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Expansion of a power-law functions from a linear combination of multidimensional vectors by hyperspherical functions

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Abstract In this paper, we consider a linear function of the form $|\mathbf{r}_1+\ldots+\mathbf{r}_{_N}|^{-\nu}\in\mathbb{R}$ $(\nu\in\mathbb{R})$ from N vectors $\mathbf{r}_s\in\mathbf{R}^D$ in D–dimensional Euclidean space, and its expansion output as a series $\sum G_{p_1,\ldots,p_N} f_{p_1}(\mathbf{r}_1)\ldots f_{p_N}(\mathbf{r}_{_N})$. As a function of $f_p(\mathbf{r})$, we consider the product of orthogonal radial functions and angular hyperspherical functions on the unit $D-1$–dimensional sphere $\mathbf{S}^{D-1}$. The choice of such functions defined by us is because for $N=2$ the coefficient G_{p_1,p_2} has the diagonal form. It is shown that with the help of a certain orthogonal complement $\mathbf{S}^D{=}\mathbf{S}^{D-1}{\oplus}\mathbf{S}^1$, this expansion can also be represented by hyperspherical functions on a D–dimensional the unit sphere $\mathbf{S}^D$.

1 Introduction

The function of the form $|\mathbf{r}_1-\mathbf{r}_2|^{-\nu}\in\mathbb{R}$ from vectors $\mathbf{r}_k\in\mathbf{R}^D$ in D–dimensional space are found in many sections of mathematics and physics. For example, in the theory of Riesz potential, hypersingular integrals, and fractional integration, where this function is included in the kernel of the integral equation of the first type [1], as well as in numerical methods similar to [2], [3]. In [4], a complete expansion was considered and applied for the case of $-\nu=p\in\mathbb{N}$ on the surface of $\mathbf{r}_k\in\mathbf{S}^{D-1}$ by hyperspheric harmonics. This function is also used in physical problems, such as problems of many bodies, aerodynamics, electrodynamics, and geophysics.

In many cases, approximation by a small parameter is used to solve some problems

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|^{\nu}}\sim\sum_p R_p\left(\frac{|\mathbf{r}_1|}{|\mathbf{r}_2|}\right)\Phi_p(\theta),\ \ |\mathbf{r}_1|<|\mathbf{r}_2|,\ \ \cos\theta=\frac{(\mathbf{r}_1\mathbf{r}_2)}{|\mathbf{r}_1||\mathbf{r}_2|}$$

or if it is possible by harmonic functions $Y_p(\theta)$

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|^{\nu}}\sim\sum_p F_p(|\mathbf{r}_1|,|\mathbf{r}_2|)Y_p(\theta)$$

where F_p, Φ_p, G_p are the functions resulting from the expansion. Unlike these expansions, we propose an exact approximation from N vectors of the form

$$|\mathbf{r}_1+\mathbf{r}_2+\ldots+\mathbf{r}_{_N}|^{-\nu}\sim\sum_{p_1,p_2,\ldots p_N} f_{p_1}(\mathbf{r}_1)f_{p_2}(\mathbf{r}_2)\ldots f_{p_N}(\mathbf{r}_{_N})$$

and the definition of such functions $f_p(\mathbf{r})\in\mathbb{R}$. In [5] it is shown that such a separation exists for two vectors in three-dimensional space $\mathbf{R}^3$. In the spherical coordinate system $\mathbf{r}_s{=}\{r_s,\theta_{1,s},\theta_{2,s}\}$ (for $\nu{\in}\mathbb{R}$, $\nu{<}3$)

$$|\mathbf{r}_s-\mathbf{r}_p|^{-\nu}=\frac{\pi^{\frac{3}{2}}\Gamma\left(\frac{3-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left(\left(r_s^2+1\right)\left(r_p^2+1\right)\right)^{\frac{3-\nu}{2}}\times$$
$$\times\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=-l}^{l}\frac{\Gamma\left(n+\frac{\nu}{2}\right)}{\Gamma\left(n+3-\frac{\nu}{2}\right)}H_{n,l,m}(\mathbf{r}_s)\overline{H}_{n,l,m}(\mathbf{r}_p) \quad (1.1)$$

$$H_{n,l,m}(\mathbf{r})=\eta_{n,l}(r)Y_{l,m}(\theta_1,\theta_2) \quad (1.2)$$

where $Y_{l,m}(\theta_1,\theta_2)$–spherical function on a unit two-dimensional sphere $\mathbf{S}^2$. The function $\eta_{n,l}(r)$ forms an orthogonal system with a weight of r^2 in the domain $0\leqslant r<\infty$, which can be expressed using polynomials of Gegenbauer C_{n-l}^{l+1} as

$$\eta_{n,l}(r)=4^{l+1}l!\sqrt{\frac{(n+1)\,(n-l)!}{\pi(n+l+1)!}}\frac{r^l}{(r^2+1)^{l+\frac{3}{2}}}C_{n-l}^{l+1}\left(\frac{r^2-1}{r^2+1}\right) \quad (1.3)$$

