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ALGEBRAIC PROPERTIES OF THE SEMIGROUP OF PSEUDO-N WEYL OPERATORS ON A HILBERT SPACE

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Abstract In this paper, for each closed subspace N of a Hilbert space H, we define the pseudo-N Weyl operator on H such that the set $\mathcal{W}_N$ of pseudo-N Weyl operators on H is a regular subsemigroup of the semigroup of bounded operators on H. Then, we characterize Green's relations and natural partial order on $\mathcal{W}_N$. Further, $\mathcal{W}_N$ has several interesting algebraic properties, including $*$-regularity, semisimplicity and unit regularity.

1 Introduction

Studying the structure of regular semigroups is very significant in semigroup theory. It is shown that the semigroup $\mathcal{B}(H)$ of bounded operators on a Hilbert space H under function composition is regular if and only if H is finite-dimensional [16]. Even if H is infinite-dimensional, $\mathcal{B}(H)$ is fascinating as it has many regular subsemigroups, such as the semigroup of Fredholm operators, the semigroup of Weyl operators, the group of invertible bounded operators and the semigroup of finite rank bounded operators. In 1990, E. Krishnan and K. S. S. Nambooripad [8] analysed the semigroup of Fredholm operators, and in 2002, Sherly Valanthara and K. S. S. Nambooripad [16] investigated the semigroup of finite rank bounded operators.

In this article, we utilise the fact that an operator T in $\mathcal{B}(H)$ is regular if and only if the range space $R(T)$ of T is closed in H [16] to construct a relevant regular subsemigroup of $\mathcal{B}(H)$ for each closed subspace of H. It is also known that a bounded operator T on H is a Weyl operator if and only if T can be written as the sum of an invertible bounded operator and a finite rank bounded operator ([8], [15]). Thus, the semigroup $\mathcal{W}$ of Weyl operators on H is given by $\mathcal{W} = \mathcal{H}_I + \mathcal{F}$, where $\mathcal{H}_I$ is the $\mathcal{H}$-class of $\mathcal{B}(H)$ containing the identity operator I on H and $\mathcal{F}$ is the semigroup of finite rank bounded operators on H. Hence, for any closed subspace N of H, we consider the $\mathcal{H}$-class $\mathcal{H}_{P_N}$ of $\mathcal{B}(H)$ containing the projection P_N of H onto N and a particular subset $\mathcal{F}_N = \{T \in \mathcal{F} : N \subseteq Z(T),\ R(T) \subseteq N^\perp\}$ of $\mathcal{F}$, where $Z(T)$ denotes the zero (null) space of T. We have that the $\mathcal{H}$-class $\mathcal{H}_{P_N}$ is a subgroup of $\mathcal{B}(H)$ and can easily see that the set $\mathcal{F}_N$ is a regular subsemigroup of $\mathcal{F}$. So for each closed subspace N of H, we define a bounded operator T on H as a pseudo-N Weyl operator if $T = A + B$ such that $A \in \mathcal{H}_{P_N}$ and $B \in \mathcal{F}_N$. That is, the set $\mathcal{W}_N$ of pseudo-N Weyl operators on H is $\mathcal{W}_N = \mathcal{H}_{P_N} + \mathcal{F}_N$.

Then, we examine various algebraic properties of $\mathcal{W}_N$. $\mathcal{W}_N$ is a regular and a $*$-regular subsemigroup of $\mathcal{B}(H)$. We also identify completely regular elements in $\mathcal{W}_N$ and characterize Green's relations and natural partial order on $\mathcal{W}_N$. Moreover, the principal factors of $\mathcal{W}_N$ are regular semigroups, and $\mathcal{W}_N$ is a completely semisimple semigroup. Finally, we show that $\mathcal{W}_N$ has additional properties if $N^\perp$ is a finite-dimensional subspace of H. Specifically, $\mathcal{W}_N$ is a strongly unit regular semigroup and the triple $\langle \mathcal{W}_N, P_N, ^* \rangle$ is a Baer *-semigroup.

2 Preliminaries

We recall some definitions and results of semigroup theory in the following. For more details, we direct the reader to ([8], [10], [16]).

Definition 2.1. ([7], [9]) Let S and Γ be two non-empty sets. Then S is a Γ-semigroup if there exists a function from $S \times \Gamma \times S$ to S written as $(x, \alpha, y) \mapsto x\alpha y$ such that $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

Let x be an element of a semigroup S. Then, x is regular if there exists $x' \in S$ such that $xx'x = x$. If every element of S is regular, then S is called a regular semigroup. An element x' in S is said to be a generalized inverse of x if $xx'x = x$ and $x'xx' = x'$.

A monoid is a semigroup with identity. If S is a semigroup without identity and $1 \notin S$, then we can extend the binary operation on S to $T = S \cup \{1\}$ by $11 = 1$ and $1x = x1 = x$ for all $x \in S$. Clearly, T is a monoid. Thus, for any semigroup S, the set S^1 defined by

$$S^1 = \begin{cases} S, & \text{if } S \text{ is a monoid} \\ T, & \text{if } S \text{ is not a monoid} \end{cases}$$

is a monoid containing S. Green's relations on a semigroup S are the following five equivalence relations [10]:

$$\begin{aligned}
\mathcal{L} &= \{(a, b) \in S \times S : \mathcal{L}(a) = S^1 a = \mathcal{L}(b)\}, \\
\mathcal{R} &= \{(a, b) \in S \times S : \mathcal{R}(a) = aS^1 = \mathcal{R}(b)\}, \\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R}, \\
\mathcal{D} &= \mathcal{L} \vee \mathcal{R}, \\
\mathcal{J} &= \{(a, b) \in S \times S : \mathcal{J}(a) = S^1 aS^1 = \mathcal{J}(b)\}.
\end{aligned}$$

Note that $\mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. For an equivalence relation ρ on a semigroup S and $a \in S$, let ρ_a denote the ρ-class containing a and S/ρ denote the set of all equivalence classes in S. If $\rho = \mathcal{L}$ or $\mathcal{R}$, then the relation $\leq$ on S/ρ defined by

$$\rho_a \leq \rho_b \iff \rho(a) \subseteq \rho(b)$$

is a partial order on S/ρ.

