Palestine Journal of Mathematics

Vol 13(4)(2024) , 657–663

INEQUALITIES INVOLVING NORM AND NUMERICAL RADIUS OF HILBERT SPACE OPERATORS

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 47A12; Secondary 47A30, 47A63.

Keywords and phrases: Bounded linear operators, numerical radius, operator norm, inequality.

The authors wish to thank the reviewers and editor for their constructive critiques and valuable insights that elevated the standard of our paper.

Abstract This paper gives several numerical radii and norm inequalities for Hilbert space operators. These inequalities improve some earlier corresponding inequalities. For an operator T, we demonstrate that

$$\omega^2(T) \le \frac{1}{2}\left\| |T|^4 + |T^*|^4 + \frac{1}{2}\left(|T|^2 + |T^*|^2\right)^2 \right\|^{\frac{1}{2}}.$$

1 Introduction

Let $(\mathbb{H}; \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The numerical range of an operator T is the subset of the complex numbers $\mathbb{C}$ given by:

$$W(T) = \{\langle Tx, x\rangle : x \in \mathbb{H}, \|x\| = 1\}.$$

The numerical radius of an operator T on $\mathbb{H}$ is shown by:

$$\omega(T) = \{|\langle Tx, x\rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

It is well known that $\omega(\cdot)$ is a norm on the C^*-algebra $\mathbb{B}(\mathbb{H})$ of all bounded linear operators $T : \mathbb{H} \to \mathbb{H}$. This norm is equivalent to the operator norm $\|T\| = \sup\limits_{\substack{x\in\mathbb{H} \\ \|x\|=1}} \|Tx\|$. The following more explicit result holds:

$$\frac{1}{2}\|T\| \le \omega(T) \le \|T\|. \tag{1.1}$$

Kittaneh has established in [8] that if $T \in \mathbb{B}(\mathbb{H})$, then

$$\omega^2(T) \le \frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\| \tag{1.2}$$

where $|T| = (T^*T)^{1/2}$. The inequality (1.2) is stronger than the second inequality in (1.1). This can be noticed by utilizing the fact that

$$\frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\| \le \frac{1}{2}\left\| |T|^2 \right\| + \frac{1}{2}\left\| |T^*|^2 \right\| = \|T\|^2.$$

In [4], El-Haddad and Kittaneh proved that

$$\omega^{2r}(T) \le \left\| (1-t)|T|^{2r} + t|T^*|^{2r} \right\|, \; 0 \le t \le 1, r \ge 1. \tag{1.3}$$

Notice that this is a generalization of the inequality (1.2). For some recent and impressive results involving inequalities for the numerical radius, visit [5], [10], [11], [12], [14], and [15].

Section 2 considerably improves the second inequality in (1.1). We also give a multiplicative refinement of this inequality.

In order to achieve the purpose of this paper, we require the following lemmas.

Lemma 1.1. *[6] If $A, B \in \mathbb{B}(\mathbb{H})$ are positive operators, then*

$$\|A+B\| \leq \frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^2}\right).$$

Lemma 1.2. *(Buzano's inequality [3]) If a, b, x are vectors in $\mathbb{H}$, then*

$$|\langle a,x\rangle|\,|\langle x,b\rangle| \leq \frac{\|a\|\,\|b\|+|\langle a,b\rangle|}{2}\|x\|^2.$$

Lemma 1.3. *[7] Let $T \in \mathbb{B}(\mathbb{H})$ and let $x, y \in \mathbb{H}$ be any vectors. If $0 \leq t \leq 1$,*

$$|\langle Tx,y\rangle|^2 \leq \left\langle |T|^{2(1-t)}x,x\right\rangle \left\langle |T^*|^{2t}y,y\right\rangle.$$

Lemma 1.4. *Let $T \in \mathbb{B}(\mathbb{H})$ be a self-adjoint operator, and let $x \in \mathbb{H}$ be a unit vector. Then,*

$$\langle Tx,x\rangle^2 \leq \left\langle T^2x,x\right\rangle.$$

Proof. By the Cauchy-Schwarz inequality, we have

$$\langle Tx,x\rangle^2 \leq \|Tx\|^2 = \langle Tx,Tx\rangle = \left\langle T^2x,x\right\rangle.$$

□

2 Main Results

We prove our numerical radius inequality, which contains several inequalities as special cases.