Similarly to the representation of (1.1) in the space $\mathbf{R}^3$, one can represent the expansion by hyperspherical functions $Y_{n,l,m}(\psi_s,\theta_{s;1},\theta_{s;2})$ on a three-dimensional sphere $\mathbf{S}^3$ as

$$|\mathbf{r}_s-\mathbf{r}_p|^{-\nu}=\frac{\pi^{\frac{3}{2}}2^{3-\nu}\Gamma\left(\frac{3-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left((1-\cos\psi_s)(1-\cos\psi_p)\right)^{\frac{\nu}{2}}\times$$
$$\times\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=-l}^{l}\frac{\Gamma\left(n+\frac{\nu}{2}\right)}{\Gamma\left(n+3-\frac{\nu}{2}\right)}Y_{n,l,m}(\psi_s,\theta_{s;1},\theta_{s;2})\overline{Y}_{n,l,m}(\psi_p,\theta_{p;1},\theta_{p;2}) \quad (1.4)$$

by performing a replacement in the form of

$$\cos\psi_s=\frac{r_s^2-1}{r_s^2+1},\ 0\leqslant\psi_s\leqslant\pi \quad (1.5a)$$

$$\eta_{n,l}(r_s)\Rightarrow(1-\cos\psi_s)^{\frac{3}{2}}\tilde{\eta}_{n,l}(\cos\psi_s) \quad (1.5b)$$

$$\tilde{\eta}_{n,l}(\cos\psi_s)=2^{l+\frac{1}{2}}\,l!\sqrt{\frac{(n+1)(n-l)!}{\pi\,(n+l+1)!}}\ \sin^l\psi_s\,C_{n-l}^{l+1}(\cos\psi_s) \quad (1.5c)$$

$$Y_{n,l,m}(\psi_s,\theta_{s;1},\theta_{s;2})=\tilde{\eta}_{n,l}(\cos\psi_s)Y_{l,m}(\theta_{s;1},\theta_{s;2}),\ \ n\geqslant l\geqslant|m| \quad (1.5d)$$

In this paper we will consider a similar expansion of (1.1) but with any number of N arbitrary D—dimensional vectors in Euclidean space. We will denote multidimensional unit vectors in D–dimensional Euclidean space as $\boldsymbol{\zeta}_s$, where s means belonging to the vector. In this case, D of the components of the vector $\boldsymbol{\zeta}_s=\{\zeta_{s;k}\}, k=1,\dots,D$ can be expressed in terms of $D-1$ of polar coordinates $\boldsymbol{\theta}_s=\{\theta_{s;i}\}$, $i=1,\dots D-1$ on the unit sphere $\mathbf{S}^{D-1}$, where for an arbitrary vector $\mathbf{r}_s$ in a spherical coordinate system $\mathbf{r}_s=r_s\boldsymbol{\zeta}_s=\{r_s,\boldsymbol{\theta}_s\}$, and the scalar product of the unit vectors α and β as the cosine of the angle between them $\cos\omega_{\alpha\beta}=(\boldsymbol{\zeta}_\alpha\boldsymbol{\zeta}_\beta)=\sum\limits_{k=1}^{D}\zeta_{\alpha;k}\zeta_{\beta;k}$. We will also not specify the choice of polar coordinates and their components on $\mathbf{S}^{D-1}$ since is not essential here. In total, there may be $\frac{(2D-2)!}{(D-1)!\,D!}$ equivalent representation, and accordingly the same number of equivalent representation exists for hyperspheric functions $Y_{l_s,\mathbf{m}_s}(\boldsymbol{\theta}_s)$, $\mathbf{m}_s=\{m_{s;1},\dots,m_{s;D-2}\}$ (here and everywhere if there is an index of s, it means belonging to the corresponding unit vector $\boldsymbol{\zeta}_s$). The theory of hyperspherical functions is well described in many literature data and works (for example, in [6, Ch.11]), and we do not consider it here.

As an example, we note an important construction of the graphical method – hyperspherical tree [7, Sec. 6.1.4]. In figure 1, thick lines show a T-tree for hyperspherical functions on the unit sphere $\mathbf{S}^{D-1}$. In this case, the separation constant l, for the hyperspherical function $Y_{l,\mathbf{m}}(\boldsymbol{\theta})$, we will denote separately. By analogy with the above example in three-dimensional space, by introducing another variable (1.5a)ψ_s, we can make an orthogonal complement from the unit sphere $\mathbf{S}^{D-1}$ to $\mathbf{S}^D$. The figure 1 of the thin line on the T-tree shows the complement to $\mathbf{S}^{D-1}$.