Proposition 2.2. [10] *Let T be a regular subsemigroup of a semigroup S. Then for $\rho = \mathcal{L}$, $\mathcal{R}$ or $\mathcal{H}$, $\rho(T) = \rho(S) \cap (T \times T)$, where $\rho(S)$ and $\rho(T)$ denote the relation ρ on the semigroups S and T respectively.*

A function $x \mapsto x^*$ on a semigroup S is an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$. A semigroup S with an involution * is called an involution semigroup and is denoted as $(S, ^*)$. A projection e of $(S, ^*)$ is an idempotent in S with $e^* = e$.

Definition 2.3. ([13], [16]) Let x be an element of an involution semigroup S. Then, x is said to be $*$-regular if x has a generalized inverse x' in S such that $(xx')^* = xx'$ and $(x'x)^* = x'x$. If every element of S is $*$-regular, then S is a $*$-regular semigroup.

Definition 2.4. ([14], [8]) An element x of a semigroup S is called completely regular if there exists $x' \in S$ such that $xx'x = x$ and $xx' = x'x$.

Let $E(S)$ be the set of idempotents of a semigroup S. For $f, e \in E(S)$, an E-chain linking f to e is a sequence $f = f_0, f_1, \ldots, f_n = e$ in $E(S)$ such that $f_{k-1}(\mathcal{L} \cup \mathcal{R})f_k$ for $k = 1, 2, \ldots, n$. The length of the E-chain $f_0, f_1, \ldots, f_n$ is defined as n, and the distance $d(f, e)$ between f and e is defined as the length of the shortest E-chain linking f to e ([3], [11], [8]).

Define the relation ω on $E(S)$ by $e\,\omega\,f \iff ef = fe = e$[10]. If $E(S)$ is non-empty, then ω is a partial order on $E(S)$.

Theorem 2.5. ([12], [8]) *The relation $\leq$ on a regular semigroup S defined by*

$$a \leq b \Longleftrightarrow \mathcal{L}_a \leq \mathcal{L}_b \text{ and } a = be \text{ for some } e \in E(\mathcal{L}_a)$$

is a partial order on S. Also, the restriction of $\leq$ to $E(S)$ is the relation ω.

The above-defined relation $\leq$ on a regular semigroup S is called the natural partial order on S.

Theorem 2.6. ([12], [8]) *Let $\leq$ be the natural partial order on a regular semigroup S. Then,*

$$a \leq b \Longleftrightarrow \mathcal{R}_a \leq \mathcal{R}_b \text{ and } a = eb \text{ for some } e \in E(\mathcal{R}_a).$$

Proposition 2.7. [10] *Let a and b be two elements of a regular semigroup S with $a \leq b$. If $a\,\mathcal{R}\,b$ or $a\,\mathcal{L}\,b$, then $a = b$.*

If S is the only ideal of a semigroup S, then S is said to be simple. A semigroup S with zero 0 is 0-simple if $S^2 \neq \{0\}$ and $S, \{0\}$ are the only ideals of S. A non-zero idempotent e of a semigroup S is a primitive if $f\,\omega\,e$ implies $f = e$ for any non-zero idempotent f in S. A simple (0-simple) semigroup S is said to be completely simple (completely 0-simple) if S contains a primitive idempotent.

Let S be a semigroup and $a \in S$. Then, $\mathcal{I}(a) = \mathcal{J}(a) - \mathcal{J}_a$ is an ideal of $\mathcal{J}(a)$ if $\mathcal{I}(a)$ is non-empty [8]. The principal factor $\mathcal{P}(a)$ of S at a is defined by

$$\mathcal{P}(a) = \begin{cases} \mathcal{J}(a)/\mathcal{I}(a), & \text{if } \mathcal{I}(a) \neq \varnothing \\ \mathcal{J}(a), & \text{if } \mathcal{I}(a) = \varnothing. \end{cases}$$

A semigroup S is semisimple if every principal factor of S is simple or 0-simple.

Proposition 2.8. [10] *If S is a regular semigroup, then S is semisimple.*

Definition 2.9. A semigroup S is said to be completely semisimple if the principal factors of S are completely simple or completely 0-simple.

An element x of a monoid S is unit regular if there exists a unit u in S such that $xux = x$. If every element of S is unit regular, then S is a unit regular semigroup ([5], [8]).

Definition 2.10. ([17], [8]) A unit regular semigroup S is called strongly unit regular if for $e, f \in E(S)$ with $e\,\mathcal{D}\,f$, there exists a unit u in S such that $f = ueu^{-1}$.

Let K be an ideal of a semigroup S and $x \in S$. Then, the right K-annihilator of $\{x\}$ is $\mathcal{R}_K(x) = \{y \in S : xy \in K\}$ and the left K-annihilator of $\{x\}$ is $\mathcal{L}_K(x) = \{y \in S : yx \in K\}$. If $k \in S$ commutes with every element of S, then k is called central.