Theorem 2.1. *If $A, B \in \mathbb{B}(\mathbb{H})$, then*

$$\omega(A\pm B) \leq \frac{1}{2}\sqrt{\left\|3\left(|A|^2+|B|^2\right)+|A^*|^2+|B^*|^2\right\|+2\left(\omega\left(A^2\right)+\omega\left(B^2\right)+2\omega\left(B^*A\right)\right)},$$

and

$$\omega(A\pm B) \leq \frac{1}{2}\sqrt{\left\|3\left(|A^*|^2+|B^*|^2\right)+|A|^2+|B|^2\right\|+2\left(\omega\left(A^2\right)+\omega\left(B^2\right)+2\omega\left(BA^*\right)\right)}.$$

More precisely,

$$\omega^2(A\pm B) \leq \frac{1}{2}\left(\omega\left(A^2\right)+\omega\left(B^2\right)\right)+\min\{\lambda,\mu\},$$

where

$$\lambda = \frac{1}{4}\left\|3\left(|A|^2+|B|^2\right)+|A^*|^2+|B^*|^2\right\|+\omega\left(B^*A\right),$$

and

$$\mu = \frac{1}{4}\left\|3\left(|A^*|^2+|B^*|^2\right)+|A|^2+|B|^2\right\|+\omega\left(BA^*\right).$$

Proof. Substituting $a = Sx$, $b = Tx$ with $x \in \mathbb{H}$, $\|x\| = 1$ in Lemma 1.2, then

$$\begin{aligned}
2\left|\langle Sx,x\rangle\right|\left|\langle x,Tx\rangle\right| &= 2\left|\langle Sx,x\rangle\right|\left|\langle Tx,x\rangle\right| \\
&\le \|Sx\|\,\|Tx\| + \left|\langle Sx,Tx\rangle\right| \\
&= \|Sx\|\,\|Tx\| + \left|\langle T^*Sx,x\rangle\right| \\
&= \sqrt{\langle Sx,Sx\rangle\,\langle Tx,Tx\rangle} + \left|\langle T^*Sx,x\rangle\right| \\
&= \sqrt{\left\langle |S|^2x,x\right\rangle\left\langle |T|^2x,x\right\rangle} + \left|\langle T^*Sx,x\rangle\right| \\
&\le \frac{1}{2}\left(\left\langle |S|^2x,x\right\rangle + \left\langle |T|^2x,x\right\rangle\right) + \left|\langle T^*Sx,x\rangle\right| \\
&\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{2}\left\langle \left(|S|^2+|T|^2\right)x,x\right\rangle + \left|\langle T^*Sx,x\rangle\right|.
\end{aligned}$$

Observe that

$$\left|\langle Sx,x\rangle\right|\left|\langle Tx,x\rangle\right| \le \frac{1}{4}\left\langle \left(|S|^2+|T|^2\right)x,x\right\rangle + \frac{1}{2}\left|\langle T^*Sx,x\rangle\right|. \tag{2.1}$$

In particular, if $T^* = S$, then

$$\left|\langle Sx,x\rangle\right|^2 \le \frac{1}{4}\left\langle \left(|S|^2+|S^*|^2\right)x,x\right\rangle + \frac{1}{2}\left|\left\langle S^2x,x\right\rangle\right|. \tag{2.2}$$

So, we have

$$\begin{aligned}
&\left|\langle (A+B)\,x,x\rangle\right|^2 \\
&\le \left(\left|\langle Ax,x\rangle\right| + \left|\langle Bx,x\rangle\right|\right)^2 \quad \text{(by the triangle inequality)} \\
&= \left|\langle Ax,x\rangle\right|^2 + \left|\langle Bx,x\rangle\right|^2 + 2\left|\langle Ax,x\rangle\right|\left|\langle Bx,x\rangle\right| \\
&\le \frac{1}{2}\left(\left|\left\langle A^2x,x\right\rangle\right| + \left|\left\langle B^2x,x\right\rangle\right| + 2\left|\langle B^*Ax,x\rangle\right|\right) \\
&\quad + \frac{1}{4}\left\langle \left(3\left(|A|^2+|B|^2\right) + |A^*|^2 + |B^*|^2\right)x,x\right\rangle \quad \text{(by (2.1) and (2.2))} \\
&\le \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right) + 2\omega\left(B^*A\right)\right) + \frac{1}{4}\left\|3\left(|A|^2+|B|^2\right) + |A^*|^2 + |B^*|^2\right\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\omega^2\left(A+B\right) &= \sup_{\|x\|=1}\left|\langle (A+B)\,x,x\rangle\right|^2 \\
&\le \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right) + 2\omega\left(B^*A\right)\right) + \frac{1}{4}\left\|3\left(|A|^2+|B|^2\right) + |A^*|^2 + |B^*|^2\right\|,
\end{aligned}$$

