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COEFFICIENT ESTIMATES FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE HORADAM POLYNOMIALS AND SUBORDINATION

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Abstract In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions defined by Horadam polynomials in the open unit disk $\mathbb{U}$. Upper bounds for the second and third coefficients of functions in this subclass are found. Also, we solve Fekete-Szegö problem of functions belonging to this family. Our results, which are presented in this paper, generalize and improve those in related works of several earlier authors.

1 Introduction

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote by $\mathcal{S}$ the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. The Koebe one-quarter Theorem [1] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Hence every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) := f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. The class consisting of bi-univalent functions are denoted by Σ.

Lewin [2] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$. Tan [4] obtained the bound for $|a_2|$ namely $|a_2| \leq 1.485$ which is the best-known estimate for functions in the class Σ. The coefficient estimate problem for each of the coefficients $|a_n| (n \in \mathbb{N} - \{1, 2\})$ is still an open problem. For a brief history and interesting examples of functions in the class Σ, see the pioneering work [5]. In fact, this widely-cited work by Srivastava et al. [5] actually revived the study of analytic and bi-univalent functions in recent years, and it has also led to a flood of papers on the subject by (for example) Srivastava et al. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Let the function f and g be analytic in $\mathbb{U}$, we say that the function f subordinate to g if there exists a schwarz function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. This subordination is denoted by $f(z) \prec g(z)$ or $f \prec g$. It is well known that(see [22]), if the function g is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) = g(\mathbb{U})$.

Hörçum and Koçer [23] considered the Horadam polynomials $h_n(x)$, which are given by the following recurrence relation(see also Horadam and Mahon[24]):

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \quad (x \in \mathbb{R},\ n \in \mathbb{N} - \{1, 2\}) \tag{1.3}$$

with $h_1(x) = a$ and $h_2(x) = bx$ for some real constants a, b, p and q.
The generating function of the horadam polynomials $h_n(x)$(see[23]) is given by

$$\Pi(x, z) = \sum_{n=1}^{\infty} h_n(x)z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}. \tag{1.4}$$

Remark 1.1. By selecting the particular values of a, b, p and q, the Horadam polynomial $h_n(x)$ reduces to several polynomials. Some of them are illustrated below:

- Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(x)$.
- Taking $a = 2$ and $b = p = q = 1$, we attain the Lucas polynomials $L_n(x)$.
- Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(x)$.
- Taking $a = b = p = 2$ and $q = 1$, we get the Pell-Lucas polynomials $Q_n(x)$.
- Taking $a = b = 1,\ p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(x)$ of the first kind.
- Taking $a = 1,\ b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(x)$ of the second kind.

More recently, Srivastava *et al.*[25] by using the above horadam polynomials, introduced the following subclass of the bi-univalent functions and obtained estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in this subclass.

Definition 1.2. (see [25]) A function $f(z) \in \Sigma$ is said to be in the class $W_\Sigma(\mu, x)$ where $(0 < \mu \leq 1, z, w \in \mathbb{U})$, if the following subordination conditions are satisfied:

$$\frac{1}{2}\left[\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\mu}}\right] \prec \prod(x, z) + 1 - a$$

and

$$\frac{1}{2}\left[\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\mu}}\right] \prec \prod(x, w) + 1 - a,$$

where a, b is a real constant and the function g is given by(1.2)..

Theorem 1.3. *(see [25]) Let f(z) given by* (1.1) *be in the class* $W_\Sigma(\mu, x)$, $(0 < \mu \leq 1, z, w \in \mathbb{U})$. *Then*

$$|a_2| \leq \frac{2\mu|bx|\sqrt{|bx|}}{\sqrt{\left|\left[(2\mu^2 + \mu + 1)b - (\mu + 1)^2 p\right]bx^2 - (\mu + 1)^2 qa\right|}}$$

and

$$|a_3| \leq \frac{4\mu^2 b^2 x^2}{(\mu+1)^2} + \frac{\mu|bx|}{\mu+1}.$$

In this paper, we introduce a new subclass $W_\Sigma(m,\mu,x)$ of bi-univalent functions of class Σ. We also obtain estimates for the initial Taylor-Maclaurin coefficients a_2 and a_3 for functions in this subclass. Our results for the bi-univalent functions of subclass $W_\Sigma(m,\mu,x)$ would generalize and improve some recent works by Srivastava *et al.* [25],

2 The new subclass $W_\Sigma(m,\mu,x)$

For introduce the subclass $W_\Sigma(m,\mu,x)$, we need to recall salagean differential operator. In [26] salagean defined the differential operator as follows($f \in \mathcal{S}$):

$$D^0 f(z) = f(z) \qquad , \qquad D^1 f(z) = Df(z) = zf'(z)$$

$$D^m f(z) = D(D^{m-1} f(z)) = z + \sum_{k=2}^{\infty} k^m a_k z^k \qquad (m \in \mathbb{N} \cup \{0\}).$$

Now we define the new subclass $W_\Sigma(m,\mu,x)$ as following.