Definition 2.11. Let k be a central idempotent of a semigroup S. Then, the pair $\langle S, k\rangle$ is called a Baer semigroup if for each $x \in S$ there exists $e, f \in E(S)$ such that $\mathcal{R}_K(x) = eS$ and $\mathcal{L}_K(x) = Sf$, where K is the principal ideal of S generated by k.

Definition 2.12. [2] A triple $\langle S, k, ^* \rangle$ is called a Baer *-semigroup if

(i) $(S, ^*)$ is an involution semigroup,

(ii) k is a central projection of S and

(iii) for each $x \in S$, a projection e exists in S such that $\mathcal{R}_K(x) = eS$, where K is the principal ideal of S generated by k.

3 Algebraic Properties of $\mathcal{W}_N$

We have that for each closed subspace N of H, the $\mathcal{H}$-class $\mathcal{H}_{P_N}$ of $\mathcal{B}(H)$ is a subgroup of $\mathcal{B}(H)$ and $\mathcal{H}_{P_N} = \{T \in \mathcal{B}(H) : R(T) = N, Z(T) = N^{\perp}\}$[16]. Let $\mathcal{F}_N = \{T \in \mathcal{F} : N \subseteq Z(T), R(T) \subseteq N^{\perp}\}$. Clearly, $\mathcal{F}_N$ is a subsemigroup of $\mathcal{F}$. Now, let us define the pseudo-N Weyl operator on H.

Definition 3.1. Let N be a closed subspace of a Hilbert space H. Then, a bounded operator T on H is said to be a pseudo-N Weyl operator if T can be expressed as the sum $T = A + B$ such that $A \in \mathcal{H}_{P_N}$ and $B \in \mathcal{F}_N$.

Let $\mathcal{W}_N$ denote the set $\mathcal{H}_{P_N} + \mathcal{F}_N$ of pseudo-N Weyl operators on H. It is obvious that $\mathcal{W}_N$ is a semigroup under function composition. Also, if $T = A+B \in \mathcal{W}_N$, then $Z(B) = N \oplus Z(T)$ and $R(T) = N \oplus R(B)$. Throughout the paper, we assume that if $T = A+B, S = C+D \in \mathcal{W}_N$, then $A, C \in \mathcal{H}_{P_N}$ and $B, D \in \mathcal{F}_N$.

Example 3.2. Let $H = l^2$ and for $n = 1, 2, 3, \ldots, e_n = (0, \ldots, 0, 1, 0, \ldots)$, where 1 occurs only in the n^{th} position. Also, let N be the closure of the subspace generated by $\{e_1, e_3, e_5, e_7, \ldots\}$. Define a bounded operator T on H by for $n = 1, 3, 5, 7, \ldots,$

$$T(e_{2n-1}) = \frac{1}{n} e_{2n+1},$$
$$T(e_{2n+1}) = \frac{1}{n} e_{2n-1},$$

and for $n = 1, 2, 3, 4, \ldots,$

$$T(e_{4n-2}) = \frac{1}{n^2} e_2,$$
$$T(e_{4n}) = \frac{1}{n^2} e_4.$$

Then, T is a pseudo-N Weyl operator on l^2.

Let $\Gamma_N = \{S \in \mathcal{B}(H) : S|_N$ is a bijection on $N, N^\perp$ is invariant under $S\}$ for a closed subspace N of H. Then for $T_1 = A_1 + B_1, T_2 = A_2 + B_2 \in \mathcal{W}_N$ and $S \in \Gamma_N$, $T_2 S T_1 = (A_2 + B_2)S(A_1 + B_1) = A_2 S A_1 + B_2 S B_1$. Since $A_2 S A_1 \in \mathcal{H}_{P_N}$ and $B_2 S B_1 \in \mathcal{F}_N$, we have the following proposition.

Proposition 3.3. *$\mathcal{W}_N$ is a Γ_N-semigroup under the usual function composition.*

For a regular element T of $\mathcal{B}(H)$, let $T^\dagger$ denote the Moore-Penrose inverse of T. We can easily see that the semigroups $\mathcal{H}_{P_N}$ and $\mathcal{F}_N$ contain the Moore-Penrose inverse of each of its elements. Hence if $T = A + B \in \mathcal{W}_N$, then $T^\dagger = A^\dagger + B^\dagger \in \mathcal{W}_N$. That is, the semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H is a regular subsemigroup of the semigroup $\mathcal{B}(H)$ of bounded operators on H.

It is known that if T is a regular element of $\mathcal{B}(H)$, then the range space $R(T^*)$ of T^* is closed in H, where T^* is the adjoint of T [16]. Thus, $\mathcal{H}_{P_N}$ and $\mathcal{F}_N$ are involution semigroups under the involution $T \mapsto T^*$. So $\mathcal{W}_N$ is an involution semigroup. Moreover, for each $T \in \mathcal{W}_N$, $TT^\dagger$ and $T^\dagger T$ are projections in $\mathcal{W}_N$. Hence, we can state the following theorem.

Theorem 3.4. *The semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H is a $*$-regular subsemigroup of $\mathcal{B}(H)$.*

For complementary closed subspaces U and V of H, let P_U^V denote the idempotent in $\mathcal{B}(H)$ with U as range space and V as zero space, and let P_U denote the projection of H onto U. It is obvious that the idempotents in $\mathcal{W}_N$ are of the form $P_N + P_U^V$ such that V is a closed subspace of H containing N and U is a finite-dimensional subspace of $N^\perp$, and the projections in $\mathcal{W}_N$ are of the form $P_N + P_U$ such that U is a finite-dimensional subspace of $N^\perp$. We now identify completely regular elements of $\mathcal{W}_N$.