and so

$$\omega^2\left(A+B\right) \le \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right) + 2\omega\left(B^*A\right)\right) + \frac{1}{4}\left\|3\left(|A|^2+|B|^2\right) + |A^*|^2 + |B^*|^2\right\|.$$

We get the first inequality if we replace B by $-B$ in the last inequality.

The second inequality can be obtained from the first inequality by replacing A and B by A^* and B^*, respectively.

The third inequality is evident due to the first and the second inequality. □

Corollary 2.2.

(i) Letting $B = A$ in Theorem 2.1, we get

$$\omega^2\left(A\right) \le \frac{1}{4}\left(\omega\left(A^2\right) + \|A\|^2\right) + \frac{1}{8}\left\|3|A|^2 + |A^*|^2\right\|.$$

(ii) Letting $B = A^*$ in Theorem 2.1, we get

$$\|\Re A\|^2 \leq \frac{1}{2}\omega\left(A^2\right) + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|,$$

where $\Re A = \frac{A+A^*}{2}$.

The following result can be obtained easily from the triangle inequality while we establish it with the inner product methods. More precisely, the direct proof is: Assume that $T \in \mathbb{B}(\mathbb{H})$ with the polar decomposition $T = U|T|$. Then

$$\omega(T) = \omega\left(T - \frac{\|T\|}{2}U + \frac{\|T\|}{2}U\right) \leq \omega\left(T - \frac{\|T\|}{2}U\right) + \frac{\|T\|}{2}\omega(U).$$

Theorem 2.3. *Let $T \in \mathbb{B}(\mathbb{H})$ with the polar decomposition $T = U|T|$. Then*

$$\omega(T) \leq \omega\left(T - \frac{\|T\|}{2}U\right) + \frac{\|T\|}{2}\omega(U).$$

Proof. It has been shown in [2, Theorem 3.3] that

$$\begin{aligned}\left|\left\langle |T|^2 x, y\right\rangle\right| &\leq \left|\left\langle |T|^2 x, y\right\rangle - \frac{\|T\|^2}{2}\langle x, y\rangle\right| + \frac{\|T\|^2}{2}|\langle x, y\rangle| \\ &\leq \frac{\|T\|^2}{2}\left(|\langle x, y\rangle| + \|x\|\,\|y\|\right).\end{aligned}$$

We can write this in the following form

$$\begin{aligned}\left|\left\langle |T|^2 x, y\right\rangle\right| &\leq \left|\left\langle |T|^2 x, y\right\rangle - \frac{\left\||T|^2\right\|}{2}\langle x, y\rangle\right| + \frac{\left\||T|^2\right\|}{2}|\langle x, y\rangle| \\ &\leq \frac{\left\||T|^2\right\|}{2}\left(|\langle x, y\rangle| + \|x\|\,\|y\|\right).\end{aligned}$$

So, by replacing $|T|^2$ by $|T|$, we get

$$\begin{aligned}|\langle |T|\, x, y\rangle| &\leq \left|\langle |T|\, x, y\rangle - \frac{\|T\|}{2}\langle x, y\rangle\right| + \frac{\|T\|}{2}|\langle x, y\rangle| \\ &\leq \frac{\|T\|}{2}\left(|\langle x, y\rangle| + \|x\|\,\|y\|\right).\end{aligned} \tag{2.3}$$