Definition 2.1. For $m \in \mathbb{N} \cup \{0\}$, $0 < \mu \leq 1$, $(z, w \in \mathbb{U})$ and $x \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the subclass $W_\Sigma(m,\mu,x)$, if the following subordination conditions are satisfied:

$$\frac{1}{2}\left[\frac{D^{m+1} f(z)}{D^m f(z)} + \left(\frac{D^{m+1} f(z)}{D^m f(z)}\right)^{\frac{1}{\mu}}\right] \prec \prod(x,z) + 1 - a \tag{2.1}$$

and

$$\frac{1}{2}\left[\frac{D^{m+1} g(w)}{D^m g(w)} + \left(\frac{D^{m+1} g(w)}{D^m g(w)}\right)^{\frac{1}{\mu}}\right] \prec \prod(x,w) + 1 - a, \tag{2.2}$$

where a, b is a real constant and the function g is given by(1.2).

Remark 2.2. There are several choices of the parameters m and μ which would provide interesting subclasses of bi-univalent functions. For example, we have the following special cases:

i. By putting $m = 0$ we have $\dfrac{D^1 f(z)}{D^0 f(z)} = \dfrac{zf'(z)}{f(z)}$. In this case the class $W_\Sigma(m,\mu,x)$ reduces to the class $W_\Sigma(\mu,x)$ in Definition 1.2 which was considered by Srivastava *et al.*[25].

ii. By putting $m = 0, a = 1,\ b = p = 2$ and $q = -1$, the class $W_\Sigma(m,\mu,x)$ reduces to the class $S_\sigma(\mu,x)$ which was considered by Altinkaya *et al.* [27].

iii. By putting $m = 0, \mu = 1$ the class $W_\Sigma(m,\mu,x)$ reduces to the class $S_\Sigma^*(x)$ which was considered by Abirami *et al.* [28].

iv. By putting $m = 1, \mu = 1$, we have $\dfrac{D^2 f(z)}{D^1 f(z)} = 1 + \dfrac{zf''(z)}{f'(z)}$. In this case the class $W_\Sigma(m,\mu,x)$ reduces to the class $K_\Sigma(x)$ which was considered by Orhan *et al.* [29].

Now, we purpose to find the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the subclass $W_\Sigma(m,\mu,x)$, which we introduced. Also we discuss the Fekete-Szegö problem for this family.

3 Coefficient bound for the function class $W_\Sigma(m,\mu,x)$

Theorem 3.1. *Let f(z) given by* (1.1) *be in the class* $W_\Sigma(m,\mu,x), (m\in\mathbb{N}\cup\{0\}, 0<\mu\le 1)$. *Then,*

$$|a_2|\le \frac{2\mu|bx|\sqrt{|bx|}}{\sqrt{\left|\left[((4.3^m-2.4^m)\mu^2+(4.3^m-3.4^m)\mu+4^m)b-(\mu+1)^2 4^m p\right]bx^2-(\mu+1)^2 4^m aq\right|}}$$

and

$$|a_3|\le \frac{4\mu^2\ b^2x^2}{4^m(1+\mu)^2}+\frac{\mu|bx|}{3^m(1+\mu)}. \tag{3.1}$$

Proof. Let $f\in W_\Sigma(m,\mu,x)$ then, there are two analytic functions $u,v:\mathbb{U}\longrightarrow\mathbb{U}$ such that $u(z)=u_1z+u_2z^2+u_3z^3+...$ and $v(z)=v_1w+v_2w^2+v_3w^3+...\quad (z,w\in\mathbb{U})$, where $u(0)=v(0)=0,\ |u(z)|<1$ and $|v(w)|<1$, such that

$$\frac{1}{2}\left[\frac{D^{m+1}f(z)}{D^mf(z)}+\left(\frac{D^{m+1}f(z)}{D^mf(z)}\right)^{\frac{1}{\mu}}\right]=\prod(x,u(z))+1-a$$