Theorem 3.5. *Let $T = A + B$ be a pseudo-N Weyl operator on H. Then T is completely regular in $\mathcal{W}_N$ if and only if $N^\perp = R(B) \oplus Z(T)$.*

Proof. Suppose that $T = A + B$ is a completely regular element of $\mathcal{W}_N$. Then there exists $T' = A^\dagger + B'$ in $\mathcal{W}_N$ such that $TT'T = T$ and $TT' = T'T$. If $TT' = P_N + P_U^V$, then $BB' = B'B = P_U^V$ gives $R(B) = R(BB') = U$ and $Z(B) = Z(B'B) = V$. Hence, $H = R(B) \oplus Z(B)$. Further, since $Z(B) = Z(T) \oplus N$, $N^\perp = R(B) \oplus Z(T)$.

Conversely, let $N^{\perp} = R(B) \oplus Z(T)$. Also, let $U = R(B)$ and $V = Z(B)$, then define an operator B_0 on U by $B_0 = B|_U$. Thus, $B' = P_U^V B_0^{-1} P_U^V$ is a generalized inverse of B in $\mathcal{F}_N$ and $(A+B)(A^{\dagger}+B') = (A^{\dagger}+B')(A+B) = P_N + P_U^V$. That is, T is a completely regular element of $\mathcal{W}_N$. □

The following proposition characterizes the partial order on the sets of $\mathcal{L}$ and $\mathcal{R}$ classes of $\mathcal{B}(H)$ containing regular elements of $\mathcal{B}(H)$.

Proposition 3.6. [16] *Let T_1 and T_2 be regular elements of $\mathcal{B}(H)$. Then*

(i) $\mathcal{L}_{T_1} \leq \mathcal{L}_{T_2} \Longleftrightarrow Z(T_2) \subseteq Z(T_1)$.

(ii) $\mathcal{R}_{T_1} \leq \mathcal{R}_{T_2} \Longleftrightarrow R(T_1) \subseteq R(T_2)$.

Therefore, by Proposition 2.2 and Proposition 3.6, we characterize the partial order on the sets of $\mathcal{L}$ and $\mathcal{R}$ classes of $\mathcal{W}_N$ in the following lemma.

Lemma 3.7. *If $T_1 = A_1 + B_1$ and $T_2 = A_2 + B_2$ are pseudo-N Weyl operators, then*

(i) $\mathcal{L}_{T_1} \leq \mathcal{L}_{T_2} \Longleftrightarrow Z(B_2) \subseteq Z(B_1)$.

(ii) $\mathcal{R}_{T_1} \leq \mathcal{R}_{T_2} \Longleftrightarrow R(B_1) \subseteq R(B_2)$.

We frequently use the above lemma without specifying it. The following proposition describes the $\mathcal{D}$ relation on $\mathcal{W}_N$.

Proposition 3.8. *Let $T_1 = A_1 + B_1$ and $T_2 = A_2 + B_2$ be two pseudo-N Weyl operators. Then, $T_1 \,\mathcal{D}\, T_2$ if and only if $\dim R(B_1) = \dim R(B_2)$.*

Proof. Let $T_1 \,\mathcal{D}\, T_2$ in $\mathcal{W}_N$, then there exists an element $T_3 = A_3 + B_3$ in $\mathcal{W}_N$ such that $T_1 \,\mathcal{R}\, T_3 \,\mathcal{L}\, T_2$. Thus, $R(B_1) = R(B_3)$ and $Z(B_3) = Z(B_2)$. Hence, $R(B_1) \cong R(B_2)$. That is, $\dim R(B_1) = \dim R(B_2)$.

Conversely, suppose that $\dim R(B_1) = \dim R(B_2)$. Then there exists an isomorphism B_0 from $R(B_1)$ to $R(B_2)$. Thus if $B_3 = P_{R(B_2)} B_0 B_1$, then $R(B_3) = R(B_2)$ and $Z(B_3) = Z(B_1)$. Hence $(A_1 + B_1) \,\mathcal{L}\, (A_2 + B_3) \,\mathcal{R}\, (A_2 + B_2)$. So $T_1 \,\mathcal{D}\, T_2$ because $A_2 + B_3 \in \mathcal{W}_N$. □

The characterization of the $\mathcal{D}$ relation among the projections in $\mathcal{W}_N$ is in the following.

Corollary 3.9. *Let $P_N + P_U$ and $P_N + P_V$ be two projections in $\mathcal{W}_N$. Then $(P_N + P_U) \,\mathcal{D}\, (P_N + P_V)$ if and only if there exists $B \in \mathcal{F}_N$ such that $B^{\dagger}B = P_U$ and $BB^{\dagger} = P_V$.*

Proof. Suppose that $(P_N + P_U) \,\mathcal{D}\, (P_N + P_V)$ in $\mathcal{W}_N$. Then, by the above proposition, $\dim U = \dim V$. Hence, if B_0 is an isomorphism from U to V and $B = P_V B_0 P_U$, then $B \in \mathcal{F}_N$ and $B^{\dagger} = P_U B_0^{-1} P_V$. So, there exists $B \in \mathcal{F}_N$ such that $B^{\dagger}B = P_U$ and $BB^{\dagger} = P_V$.