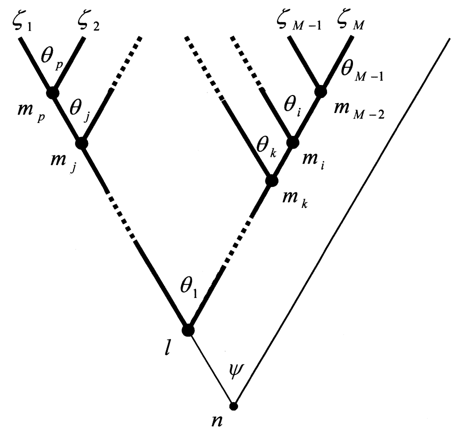

Figure 1. Hyperspherical T-tree on the D–dimensional unit sphere $\mathbf{S}^D = \mathbf{S}^{D-1} \oplus \mathbf{S}^1$. The space on $\mathbf{S}^{D-1}$ is represented as a T-tree by thick lines with a separation constant l, $\mathbf{S}^1$–with thin lines with a separation constant n.

We will denote the hyperspherical function on $\mathbf{S}^D$ as $Y_{n,l,\mathbf{m}}(\psi, \boldsymbol{\theta})$ with separately designated separation constants n and l.

The surface element on $\mathbf{S}^D$ we will denote by $d\boldsymbol{\Omega}_{\boldsymbol{\psi},\boldsymbol{\theta}}$, and on $\mathbf{S}^{D-1}$ as $d\boldsymbol{\Omega}_{\boldsymbol{\theta}}$. The relationship between them

$$d\boldsymbol{\Omega}_{\boldsymbol{\psi},\boldsymbol{\theta}} = \sin^{D-1}\psi \, d\psi \, d\boldsymbol{\Omega}_{\boldsymbol{\theta}}$$

In all sections of this work, the integral over the surface of the unit hypersphere $\int d\boldsymbol{\Omega}_{\boldsymbol{\theta}} f(\boldsymbol{\zeta})$ it is implied that the integration is taken over the entire $D-1$ dimensional space. For an arbitrary system of hyperspherical coordinates, the volume element in $\mathbf{R}^D$ and the area of a D–dimensional unit sphere are represented by the relations respectively as [6, Ch.11]

$$d\mathbf{r} = dV_D = r^{D-1} dr \, d\boldsymbol{\Omega}_{\boldsymbol{\theta}}, \quad \int d\boldsymbol{\Omega}_{\psi,\boldsymbol{\theta}} = S_D = \frac{2\pi^{\frac{D+1}{2}}}{\Gamma\left(\frac{D+1}{2}\right)}, \quad \int d\boldsymbol{\Omega}_{\boldsymbol{\theta}} = S_{D-1}$$

So, for example, for hyperspherical functions satisfying the orthogonality condition

$$\int d\boldsymbol{\Omega}_{\boldsymbol{\theta}} Y_{l_s,\mathbf{m}_s}(\boldsymbol{\theta}) \overline{Y}_{l_p,\mathbf{m}_p}(\boldsymbol{\theta}) = \delta_{l_s,l_p} \delta_{\mathbf{m}_s,\mathbf{m}_p}$$
$$\int d\boldsymbol{\Omega}_{\boldsymbol{\psi},\boldsymbol{\theta}} Y_{n_s,l_s,\mathbf{m}_s}(\psi,\boldsymbol{\theta}) \overline{Y}_{n_p,l_p,\mathbf{m}_p}(\boldsymbol{\psi},\boldsymbol{\theta}) = \delta_{n_s,n_p} \delta_{l_s,l_p} \delta_{\mathbf{m}_s,\mathbf{m}_p} \tag{1.6}$$

$$\delta_{\mathbf{m}_s,\mathbf{m}_p} = \delta_{m_{s;1},m_{p;1}} \delta_{m_{s;2},m_{p;2}} \ldots \delta_{m_{s;D-2},m_{p;D-2}}$$

where δ_{p_1,p_2} is the Kronecker symbol.

Everywhere $C^{\alpha}_{\mu}(z)$ —Gegenbauer function. $\Gamma(...)$–Gamma function. The generalized hypergeometric function is as

$${}_pF_q \left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| \, z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}$$

where the symbols $(a)_k$ denote the Pochhammer's symbol

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$$

In our notation $\mathbf{n}=\{n_1, \ldots, n_N\}$, $\mathbf{l}=\{l_1, \ldots, l_N\}$, $\mathbf{m_k}=\{\mathbf{m}_1, \ldots, \mathbf{m}_N\}$, and the total summation means as

$$\sum_{\mathbf{n},\mathbf{l},\mathbf{m_k}} = \sum_{n_1=0}^{\infty} \sum_{l_1=0}^{n_1} \sum_{\mathbf{m}_1} \cdots \sum_{n_N=0}^{\infty} \sum_{l_N=0}^{n_N} \sum_{\mathbf{m}_N}$$