and

$$\frac{1}{2}\left[\frac{D^{m+1}g(w)}{D^mg(w)}+\left(\frac{D^{m+1}g(w)}{D^mg(w)}\right)^{\frac{1}{\mu}}\right]=\prod(x,v(w))+1-a$$

or equivallently,

$$\frac{1}{2}\left[\frac{D^{m+1}f(z)}{D^mf(z)}+\left(\frac{D^{m+1}f(z)}{D^mf(z)}\right)^{\frac{1}{\mu}}\right]=1+h_1(x)-a+h_2(x)u(z)+h_3(x)u^2(z)+... \tag{3.2}$$

and

$$\frac{1}{2}\left[\frac{D^{m+1}g(w)}{D^mg(w)}+\left(\frac{D^{m+1}g(w)}{D^mg(w)}\right)^{\frac{1}{\mu}}\right]=1+h_1(x)-a+h_2(x)v(w)+h_3(x)v^2(w)+... \tag{3.3}$$

From these last equations (3.2) ,(3.3), we obtain

$$\frac{1}{2}\left[\frac{D^{m+1}f(z)}{D^mf(z)}+\left(\frac{D^{m+1}f(z)}{D^mf(z)}\right)^{\frac{1}{\mu}}\right]=1+h_2(x)u_1z+[h_2(x)u_2+h_3(x)u_1^2]z^2+... \tag{3.4}$$

and

$$\frac{1}{2}\left[\frac{D^{m+1}g(w)}{D^mg(w)}+\left(\frac{D^{m+1}g(w)}{D^mg(w)}\right)^{\frac{1}{\mu}}\right]=1+h_2(x)v_1w+[h_2(x)v_2+h_3(x)v_1^2]w^2+... \tag{3.5}$$

It is well known that if, $|u(z)| < 1$ and $|v(w)| < 1$ for $z, w \in \mathbb{U}$, then $|u_i| \leq 1$ and $|v_i| \leq 1$ for all $i \in \mathbb{N}$.

Comparing the corresponding coefficients in (3.4) and (3.5) we have

$$\frac{\mu+1}{2\mu}2^m a_2 = h_2(x)u_1, \tag{3.6}$$

$$\frac{\mu+1}{2\mu}(3^m 2a_3 - 4^m a_2^2) + \frac{1-\mu}{4\mu^2}4^m a_2^2 = h_2(x)u_2 + h_3(x)u_1^2, \tag{3.7}$$

$$-\frac{\mu+1}{2\mu}2^m a_2 = h_2(x)v_1, \tag{3.8}$$

and

$$\frac{\mu+1}{2\mu}\left[(4.3^m - 4^m)a_2^2 - 3^m a_3\right] + \frac{1-\mu}{4\mu^2}4^m a_2^2 = h_2(x)v_2 + h_3(x)v_1^2. \tag{3.9}$$

From (3.6) and (3.8) we can see that

$$u_1 = -v_1 \tag{3.10}$$

and

$$\frac{(\mu+1)^2}{2\mu^2}4^m a_2^2 = [h_2(x)]^2(u_1^2 + v_1^2). \tag{3.11}$$

If we add (3.7) and (3.9) we have

$$\frac{(4.3^m - 2.4^m)\mu^2 + (4.3^m - 3.4^m)\mu + 4^m}{2\mu^2}a_2^2$$

$$= h_2(x)(u_2 + v_2) + h_3(x)(u_1^2 + v_1^2), \tag{3.12}$$

by using (3.11) and (3.12), we get

$$\left[\left((4.3^m - 2.4^m)\mu^2 + (4.3^m - 3.4^m)\mu + 4^m\right)[h_2(x)]^2 - (\mu+1)^2 4^m h_3(x)\right]a_2^2$$

$$= 2\mu^2[h_2(x)]^3(u_2 + v_2)$$

which gives

$$|a_2| \leq \frac{2\mu|bx|\sqrt{|bx|}}{\sqrt{\left|\left[\left((4.3^m - 2.4^m)\mu^2 + (4.3^m - 3.4^m)\mu + 4^m\right)b - (\mu+1)^2 4^m p\right]bx^2 - (\mu+1)^2 4^m aq\right|}}.$$

Next if we substract (3.9) from (3.7), we can easily see that

$$\frac{2(\mu+1)3^m}{\mu}(a_3 - a_2^2) = h_2(x)(u_2 - v_2) + h_3(x)(u_1^2 - v_1^2). \tag{3.13}$$