If we replace y with U^*y, in (2.3), we get

$$\begin{aligned}|\langle Tx, y\rangle| &\leq \left|\left\langle\left(T - \frac{\|T\|}{2}U\right)x, y\right\rangle\right| + \frac{\|T\|}{2}|\langle Ux, y\rangle| \\ &\leq \frac{\|T\|}{2}\left(|\langle Ux, y\rangle| + \|x\|\,\|U^*y\|\right).\end{aligned}$$

In particular, for any unit vector $x \in \mathbb{H}$

$$\begin{aligned}|\langle Tx, x\rangle| &\leq \left|\left\langle\left(T - \frac{\|T\|}{2}U\right)x, x\right\rangle\right| + \frac{\|T\|}{2}|\langle Ux, x\rangle| \\ &\leq \omega\left(T - \frac{\|T\|}{2}U\right) + \frac{\|T\|}{2}\omega(U)\end{aligned}$$

i.e.,

$$|\langle Tx, x\rangle| \leq \omega\left(T - \frac{\|T\|}{2}U\right) + \frac{\|T\|}{2}\omega(U).$$

By taking supremum over all unit vector $x \in \mathbb{H}$, we infer that

$$\omega(T) \le \omega\left(T - \frac{\|T\|}{2}U\right) + \frac{\|T\|}{2}\omega(U),$$

as desired. □

Lemma 2.4. *Let $T \in \mathbb{B}(\mathbb{H})$. Then*

$$\left\| |T| - \frac{\|T\|}{2}I \right\| = \frac{\|T\|}{2},$$

where $I \in \mathbb{B}(\mathbb{H})$ is the identity operator.

Proof. By (2.3), we can see that

$$\left| \left\langle \left(|T| - \frac{\|T\|}{2}I \right) x, x \right\rangle \right| \le \frac{\|T\|}{2}$$

for any unit vector $x \in \mathbb{H}$. Therefore,

$$\left\| |T| - \frac{\|T\|}{2}I \right\| = \omega\left(|T| - \frac{\|T\|}{2}I \right) \le \frac{\|T\|}{2}. \tag{2.4}$$

On the other hand,

$$\|T\| = \| \, |T| \, \| = \left\| |T| - \frac{\|T\|}{2}I + \frac{\|T\|}{2}I \right\| \le \left\| |T| - \frac{\|T\|}{2}I \right\| + \frac{\|T\|}{2}. \tag{2.5}$$

Combining (2.4) and (2.5) implies the desired result. □

Lemma 2.5. *If $T \in \mathbb{B}(\mathbb{H})$ with the polar decomposition $T = U|T|$. Then*

$$\left\| T - \frac{\|T\|}{2}U \right\| = \frac{\|T\|}{2}.$$

Proof. Indeed,

$$\begin{aligned}
\|T\| &\le \left\| T - \frac{\|T\|}{2}U \right\| + \frac{\|T\|}{2} \\
&= \left\| U\left(|T| - \frac{\|T\|}{2} \right) \right\| + \frac{\|T\|}{2} \\
&\le \left\| |T| - \frac{\|T\|}{2} \right\| + \frac{\|T\|}{2} \\
&= \|T\| \quad \text{(by Lemma 2.4).}
\end{aligned}$$

□

Remark 2.6. It follows from Theorem 2.3 that

$$\begin{aligned}
\omega(T) &\le \omega\left(T - \frac{\|T\|}{2}U\right) + \frac{\|T\|}{2}\omega(U) \\
&\le \left\| T - \frac{\|T\|}{2}U \right\| + \frac{\|T\|}{2}\omega(U) \quad \text{(by the RHS of (1.1))} \\
&= (1 + \omega(U))\frac{\|T\|}{2} \quad \text{(by Lemma 2.5).}
\end{aligned}$$

Finally, we close this section by giving the following new bound using Lemmas 1.3 and 1.4.

