = -2e_2,$$
$$\tilde{R}(e_1, e_2)e_2 = -4e_1, \ \ \tilde{R}(e_3, e_1)e_2 = 0, \quad \tilde{S}(e_1, e_1) = -6,$$
$$\tilde{S}(e_2, e_2) = -6, \quad \tilde{S}(e_3, e_3) = -4, \quad \tilde{r} = -16$$

and other components can be calculated by symmetric and skew-symmetric properties. We can easily observe that the equations (3.7), 3.8), (3.9), (3.10), (3.11), (3.12) and Corollary 4.9 are verified.

Let X and Y be the vector fields of M^3, then $X = X^1 e_1 + X^2 e_2 + X^3 e_3$ and $Y = Y^1 e_1 + Y^2 e_2 + Y^3 e_3$, where X^i and Y^i are scalars for $i = 1, 2, 3$. It is no hard to find that

$$(\tilde{\mathcal{L}}_\xi g)(X, Y) = 4\{X^1 Y^1 + X^2 Y^2\}$$

and

$$\tilde{S}(X, Y) = -6X^1 Y^1 - 6X^2 Y^2 - 4X^3 Y^3.$$

Now,

$$(\tilde{\mathcal{L}}_\xi g)(X, Y) + 2\tilde{S}(X, Y) = -8(X^1 Y^1 + X^2 Y^2 + X^3 Y^3).$$

Since $g(X, Y) = X^1 Y^1 + X^2 Y^2 + X^3 Y^3$, therefore

$$(\tilde{\mathcal{L}}_\xi g)(X, Y) + 2\tilde{S}(X, Y) = -2\lambda g(X, Y)$$

for $\lambda = 4 > 0$. Thus the Ricci soliton (ξ, g, λ) on (M^3, g) is expanding and hence the statement of the Theorem 3.1.

7 Conclusion remarks

An affine connection is typically given in the form of a covariant derivative, which gives a means for taking directional derivatives of vector fields, measuring the deviation of a vector field from being parallel in a given direction. Connections are of central importance in modern geometry in large part because they allow a comparison between the local geometry at one point and the local geometry at another point. Differential geometry embraces several variations on the connection theme, which fall into two major groups: the infinitesimal and the local theory. This manuscript provides the solutions of the Eisenhart problems on Kenmotsu manifolds admitting a semi-symmetric metric connection. Consequently, we proved many interesting results of Kenmotsu manifolds. We define the Ricci solitons with respect to the semi-symmetric metric connection on Kenmotsu manifolds, and proved some of its results. This manuscript may be helpful in the future study of different solitons on contact metric manifolds.

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