2 Representation of the form (1.1) for $N > 2$ multidimensional vectors

Theorem 2.1. *For arbitrary D–dimensional vectors* $\mathbf{r}_s = \{r_s, \boldsymbol{\theta}_s\}$, $s{=}1,2,\ldots,N$, *in a spherical coordinate system, as well as real numbers* $0{<}\nu{<}D$, *a linear function* $|\mathbf{r}_1{+}\ldots{+}\mathbf{r}_N|^{-\nu} \in\mathbb{R}$ *has a expansion of the form*

$$\frac{1}{|\mathbf{r}_1+\ldots+\mathbf{r}_N|^{\nu}} = \frac{\Gamma\left(\frac{D-\nu}{2}\right)}{\pi^{\frac{D}{2}}\Gamma\left(\frac{\nu}{2}\right)} \sum_{\mathbf{n},\mathbf{l},\mathbf{m_k}} G_{\mathbf{n},\mathbf{l},\mathbf{m_k}} \prod_{p=1}^{N}\left((r_p^2+1)^{\frac{D-\nu}{2}} H_{n_p,l_p,\mathbf{m}_p}(\mathbf{r}_p)\right)$$
$$G_{\mathbf{n},\mathbf{l},\mathbf{m_k}} = \int_0^\infty du\, u^{\nu-1}\int d\Omega_{\boldsymbol{\theta}} \prod_{s=1}^{N} \Xi^{(\nu,D)}_{n_s,l_s,\mathbf{m}_s}(u,\boldsymbol{\theta}) \tag{2.1}$$

where

$$\Xi^{(\nu,D)}_{n,l,\mathbf{m}}(u,\boldsymbol{\theta}) = \xi^{(\nu,D)}_{n,l}(u)\overline{Y}_{l,\mathbf{m}}(\boldsymbol{\theta}) \tag{2.2}$$

function

$$\xi^{(\nu,D)}_{n,l}(u) = \frac{i^l 2\pi^{\frac{D}{2}} u^l}{\Gamma\left(l{+}\frac{D}{2}\right)\Gamma\left(l{+}D{-}\frac{\nu}{2}\right)}\sqrt{\frac{\left(n{+}\frac{D-1}{2}\right)(n{+}l{+}D{-}2)!}{(n{-}l)!}}\times$$
$$\times\int_0^\infty dz\, e^{-z-\frac{u^2}{z}} z^{\frac{D-\nu}{2}-1} {}_2F_2\left[\begin{matrix} -n+l\,,\; n+l+D-1 \\ l+\dfrac{D}{2}\,,\; l+D-\dfrac{\nu}{2}\end{matrix}\middle|\, z\right] \tag{2.3}$$

forms an orthogonal system with a weight of $u^{\nu-1}$ *in the domain of* $u\in[0\ldots\infty)$

$$\int_0^\infty du\, u^{\nu-1}\xi^{(\nu,D)}_{n_1,l}(u)\xi^{(\nu,D)}_{n_2,l}(u) = \frac{(-1)^l\pi^D\Gamma\left(n_1+\frac{\nu}{2}\right)}{\Gamma\left(n_1+D-\frac{\nu}{2}\right)}\delta_{n_1,n_2} \tag{2.4}$$

and

$$H_{n,l,\mathbf{m}}(\mathbf{r}) = \eta^{(D)}_{n,l}(r)Y_{l,\mathbf{m}}(\boldsymbol{\theta}) \tag{2.5}$$

where

$$\eta^{(D)}_{n,l}(r) = 2^{2l+D-1}\Gamma\left(l{+}\frac{D{-}1}{2}\right)\sqrt{\frac{(n{-}l)!\,\left(n{+}\frac{D-1}{2}\right)}{\pi\,(n{+}l{+}D{-}2)!}}\,\frac{r^l}{(r^2{+}1)^{l+\frac{D}{2}}}C^{l+\frac{D-1}{2}}_{n-l}\left(\frac{r^2{-}1}{r^2{+}1}\right) \tag{2.6}$$

forms an orthogonal system with weight r^{D-1}

$$\int_0^\infty dr\, r^{D-1}\eta^{(D)}_{n_1,l}(r)\eta^{(D)}_{n_2,l}(r) = \delta_{n_1,n_2} \tag{2.7}$$