Conversely, assume that there exists $B \in \mathcal{F}_N$ such that $B^{\dagger}B = P_U$ and $BB^{\dagger} = P_V$. Then $R(B^{\dagger}) = U$ and $R(B) = V$. Also since $R(B) \cong R(B^{\dagger})$, $\dim U = \dim V$. Hence, $(P_N + P_U) \,\mathcal{D}\, (P_N + P_V)$. □

Next, we proceed to show that the distance between any two $\mathcal{D}$-related idempotents in $\mathcal{W}_N$ is at most 3.

Proposition 3.10. *Let P_1 and P_2 be two idempotents of $\mathcal{W}_N$ with $P_1 \,\mathcal{D}\, P_2$. Then there exists an E-chain $P_1 \,\mathcal{R}\, P_3 \,\mathcal{L}\, P_4 \,\mathcal{R}\, P_2$ in $\mathcal{W}_N$.*

Proof. Let $P_1 = P_N + P_{U_1}^{V_1}$ and $P_2 = P_N + P_{U_2}^{V_2}$ be two idempotents in $\mathcal{W}_N$. If $P_1 \,\mathcal{D}\, P_2$, then by Proposition 3.8, $\dim U_1 = \dim U_2$. Let $\dim (U_1 \cap U_2) = m$ and $\dim U_1 = m + n$. Also, let

$$\begin{aligned} U_1 \cap U_2 &= \operatorname{span}\{c_1, c_2, \ldots, c_m\}, \\ U_1 &= \operatorname{span}\{c_1, c_2, \ldots, c_m, a_1, a_2, \ldots, a_n\} \text{ and} \\ U_2 &= \operatorname{span}\{c_1, c_2, \ldots, c_m, b_1, b_2, \ldots, b_n\}. \end{aligned}$$

So if $W = \operatorname{span}\{a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\}$, then $U_1 + U_2 = U_1 \oplus W = U_2 \oplus W$. That is,

$$H = U_1 \oplus W \oplus (U_1 + U_2)^{\perp} = U_2 \oplus W \oplus (U_1 + U_2)^{\perp}.$$

Thus, if $V = W \oplus (U_1 + U_2)^{\perp}$, then V is a closed subspace of H containing N and $H = U_1 \oplus V = U_2 \oplus V$. Hence,

$$(P_N + P_{U_1}^{V_1})\ \mathcal{R}\ (P_N + P_{U_1}^{V})\ \mathcal{L}\ (P_N + P_{U_2}^{V})\ \mathcal{R}\ (P_N + P_{U_2}^{V_2})$$

is an E-chain in $\mathcal{W}_N$. □

The following lemma identifies the natural partial order on $\mathcal{W}_N$.

Lemma 3.11. *Let $T_1 = A_1 + B_1$ and $T_2 = A_2 + B_2$ be two pseudo-N Weyl operators. Then $T_1 \leq T_2$ if and only if $Z(B_2) \subseteq Z(B_1)$, $A_1 = A_2$ and on a complement of $Z(B_1)$ contained in $N^{\perp}$, $B_1 = B_2$.*

Proof. Let $T_1 \leq T_2$ in $\mathcal{W}_N$, then by Theorem 2.5 $\mathcal{L}_{T_1} \leq \mathcal{L}_{T_2}$ and there exists $P \in E(\mathcal{L}_{T_1})$ such that $T_1 = T_2P$. Hence, $Z(B_2) \subseteq Z(B_1)$ and also if $P = P_N + P_U^V$, then $Z(B_1) = V$ and $A_1 + B_1 = A_2 + B_2P_U^V$. So $A_1 = A_2$ and on U, $B_1 = B_2$. That is, $Z(B_2) \subseteq Z(B_1)$, $A_1 = A_2$ and on a complement of $Z(B_1)$ contained in $N^{\perp}$, $B_1 = B_2$.

Conversely, suppose that $Z(B_2) \subseteq Z(B_1)$, $A_1 = A_2$ and on a complement of $Z(B_1)$ contained in $N^{\perp}$, $B_1 = B_2$. Also, let $V = Z(B_1)$, and if U is a complement of V contained in $N^{\perp}$ with $B_1 = B_2$ on U, then $B_1 = B_2P_U^V$. Thus, $\mathcal{L}_{T_1} \leq \mathcal{L}_{T_2}$ and $A_1 + B_1 = (A_2 + B_2)(P_N + P_U^V)$ for $P_N + P_U^V \in E(\mathcal{L}_{T_1})$. Hence the proof. □

Now, we check the existence of maximal and minimal elements in non-empty subsets of $\mathcal{W}_N$.

Proposition 3.12. *Every non-empty subset of $\mathcal{W}_N$ has a minimal element.*

Proof. Let $\mathcal{S}$ be a non-empty subset of $\mathcal{W}_N$, and let $n = \min\{\dim R(B) : T = A + B \in \mathcal{S}\}$. Then there exists an element $T = A+B$ in $\mathcal{S}$ such that $\dim R(B) = n$. If $T_1 = A_1+B_1 \in \mathcal{S}$ with $T_1 \leq T$, then $\mathcal{R}_{T_1} \leq \mathcal{R}_T$ by Theorem 2.6. Thus, $R(B_1) \subseteq R(B)$ implies $\dim R(B_1) \leq n$. But by the definition of n, $\dim R(B_1) = n$. Hence, $R(B_1) = R(B)$ gives $T_1\, \mathcal{R}\, T$. So by Proposition 2.7, $T_1 = T$. That is, T is a minimal element of $\mathcal{S}$. □

Similarly, we can say the following proposition.

Proposition 3.13. *Let $\mathcal{S}$ be a non-empty subset of $\mathcal{W}_N$. Then, $\mathcal{S}$ has a maximal element if the set $\{\dim R(B) : T = A + B \in \mathcal{S}\}$ has maximum.*

Hence, we obtain the corollary below.