In view of (3.10) and (3.11), we get from(3.13)

$$a_3 = \frac{2\mu^2[h_2(x)]^2(u_1^2+v_1^2)}{4^m(\mu+1)^2} + \frac{\mu h_2(x)(u_2-v_2)}{2(\mu+1)3^m}.$$

Thus applying (1.3), we obtain

$$|a_3| \le \frac{4\mu^2b^2x^2}{4^m(\mu+1)^2} + \frac{\mu|bx|}{(\mu+1)3^m}.$$

This completes the proof of Theorem 3.1. □

4 The Fekete-Szegö problem for the class $W_\Sigma(m,\mu,x)$

Theorem 4.1. *Let f(z) given by* (1.1) *be in the class* $W_\Sigma(m,\mu,x)$ *where* $\nu \in \mathbb{R}$. *Then*

$$|a_3 - \nu {a_2}^2| \le \begin{cases} \dfrac{\mu|bx|}{(\mu+1)3^m} & (|\nu-1| \le T) \\[2ex] \dfrac{4\mu^2|1-\nu||bx|^3}{P} & (|\nu-1| \ge T), \end{cases}$$

where:

$$P = \left|\left[\left((4.3^m-2.4^m)\mu^2+(4.3^m-3.4^m)\mu+4^m\right)b-(\mu+1)^24^mp\right]bx^2-(\mu+1)^24^maq\right|,$$

$$T = \frac{1}{4\mu(\mu+1)3^m}\left|\left((4.3^m-2.4^m)\mu^2+(4.3^m-3.4^m)\mu+4^m\right)-(\mu+1)^24^m\left[\frac{pbx^2+aq}{b^2x^2}\right]\right|.$$

Proof. It follows from (3.12) and (3.13) that

$$a_3-\nu{a_2}^2 = \frac{2\mu^2[h_2(x)]^3(1-\nu)(u_2+v_2)}{\left[(4.3^m-2.4^m)\mu^2+(4.3^m-3.4^m)\mu+4^m\right]\left[h_2(x)\right]^2-(\mu+1)^24^m.h_3(x)}$$

$$+\frac{\mu h_2(x)(u_2-v_2)}{2(\mu+1)3^m}$$

$$= h_2(x)\left[\left(\psi(\nu,x)+\frac{\mu}{2(\mu+1)3^m}\right)u_2+\left(\psi(\nu,x)-\frac{\mu}{2(\mu+1)3^m}\right)v_2\right],$$

where

$$\psi(\nu,x) = \frac{2\mu^2[h_2(x)]^2(1-\nu)}{\left[(4.3^m-2.4^m)\mu^2+(4.3^m-3.4^m)\mu+4^m\right]\left[h_2(x)\right]^2-(\mu+1)^24^m.h_3(x)}.$$

According to (1.3), we find that

$$|a_3-\nu{a_2}^2| = \begin{cases} \dfrac{\mu|b(x)|}{(\mu+1)3^m} & \left(0\le|\psi(\nu,x)|\le\dfrac{\mu}{2(\mu+1)3^m}\right) \\[2ex] 2|h_2(x)|.|\psi(\nu,x)| & \left(|\psi(\nu,x)|\ge\dfrac{\mu}{2(\mu+1)3^m}\right). \end{cases}$$

After some computations we have proved the statement of Theorem 4.1. □

5 A Set of Corollaries and Consequences

By taking $m = 0$ in Theorem 3.1, we have the following result which obtained by Srivastava *et al.* [25, Theorem 2.].

Corollary 5.1. *If $f(z)$ of the form (1.1) be in the class $W_\Sigma(\mu, x)$. Then,*

$$|a_2| \leq \frac{2\mu|bx|\sqrt{|bx|}}{\sqrt{|[(2\mu^2+\mu+1)b-(\mu+1)^2p]bx^2-(\mu+1)^2aq|}}$$

and

$$|a_3| \leq \frac{4\mu^2b^2x^2}{(\mu+1)^2} + \frac{\mu|bx|}{\mu+1}.$$

By taking $a = 1,\ b = p = 2$ and $q = -1$ in Corollary 5.1, we have the following result which obtained by Altinkaya *et al.* [27, Theorem 2.1].