The replacement of the variable r_s(1.5a), the expansion of (2.1) can be represented as similar to (1.4) but on a unit D–dimensional sphere $\mathbf{S}^D$. With the help of the orthogonal complement specified in the introduction 1, such a transition will be carried out by substitutions

$$Y_{n,l,\mathbf{m}}(\psi,\boldsymbol{\theta}) = \tilde{\eta}^{(D)}_{n,l}(\cos\psi)Y_{l,\mathbf{m}}(\boldsymbol{\theta}) \tag{2.8a}$$

$$\tilde{\eta}^{(D)}_{n,l}(\cos\psi){=}2^{l+\frac{D-2}{2}}\,\Gamma\left(l{+}\frac{D{-}1}{2}\right)\sqrt{\frac{\left(n{+}\frac{D-1}{2}\right)(n{-}l)!}{\pi\,(n{+}l{+}D{-}2)!}}\,\sin^l\!\psi\, C^{l+\frac{D-1}{2}}_{n-l}(\cos\psi) \tag{2.8b}$$

$$H_{n,l,\mathbf{m}}(\mathbf{r}) \Leftrightarrow (1-\cos\psi)^{\frac{D}{2}}Y_{n,l,\mathbf{m}}(\psi,\boldsymbol{\theta}) \tag{2.8c}$$

$$(r^2+1)^{\frac{D-\nu}{2}}H_{n,l,\mathbf{m}}(\mathbf{r}) \Leftrightarrow 2^{\frac{D-\nu}{2}}(1-\cos\psi)^{\frac{\nu}{2}}Y_{n,l,\mathbf{m}}(\psi,\boldsymbol{\theta}) \tag{2.8d}$$

In this case, the expansion of (2.1) by D–dimensional hyperspherical functions will take the form

$$\frac{1}{|\mathbf{r}_1+\ldots+\mathbf{r}_N|^{\nu}}=\frac{\Gamma\left(\frac{D-\nu}{2}\right)}{\pi^{\frac{D}{2}}\Gamma\left(\frac{\nu}{2}\right)}\sum_{\mathbf{n},\mathbf{l},\mathbf{m_k}}G_{\mathbf{n},\mathbf{l},\mathbf{m_k}}\prod_{p=1}^{N}\Big(2^{\frac{D-\nu}{2}}(1-\cos\psi_p)^{\frac{\nu}{2}}Y_{n_p,l_p,\mathbf{m}_p}(\psi_p,\boldsymbol{\theta}_p)\Big) \tag{2.9}$$

That is, in this expansion, we have moved from the D-dimensional Euclidean space $\mathbf{R}^D$, to the space $\mathbf{S}^D$ of the unit D-dimensional sphere. When replacing (2.8), the expansion $|\mathbf{r}_1+\ldots+\mathbf{r}_N|^{-\nu}$ can be represented as (2.1) or (2.9). The expressions (1.1)-(1.4) are a special case of the theorem 2.1 and the expressions (2.9) for D=3, N=2.

Example. For N=2 and D–dimensional vectors $\mathbf{r}_s$ from (2.1) taking into account (2.4) and (1.6) we get

$$G_{\mathbf{n},\mathbf{l},\mathbf{m_k}}=\pi^D\frac{\Gamma\left(n_1+\frac{\nu}{2}\right)}{\Gamma\left(n_1+D-\frac{\nu}{2}\right)}\delta_{n_1,n_2}\delta_{l_1,l_2}\delta_{\mathbf{m}_1,\mathbf{m}_2}=\pi^D g_{n_1}\delta_{n_1,n_2}\delta_{l_1,l_2}\delta_{\mathbf{m}_1,\mathbf{m}_2}$$

$$\sum_{\mathbf{n},\mathbf{l},\mathbf{m_k}}(\ldots)=\sum_{n,l,\mathbf{m}}(\ldots)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{\mathbf{m}}(\ldots)$$

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|^{\nu}}=\frac{\pi^{\frac{D}{2}}\Gamma\left(\frac{D-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left((r_1^2+1)\,(r_2^2+1)\right)^{\frac{D-\nu}{2}}\sum_{n,l,\mathbf{m}}g_nH_{n,l,\mathbf{m}}(\mathbf{r}_1)\overline{H_{n,l,\mathbf{m}}}(\mathbf{r}_2)=$$
$$=\frac{\pi^{\frac{D}{2}}2^{D-\nu}\Gamma\left(\frac{D-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left((1-\cos\psi_1)(1-\cos\psi_2)\right)^{\frac{\nu}{2}}\sum_{n,l,\mathbf{m}}g_nY_{n,l,\mathbf{m}}(\psi_1,\boldsymbol{\theta}_1)\overline{Y_{n,l,\mathbf{m}}}(\psi_2,\boldsymbol{\theta}_2)$$

and also

$$\frac{1}{|\mathbf{r}_1+\mathbf{r}_2|^{\nu}}=\frac{\pi^{\frac{D}{2}}\Gamma\left(\frac{D-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left((r_1^2+1)\,(r_2^2+1)\right)^{\frac{D-\nu}{2}}\sum_{n,l,\mathbf{m}}g_nH_{n,l,\mathbf{m}}(\mathbf{r}_1)H_{n,l,\mathbf{m}}(\mathbf{r}_2)=$$
$$=\frac{\pi^{\frac{D}{2}}2^{D-\nu}\Gamma\left(\frac{D-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left((1-\cos\psi_1)(1-\cos\psi_2)\right)^{\frac{\nu}{2}}\sum_{n,l,\mathbf{m}}g_nY_{n,l,\mathbf{m}}(\psi_1,\boldsymbol{\theta}_1)Y_{n,l,\mathbf{m}}(\psi_2,\boldsymbol{\theta}_2) \tag{2.10}$$