Corollary 3.14. *If $N^{\perp}$ is a finite-dimensional subspace of H, then every non-empty subset of $\mathcal{W}_N$ has both maximal and minimal elements.*

Next, we find the principal ideals of the semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H.

Proposition 3.15. *Let $T = A + B$ be a pseudo-N Weyl operator with $n = \dim R(B)$. Then, the principal ideal generated by T is $\mathcal{J}(T) = \{S = C + D \in \mathcal{W}_N : \dim R(D) \leq n\}$.*

Proof. Let $\mathcal{I}_n = \{S = C + D \in \mathcal{W}_N : \dim R(D) \leq n\}$. Since $\mathcal{I}_n$ is an ideal of $\mathcal{W}_N$ containing T, $\mathcal{J}(T) \subseteq \mathcal{I}_n$.

To prove the reverse inclusion, let $S = C + D \in \mathcal{I}_n$, and if U is a subspace of $N^{\perp}$ with $R(D) \subseteq U$ and $\dim U = n$, then $T\, \mathcal{D}\, (P_N + P_U)$ by Proposition 3.8. Also, since $\mathcal{D} \subseteq \mathcal{J}$, $\mathcal{J}(T) = \mathcal{J}(P_N + P_U)$. Thus, $S \in \mathcal{J}(T)$ because $S = (P_N + P_U)S \in \mathcal{J}(P_N + P_U)$. Hence the proof. □

The above proposition says that if $T = A + B, S = C + D \in \mathcal{W}_N$, then $T\, \mathcal{J}\, S$ if and only if $\dim R(B) = \dim R(D)$. So, we can state the following.

Corollary 3.16. *If $T = A+B$ is a pseudo-N Weyl operator with $n = \dim R(B)$, then the $\mathcal{J}$-class of $\mathcal{W}_N$ containing T is $\mathcal{J}_T = \{S = C + D \in \mathcal{W}_N : \dim R(D) = n\}$. In particular, $\mathcal{D} = \mathcal{J}$ in the semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H.*

The following theorem reveals that every proper ideal of $\mathcal{W}_N$ is a principal ideal of $\mathcal{W}_N$. In addition, $\mathcal{W}_N$ is a principal ideal if and only if $N^{\perp}$ is a finite-dimensional subspace of H.

Theorem 3.17. *Let $\mathcal{I}$ be an ideal of the semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H. Then $\mathcal{I} = \mathcal{W}_N$ or $\mathcal{I} = \mathcal{I}_n$ for some non-negative integer $n \leq \dim N^{\perp}$.*

Proof. Let $\mathcal{I}$ be an ideal of $\mathcal{W}_N$, and let $M = \{\dim R(B) : T = A + B \in \mathcal{I}\}$. Then M may or may not have an upper bound. We have to prove that $\mathcal{I} = \mathcal{I}_n$ if $n = \max M$ and $\mathcal{I} = \mathcal{W}_N$ if M has no upper bound. Assume that M has an upper bound and $n = \max M$. Then there exists an element $T = A + B$ in $\mathcal{I}$ such that $\dim R(B) = n$. Thus, $\mathcal{I}_n = \mathcal{J}(T) \subseteq \mathcal{I}$. For the opposite inclusion, let $S = C + D \in \mathcal{I}$. Then, since $n = \max M$, $\dim R(D) \leq n$ gives $S \in \mathcal{I}_n$. Hence, $\mathcal{I} = \mathcal{I}_n$ if $n = \max M$.

So, suppose that M has no upper bound. Then, for any $T = A + B \in \mathcal{W}_N$, there exists $S = C + D \in \mathcal{I}$ such that $\dim R(B) \leq \dim R(D)$. Hence, $T \in \mathcal{J}(S) \subseteq \mathcal{I}$. Thus, $\mathcal{I} = \mathcal{W}_N$ if M has no upper bound. □

Let us prove that the semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H is completely semisimple. For that, we first figure out the principal factors of $\mathcal{W}_N$. Let $T = A + B$ be a pseudo-N Weyl operator. We have $\mathcal{I}(T) = \mathcal{J}(T) - \mathcal{J}_T$. So if $B = 0$, the zero operator on H, then $\mathcal{I}(T) = \varnothing$ implies

$$\mathcal{P}(T) = \mathcal{J}(T) = \mathcal{H}_{P_N}.$$

And if $B \neq 0$ with $n = \dim R(B)$, then $\mathcal{I}(T) = \mathcal{I}_{n-1}$ gives

$$\mathcal{P}(T) = \mathcal{I}_n/\mathcal{I}_{n-1} = \{\mathcal{I}_{n-1}\} \cup \{\{S = C + D\} : S \in \mathcal{W}_N, \dim R(D) = n\}.$$

Also, since $\mathcal{W}_N$ is a regular semigroup, $\mathcal{W}_N$ is semisimple by Proposition 2.8. Hence, $\mathcal{P}(T)$ is simple if $B = 0$ and $\mathcal{P}(T)$ is 0-simple if $B \neq 0$. Moreover, we show that the principal factors of $\mathcal{W}_N$ are regular semigroups in the following.