Theorem 2.7. *Let $T \in \mathbb{B}(\mathbb{H})$ and let $0 \le t \le 1$. Then*

$$\omega^2(T) \le \frac{1}{2}\left\| |T|^4 + |T^*|^4 + \frac{1}{2}\left(|T|^2 + |T^*|^2\right)^2 \right\|^{\frac{1}{2}}.$$

Proof. We mimic some ideas presented in [9, 13]. If $0 \leq t \leq 1/2$, then

$$\begin{aligned}
&\left((1-t)\,|T|^2 + t\,|T^*|^2\right)^2 \\
&= \left((1-2t)\,|T|^2 + 2t\left(\frac{|T|^2+|T^*|^2}{2}\right)\right)^2 \\
&\leq (1-2t)\,|T|^4 + 2t\left(\frac{|T|^2+|T^*|^2}{2}\right)^2 \\
&\qquad \text{(since } f(t)=t^2 \text{ is operator convex; see Theorem 1.5.8 in [1])} \\
&= (1-t)\,|T|^4 + t|T^*|^4 - 2r\left(\frac{|T|^4+|T^*|^4}{2} - \left(\frac{|T|^2+|T^*|^2}{2}\right)^2\right).
\end{aligned}$$

A similar discussion holds for $1/2 \leq t \leq 1$. So we obtain

$$\left((1-t)\,|T|^2 + t\,|T^*|^2\right)^2 \leq (1-t)\,|T|^4 + t|T^*|^4 - 2r\left(\frac{|T|^4+|T^*|^4}{2} - \left(\frac{|T|^2+|T^*|^2}{2}\right)^2\right).$$

On the other hand,

$$\begin{aligned}
|\langle Tx,x\rangle|^4 &\leq \left(\left\langle |T|^{2(1-t)}x,x\right\rangle \left\langle |T^*|^{2t}x,x\right\rangle\right)^2 \quad \text{(by Lemma 1.3)} \\
&\leq \left(\left\langle |T|^2x,x\right\rangle^{1-t}\left\langle |T^*|^2x,x\right\rangle^{t}\right)^2 \quad \text{(by [4, Lemma 3])} \\
&\leq \left((1-t)\left\langle |T|^2x,x\right\rangle + t\left\langle |T^*|^2x,x\right\rangle\right)^2 \\
&\qquad \text{(by the weighted arithmetic-geometric mean inequality)} \\
&= \left\langle \left((1-t)\,|T|^2 + t|T^*|^2\right)x,x\right\rangle^2 \\
&\leq \left\langle \left((1-t)\,|T|^2 + t|T^*|^2\right)^2 x,x\right\rangle \quad \text{(by Lemma 1.4)}
\end{aligned}$$

for any unit vector $x \in \mathbb{H}$. Hence,

$$|\langle Tx,x\rangle|^4 \leq \left\langle \left((1-t)\,|T|^4 + t|T^*|^4 - 2r\left(\frac{|T|^4+|T^*|^4}{2} - \left(\frac{|T|^2+|T^*|^2}{2}\right)^2\right)\right)x,x\right\rangle. \tag{2.6}$$

If we take integral over $0 \leq t \leq 1$ in the inequality (2.6), we get

$$\begin{aligned}
|\langle Tx,x\rangle|^4 &\leq \left\langle \left(\frac{1}{2}\left(|T|^4+|T^*|^4\right) - \frac{1}{2}\left(\frac{|T|^4+|T^*|^4}{2} - \left(\frac{|T|^2+|T^*|^2}{2}\right)^2\right)\right)x,x\right\rangle \\
&= \frac{1}{4}\left\langle \left(|T|^4+|T^*|^4 + \frac{1}{2}\left(|T|^2+|T^*|^2\right)^2\right)x,x\right\rangle \\
&\leq \frac{1}{4}\left\| |T|^4+|T^*|^4 + \frac{1}{2}\left(|T|^2+|T^*|^2\right)^2\right\|.
\end{aligned}$$

Namely,

$$|\langle Tx,x\rangle|^4 \leq \frac{1}{4}\left\| |T|^4+|T^*|^4 + \frac{1}{2}\left(|T|^2+|T^*|^2\right)^2\right\|.$$

By taking supremum over all unit vectors $x \in \mathbb{H}$, we obtain

$$\omega^4(T) \leq \frac{1}{4}\left\| |T|^4 + |T^*|^4 + \frac{1}{2}\left(|T|^2 + |T^*|^2\right)^2 \right\|$$

or

$$\omega^2(T) \leq \frac{1}{2}\left\| |T|^4 + |T^*|^4 + \frac{1}{2}\left(|T|^2 + |T^*|^2\right)^2 \right\|^{\frac{1}{2}}.$$

□

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Received: 2023-07-18
Accepted: 2023-10-22