where ψ_s is defined in (1.5a). In particular for $x,y\in\mathbb{R},\ \nu<2$

$$\frac{1}{|x-y|^{\nu}}=\frac{\Gamma\left(1-\frac{\nu}{2}\right)}{2\Gamma\left(\frac{\nu}{2}\right)}\left((x^2+1)(y^2+1)\right)^{1-\frac{\nu}{2}}\sum_{n=0}^{\infty}\sum_{l=-n}^{+n}\frac{\Gamma\left(n+\frac{\nu}{2}\right)}{\Gamma\left(n+2-\frac{\nu}{2}\right)}\eta^{(2)}_{n,|l|}(x)\eta^{(2)}_{n,|l|}(y)=$$
$$=\frac{2^{1-\nu}\Gamma\left(1-\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left((1-\cos\psi_x)(1-\cos\psi_y)\right)^{\frac{\nu}{2}}\sum_{n=0}^{\infty}\sum_{l=-n}^{+n}\frac{\Gamma\left(n+\frac{\nu}{2}\right)}{\Gamma\left(n+2-\frac{\nu}{2}\right)}\Theta_{n,|l|}(\psi_x)\Theta_{n,|l|}(\psi_y)$$

where

$$\Theta_{n,l}(z)=\sqrt{\frac{(n-l)!\,(n+\frac{1}{2})}{(n+l)!}}P_n^l(\cos z),\quad \int_{-1}^{1}dz\,\Theta_{n_1,l}(z)\Theta_{n_2,l}(z)=\delta_{n_1,n_2}$$

and $P_n^l(\cos z)$– Legendre function.

Consider the case when $\mathbf{r}_k\in\mathbf{S}^{D-1}$, $|\mathbf{r}_k|=1$. From (2.5) and (2.6)

$$H_{n,l,\mathbf{m}}(\mathbf{r})=\eta^{(D)}_{n,l}(1)Y_{l,\mathbf{m}}(\boldsymbol{\theta})$$

$$\eta^{(D)}_{n,l}(1) = 2^{l+\frac{D}{2}-1}\Gamma\Big(l+\frac{D-1}{2}\Big)\sqrt{\frac{(n-l)!\ (n+\frac{D-1}{2})}{\pi\,(n+l+D-2)!}}C^{l+\frac{D-1}{2}}_{n-l}(0) =$$

$$= 2^{n+\frac{D}{2}-1}\frac{\Gamma\big(\frac{n+l+D-1}{2}\big)}{\Gamma\big(\frac{-n+l+1}{2}\big)}\sqrt{\frac{\big(n+\frac{D-1}{2}\big)}{(n-l)!\,(n+l+D-2)!}}$$

After summing by n we have the final result

$$a_l(\nu,D) = \frac{\pi^{\frac{D}{2}}\Gamma\big(\frac{D-\nu}{2}\big)\,2^{D-\nu}}{\Gamma\big(\frac{\nu}{2}\big)}\sum_{n=0}^{\infty} g_n\,\eta^{(D)}_{n,l}(1)\,\eta^{(D)}_{n,l}(1) =$$

$$= S_{D-1}\frac{2^{D-\nu-2}\,\Gamma\big(\frac{D}{2}\big)\,\Gamma\big(\frac{D-\nu-1}{2}\big)\,\Gamma\big(l+\frac{\nu}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{\nu}{2}\big)\,\Gamma\big(l+D-1-\frac{\nu}{2}\big)} \tag{2.11}$$

In this infinite sum, the nonzero terms of the series correspond for $n = l+2k$, $k = 0\ldots\infty$. Thus

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|^{\nu}} = \sum_{l,\mathbf{m}} a_l(\nu,D)Y_{l,\mathbf{m}}(\boldsymbol{\theta}_1)\overline{Y_{l,\mathbf{m}}}(\boldsymbol{\theta}_2), \qquad \text{for} \quad |\mathbf{r}_1| = |\mathbf{r}_2| = 1$$

here $Y_{l,\mathbf{m}}(\boldsymbol{\theta}_i)$ is a hyperspherical function on $\mathbf{S}^{D-1}$. Note that in [4, Sec.3], a similar result of the decomposition of (2.11) was used for $\nu = -p \in \mathbb{N}$.