Lemma 3.18. *If $T = A + B$ is a pseudo-N Weyl operator, then the principal factor $\mathcal{P}(T)$ of $\mathcal{W}_N$ at T is a regular semigroup.*

Proof. Let $T = A+B$ be a pseudo-N Weyl operator on H. If $B = 0$, then $\mathcal{P}(T) = \mathcal{H}_{P_N}$ implies $\mathcal{P}(T)$ is a regular semigroup. So let $B \neq 0$, and let $n = \dim R(B)$. Then, $\mathcal{I}_{n-1}$ is the zero element of $\mathcal{P}(T)$, and for each $\{S\}$ in $\mathcal{P}(T)$, there exists $\{S^{\dagger}\}$ in $\mathcal{P}(T)$ such that $\{S\}\{S^{\dagger}\}\{S\} = \{SS^{\dagger}S\} = \{S\}$. Thus, $\mathcal{P}(T)$ is a regular semigroup if $B \neq 0$. Hence the proof. □

So, to prove that $\mathcal{W}_N$ is completely semisimple, it is enough to prove that the principal factor $\mathcal{P}(T)$ of $\mathcal{W}_N$ at $T = A + B$ has a primitive element if $B \neq 0$. Because if $B = 0$, then $\mathcal{P}(T) = \mathcal{H}_{P_N}$ is a group. So, assume that $B \neq 0$. Let $U = R(B)$, then $\{P_N + P_U\}$ is a primitive element of $\mathcal{P}(T)$. Because if $\{P_N + P_{U_1}^{V_1}\} \leq \{P_N + P_U\}$ in $\mathcal{P}(T)$, then by Theorem 2.5, Theorem 2.6 and Lemma 3.18,

$$\mathcal{L}_{\{P_N+P_{U_1}^{V_1}\}} \leq \mathcal{L}_{\{P_N+P_U\}} \text{ and } \mathcal{R}_{\{P_N+P_{U_1}^{V_1}\}} \leq \mathcal{R}_{\{P_N+P_U\}} \text{ in } \mathcal{P}(T).$$

Hence,

$$\mathcal{L}_{P_N+P_{U_1}^{V_1}} \leq \mathcal{L}_{P_N+P_U} \text{ and } \mathcal{R}_{P_N+P_{U_1}^{V_1}} \leq \mathcal{R}_{P_N+P_U} \text{ in } \mathcal{W}_N.$$

Thus, $U^{\perp} \subseteq V_1$ and $U_1 \subseteq U$. But by the property of elements in $\mathcal{P}(T)$, $\dim U = \dim U_1$. Hence, $U = U_1$ gives $U^{\perp} = V_1$. That is, $\{P_N + P_{U_1}^{V_1}\} = \{P_N + P_U\}$.

Actually, a slight change in the above discussion shows that every non-zero idempotent in $\mathcal{P}(T)$ is a primitive element of $\mathcal{P}(T)$. Thus, we can now formulate our result below.

Theorem 3.19. *The semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H is a completely semisimple semigroup.*

4 Properties of $\mathcal{W}_N$ if $N^\perp$ is finite-dimensional

From now on, we assume that $N^\perp$ is a finite-dimensional subspace of H. Hence, the semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H is a monoid. Let $\mathcal{G}_N$ denote the group of units in $\mathcal{W}_N$. That is,

$$\mathcal{G}_N = \{T = A + B \in \mathcal{W}_N : R(B) = N^\perp\}.$$

Now, we intend to show that the monoid $\mathcal{W}_N$ of pseudo-N Weyl operators on H is a strongly unit regular semigroup. For this, we first prove that $\mathcal{W}_N$ is a unit regular semigroup in the lemma below.

Lemma 4.1. *The semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H is a unit regular semigroup if $N^\perp$ is a finite-dimensional subspace of H.*

Proof. Let $T = A + B \in \mathcal{W}_N$, then the assertion follows if there exists $G \in \mathcal{G}_N$ such that $TGT = T$. Let $V = Z(B)$, $U = R(B)$ and $W = Z(T)$, then $V = N \oplus W$ and $N^\perp = W \oplus V^\perp$. We define the operator A_0 on N by $A_0 = A|_N$ and define the operator B_0 from $V^\perp$ to U by $B_0 = B|_{V^\perp}$. Then, A_0 and B_0 are isomorphisms. Also, let X be a subspace of $N^\perp$ such that $N^\perp = X \oplus U$. But since $N^\perp$ is a finite-dimensional subspace of H and $V^\perp \cong U$, $W \cong X$. So choose an isomorphism C_0 from W to X, and define G on H by

$$\begin{aligned} G &= A + B + P_X C_0 P_W^{V^\perp + N} \\ &= P_N A_0 P_N + P_U B_0 P_{V^\perp} + P_X C_0 P_W^{V^\perp + N}. \end{aligned}$$

Thus, $G \in \mathcal{G}_N$ and

$$G^{-1} = P_N A_0^{-1} P_N + P_{V^\perp} B_0^{-1} P_U^{X+N} + P_W C_0^{-1} P_X^{U+N}.$$

Moreover,

$$\begin{aligned} TG^{-1}T &= T(P_N A_0^{-1} P_N + P_{V^\perp} B_0^{-1} P_U^{X+N} + P_W C_0^{-1} P_X^{U+N})(P_N A_0 P_N + P_U B_0 P_{V^\perp}) \\ &= (P_N A_0 P_N + P_U B_0 P_{V^\perp})(P_N + P_{V^\perp}) \\ &= P_N A_0 P_N + P_U B_0 P_{V^\perp} \\ &= A + B = T. \end{aligned}$$

Hence the proof. □

Theorem 4.2. *Let $N^\perp$ be a finite-dimensional subspace of H. Then, the monoid $\mathcal{W}_N$ of pseudo-N Weyl operators on H is a strongly unit regular semigroup.*