Corollary 5.2. *If $f(z)$ given by (1.1) be in the class $S_\sigma(\mu, x)$. Then,*

$$|a_2| \leq \frac{4\mu x\sqrt{2x}}{\sqrt{4(\mu^2-\mu)x^2+(\mu+1)^2}}$$

and

$$|a_3| \leq \frac{16\mu^2x^2}{(\mu+1)^2} + \frac{2\mu x}{\mu+1}.$$

By taking $\mu = 1$ in Theorem 3.1, we have the following result.

Corollary 5.3. *Let $f(z)$ given by (1.1) be in the class $W_\Sigma(m, x)$. Then,*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[3^m(2b)-4^m(b+p)]bx^2-4^maq|}}$$

and

$$|a_3| \leq \frac{b^2x^2}{4^m} + \frac{|bx|}{2.3^m}.$$

By taking $m = 1$ in Corollary 5.3, we obtain the following result which obtained by Orhan *et al.* [29, Corollary 1].

Corollary 5.4. *If $f(z)$ of the form (1.1) be in the class $K_\Sigma(x)$. Then,*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(2b-4p)bx^2-4aq|}}$$

and

$$|a_3| \leq \frac{b^2x^2}{4} + \frac{|bx|}{6}.$$

By taking $\mu = 1$ in Theorem 4.1 we have the following result.

Corollary 5.5. *Let f given by* (1.1) *be in the class* $W_\Sigma(m,x)$*, where* $\nu\in\mathbb{R}$*. Then,*

$$|a_3-\nu {a_2}^2|\leq\begin{cases}\dfrac{|bx|}{2.3^m} & \big(|\nu-1|\leq S\big)\\[2ex] \dfrac{|1-\nu||bx|^3}{\big|\big[(2.3^m-4^m)b-4^mp\big]bx^2-4^maq\big|} & \big(|\nu-1|\geq S\big),\end{cases}$$

where:

$$S=\frac{\big|\big[(2.3^m-4^m)b-4^mp\big]bx^2-4^maq\big|}{2.3^mb^2x^2}.$$

By putting $m=0$ in Corollary 5.5 we have the following result which obtained by Srivastava *et al.* [25, Corollary 3].

Corollary 5.6. *Let f given by* (1.1) *be in the class* $W_\Sigma(x)$*, where* $\nu\in\mathbb{R}$*. Then,*

$$|a_3-\nu {a_2}^2|\leq\begin{cases}\dfrac{|bx|}{2} & \Big(|\nu-1|\leq\dfrac{|(b-p)bx^2-aq|}{2b^2x^2}\Big)\\[2ex] \dfrac{|1-\nu||bx|^3}{|(b-p)bx^2-aq|} & \Big(|\nu-1|\geq\dfrac{|(b-p)bx^2-aq|}{2b^2x^2}\Big).\end{cases}$$

By taking $m=1$ in Corollary 5.5 we have the following result which obtained by Orhan *et al.* [29, Corollary 1].

Corollary 5.7. *Let f given by* (1.1) *be in the class* $K_\Sigma(x)$*, where* $\nu\in\mathbb{R}$*. Then,*

$$|a_3-\nu {a_2}^2|\leq\begin{cases}\dfrac{|bx|}{6} & \Big(|\nu-1|\leq\dfrac{\big|(2b-4p)bx^2-4aq\big|}{6b^2x^2}\Big)\\[2ex] \dfrac{|1-\nu||bx|^3}{\big|(2b-4p)bx^2-4aq\big|} & \Big(|\nu-1|\geq\dfrac{\big|(2b-4p)bx^2-4aq\big|}{6b^2x^2}\Big).\end{cases}$$

Putting $\nu=1$ in Theorem 4.1, we have the following results.

Corollary 5.8. *Let* $f\in\mathcal{A}$ *be in the family* $W_\Sigma(m,\mu,x)$*. Then*

$$|a_3-{a_2}^2|\leq\frac{\mu|bx|}{(\mu+1)3^m}.$$

If we set $b=2, m=0$ in Corollary 5.8, we have the following result which obtained by Altinkaya *et al.* [27, Corollary 3.2.].

Corollary 5.9. *If the function* $f\in\mathcal{A}$ *be in the family* $S_\sigma(\mu,x)$*.Then*

$$|a_3-{a_2}^2|\leq\frac{2\mu x}{\mu+1}.$$

6 Conclusion remarks

This paper aims to obtain upper bounds for the second and third coefficients of functions in the subclass $W_\Sigma(m,\mu,x)$ of analytic and bi-univalent functions defined by Horadam polynomials in the open unit disk $\mathbb{U}$. Also, we solve Fekete-Szegö problem of functions belonging to this family. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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