Proof. For output (2.1), we will proceed from [8], where it was shown that

$$\frac{1}{|\mathbf{r}_1+\ldots+\mathbf{r}_N|^{\nu}} = \sum_{l_1,\ldots,l_N=0}^{\infty} V_{l_1,\ldots,l_N}(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_N)\,R^{(\nu,D)}_{l_1,\ldots,l_N}(r_1,\ldots,r_N) \tag{2.12a}$$

The function $R^{(\nu,D)}_{l_1,\ldots,l_N}$ can be represented through the Lauricella's hypergeometric function of $N-1$ variables $F^{(N-1)}_C$ [9, Ch.7, eq.7.2.4(14)] as (here $r_N = \max(r_1, r_2,\ldots,r_N)$)

$$R^{(\nu,D)}_{l_1,\ldots,l_N}(r_1,\ldots,r_N) = \frac{(-1)^{l_N}}{(r_N)^{\nu}}\frac{\big(\frac{\nu}{2}\big)_l\big(\frac{\nu-D+2}{2}\big)_{l-l_N}}{\prod\limits_{p=1}^{N-1}\Big(\frac{D}{2}\Big)_{l_p}}\times$$

$$\times\prod_{p=1}^{N-1}\Big(\frac{r_p}{r_N}\Big)^{l_p}F^{(N-1)}_C\left[\begin{matrix} l+\frac{\nu}{2},\ \frac{\nu-D+2}{2}+l-l_N \\ l_1+\frac{D}{2},\ldots,l_{N-1}+\frac{D}{2}\end{matrix}\,\middle|\,\Big(\frac{r_1}{r_N}\Big)^2,\ldots,\Big(\frac{r_{N-1}}{r_N}\Big)^2\right] \tag{2.12b}$$

and

$$V_{l_1,\ldots,l_N}(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_N) = \int\frac{d\Omega_{\boldsymbol{\theta}}}{S_{D-1}}\prod_{i=1}^{N}\frac{l_i+\frac{D}{2}-1}{\frac{D}{2}-1}C^{\frac{D}{2}-1}_{l_i}((\boldsymbol{\zeta}_i\boldsymbol{\zeta})) =$$

$$= \big(S_{D-1}\big)^{N-1}\sum_{\mathbf{m}_1,\ldots,\mathbf{m}_N}\prod_{p=1}^{N}Y_{l_p,\mathbf{m}_p}(\boldsymbol{\theta}_p)\int d\Omega_{\boldsymbol{\theta}}\prod_{p=1}^{N}\overline{Y}_{l_p,\mathbf{m}_p}(\boldsymbol{\theta}) \tag{2.12c}$$

where in the last expression we use the Gegenbauer addition theorem by hyperspherical functions

$$C^{\frac{D}{2}-1}_{l}((\boldsymbol{\zeta}_1\boldsymbol{\zeta}_2)) = S_{D-1}\frac{\frac{D}{2}-1}{l+\frac{D}{2}-1}\sum_{\mathbf{m}}Y_{l,\mathbf{m}}(\boldsymbol{\theta}_1)\overline{Y}_{l,\mathbf{m}}(\boldsymbol{\theta}_2) \tag{2.13}$$

Using the expression [10, Ch.2, eq.2.12.44(7)] of the integral from products for the Bessel function $J_{\lambda_k}(c_k u)$

$$\int_0^\infty du\, u^{\alpha-1} \prod_{k=1}^{N} J_{\lambda_k}(c_k u) = \frac{2^{\alpha-1}\,(c_{_N})^{\lambda_N-\beta}\,\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\lambda_{_N} - \frac{\beta}{2} + 1\right)} \times$$

$$\times \prod_{k=1}^{N-1} \frac{(c_k)^{\lambda_k}}{\Gamma(\lambda_k+1)}\, F_C^{(N-1)} \left[\begin{matrix} \frac{\beta}{2}\,,\ \frac{\beta}{2} - \lambda_{_N} \\ \lambda_1+1\,,\dots,\ \lambda_{_{N-1}}+1 \end{matrix} \middle| \left(\frac{c_1}{c_{_N}}\right)^2, \dots, \left(\frac{c_{_{N-1}}}{c_{_N}}\right)^2 \right] \qquad (2.14)$$

where for real parameter $\alpha,\ \lambda_k,\ c_k$

$$N \geqslant 2; \quad c_k > 0,\ k = 1,\dots,N; \quad c_{_N} > c_1 + \dots + c_{_{N-1}};$$
$$-(\lambda_1 + \dots + \lambda_{_N}) < \alpha < \frac{N}{2} + 1; \quad \beta = \alpha + \lambda_1 + \dots + \lambda_{_N}$$