Proof. Let $N^\perp$ be a finite-dimensional subspace of H, and let $P_1, P_2 \in E(\mathcal{W}_N)$ with $P_1 \mathcal{D} P_2$. Then, by the above lemma, $\mathcal{W}_N$ is a unit regular semigroup. So, to prove that $\mathcal{W}_N$ is a strongly unit regular semigroup, it is enough to find an element G in $\mathcal{G}_N$ such that $P_2 = GP_1G^{-1}$. Let $P_1 = P_N + P_{U_1}^{V_1}$ and $P_2 = P_N + P_{U_2}^{V_2}$, then $U_1 \cong U_2$ by Proposition 3.8. Set $W_1 = Z(P_1)$ and $W_2 = Z(P_2)$, then $N^\perp = U_1 \oplus W_1$ and $N^\perp = U_2 \oplus W_2$. So $W_1 \cong W_2$ because $U_1 \cong U_2$ and $N^\perp$ is a finite-dimensional subspace of H. Hence, there is an isomorphism B from U_1 to U_2 and an isomorphism C from W_1 to W_2. Now, we define G on H by

$$G = P_N + P_{U_2} B P_{U_1}^{V_1} + P_{W_2} C P_{W_1}^{U_1+N}.$$

Then, $G \in \mathcal{G}_N$ and

$$G^{-1} = P_N + P_{U_1} B^{-1} P_{U_2}^{V_2} + P_{W_1} C^{-1} P_{W_2}^{U_2+N}.$$

Furthermore,

$$\begin{aligned} GP_1G^{-1} &= G(P_N + P_{U_1}^{V_1})(P_N + P_{U_1} B^{-1} P_{U_2}^{V_2} + P_{W_1} C^{-1} P_{W_2}^{U_2+N}) \\ &= (P_N + P_{U_2} B P_{U_1}^{V_1} + P_{W_2} C P_{W_1}^{U_1+N})(P_N + P_{U_1} B^{-1} P_{U_2}^{V_2}) \\ &= P_N + P_{U_2}^{V_2} = P_2. \end{aligned}$$

Thus, if $N^\perp$ is a finite-dimensional subspace of H, then $\mathcal{W}_N$ is a strongly unit regular semigroup. □

We will finally show that the triple $\langle \mathcal{W}_N, P_N, ^* \rangle$ is a Baer *-semigroup if $N^\perp$ is a finite-dimensional subspace of H.

Proposition 4.3. *If $N^\perp$ is a finite-dimensional subspace of H, then the pair $\langle \mathcal{W}_N, P_N \rangle$ is a Baer semigroup.*

Proof. Let $N^\perp$ be a finite-dimensional subspace of H. From Proposition 3.15, we have that $\mathcal{I}_0 = \mathcal{H}_{P_N}$ is the principal ideal of $\mathcal{W}_N$ generated by the central projection P_N of H onto N. Let $T = A + B$ be a pseudo-N Weyl operator. Then,

$$\begin{aligned}\mathcal{R}_{\mathcal{I}_0}(T) &= \{S = C + D \in \mathcal{W}_N : TS \in \mathcal{I}_0\}\\ &= \{S = C + D \in \mathcal{W}_N : AC + BD \in \mathcal{I}_0\}\\ &= \{S = C + D \in \mathcal{W}_N : BD = 0\}\\ &= (P_N + P_{Z(T)})\mathcal{W}_N.\end{aligned}$$

Also, since $Z(T) \subseteq N^\perp$, $P_N + P_{Z(T)}$ is a projection in $\mathcal{W}_N$. Moreover,

$$\begin{aligned}\mathcal{L}_{\mathcal{I}_0}(T) &= \{S = C + D \in \mathcal{W}_N : ST \in \mathcal{I}_0\}\\ &= \{S = C + D \in \mathcal{W}_N : DB = 0\}\\ &= \mathcal{W}_N(P_N + P_{R(T)^\perp}),\end{aligned}$$

and $P_N + P_{R(T)^\perp}$ is a projection in $\mathcal{W}_N$ because $R(T)^\perp \subseteq N^\perp$. Hence, the pair $\langle \mathcal{W}_N, P_N \rangle$ is a Baer semigroup. □

Summarizing, we have

(i) $(\mathcal{W}_N, ^*)$ is an involution semigroup.

(ii) P_N is a central projection of $\mathcal{W}_N$.

(iii) For each $T \in \mathcal{W}_N$, there exists a projection $P_N + P_{Z(T)}$ in $\mathcal{W}_N$ such that $\mathcal{R}_{\mathcal{I}_0}(T) = (P_N + P_{Z(T)})\mathcal{W}_N$.

As a result, we can say the following theorem.

Theorem 4.4. *The triple $\langle \mathcal{W}_N, P_N, ^* \rangle$ is a Baer *-semigroup if $N^\perp$ is a finite-dimensional subspace of H.*

5 Conclusion remarks

In this article, we defined the pseudo-N Weyl operator on a Hilbert space H for each closed subspace N of H in such a way that the semigroup $\mathcal{W}_N$ of pseudo-N Weyl operators on H is a regular subsemigroup of the semigroup $\mathcal{B}(H)$ of bounded operators on H. Then, we identified various algebraic properties of $\mathcal{W}_N$, such as $\mathcal{W}_N$ is $*$-regular and completely semisimple. Also, if $N^\perp$ is a finite-dimensional subspace of H, then $\mathcal{W}_N$ is strongly unit regular, and the triple $\langle \mathcal{W}_N, P_N, ^* \rangle$ is a Baer *-semigroup. Besides, we can study the structure of $\mathcal{W}_N$ using the cross-connection theory in the future.

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