For the expression (2.12b) with values

$$c_k = 2r_k,\ \lambda_k = l_k + \frac{D}{2} - 1,\ k = 1,\dots,N;$$
$$\alpha = \nu - N\left(\frac{D}{2} - 1\right) = \nu - \sum_{k=1}^{N}\left(\frac{D}{2} - 1\right);$$
$$\left(\beta = l_1 + \dots + l_{_N} + \nu = 2l + \nu;\ 0 < \nu < \frac{N(D-1)}{2} + 1\right)$$

will be presented in the following form

$$R^{(\nu,D)}_{l_1,\dots,l_{_N}}(r_1,\dots,r_{_N}) = \frac{2\,\Gamma\left(\frac{D-\nu}{2}\right)}{(S_{_{D-1}})^N \Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{D}{2}\right)} \int_0^\infty du\, u^{\nu-1} \prod_{k=1}^{N} \varphi^{(D)}_{l_k}(r_k u) \qquad (2.15)$$

where we introduce a function of the form

$$\varphi^{(D)}_l(x) = \frac{i^l\, 2\,\pi^{\frac{D}{2}}\, J_{l+\frac{D}{2}-1}(2x)}{(x)^{\frac{D}{2}-1}} \qquad (2.16)$$

Unlike (2.12b), the expression (2.15) is symmetric over all $r_k,\ k{=}1,\dots,N$.

Orthogonalization $R^{(\nu,D)}_{l_1,\dots,l_{_N}}(r_1,\dots,r_{_N})$

Let $\phi_n(r)$ be an arbitrary orthogonal system of functions in the domain of value $r{\in}E$ with weight $\rho(r){>}0$.

$$\int_E dr\, \rho(r)\phi_n(r)\phi_m(r) = \delta_{n,m} \qquad (2.17)$$

Represent the expression (2.15) in series of these functions

$$R^{(\nu,D)}_{\mathbf{l}}(r_1,\dots,r_{_N}) = \sum_{\mathbf{n}} A_{\mathbf{l},\mathbf{n}} \prod_{k=1}^{N} \phi^{(D)}_{n_k}(r_k) \qquad (2.18)$$

then from (2.17) and (2.18) the coefficients $A_{\mathbf{n},\mathbf{l}}$ will be in the form

$$A_{\mathbf{n},\mathbf{l}} = \int_E dr_1 \rho(r_1)\phi^{(D)}_{n_1}(r_1) \dots \int_E dr_{_N}\rho(r_{_N})\phi^{(D)}_{n_N}(r_{_N}) R^{(\nu,D)}_{\mathbf{l}}(r_1,\dots,r_{_N})$$

If enter a function of the form (using (2.16))

$$\xi^{(\nu,D)}_{n,l}(u) = \int_E dr \rho(r)\,\phi_n(r)\varphi^{(D)}_l(ru) = i^l\, 2\,\pi^{\frac{D}{2}} \int_E dr \rho(r)\phi_n(r)\, \frac{J_{l+\frac{D}{2}-1}(2ru)}{(ru)^{\frac{D}{2}-1}} \qquad (2.19)$$

we obtain the coefficient of the following form

$$A_{\mathbf{n},\mathbf{l}} = \frac{2\,\Gamma\left(\frac{D-\nu}{2}\right)}{(S_{_{D-1}})^N\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{D}{2}\right)}\int\limits_0^\infty du\, u^{\nu-1}\prod_{k=1}^{N}\xi^{(\nu,D)}_{n_k,l_k}(u) \tag{2.20}$$

We find such functions $\phi_n(r)$ with the domain E, in which the functions $\xi^{(\nu,D)}_{n,l}(u)$ in (2.19) will also form an orthogonal system of the function with weight $u^{\nu-1}$. Indeed

$$\begin{aligned}
&\int\limits_0^\infty du\, u^{\nu-1}\xi^{(\nu,D)}_{n_1,l}(u)\xi^{(\nu,D)}_{n_2,l}(u) = \\
&\quad = \int_E dr_1\,\rho(r_1)\phi_{n_1}(r_1)\int_E dr_2\,\rho(r_2)\phi_{n_2}(r_2)\int\limits_0^\infty du\, u^{\nu-1}\varphi^{(D)}_l(r_1u)\varphi^{(D)}_l(r_2u) = \\
&\quad = (-1)^l\,4\,\pi^D\int_E dr_1\,\rho(r_1)\phi_{n_1}(r_1)\int_E dr_2\,\rho(r_2)\phi_{n_2}(r_2)\times \\
&\quad\times\int\limits_0^\infty du\, u^{\nu-1}\frac{J_{l+\frac{D}{2}-1}(2r_1u)}{(r_1u)^{\frac{D}{2}-1}}\frac{J_{l+\frac{D}{2}-1}(2r_2u)}{(r_2u)^{\frac{D}{2}-1}}
\end{aligned} \tag{2.21}$$

The integral by u can be expressed at $N=2$ from (2.14) or found from the general expression from (2.12b) and represented as

$$\begin{aligned}
&\int\limits_0^\infty du\, u^{\nu-1}\frac{J_{l+\frac{D}{2}-1}(2r_1u)}{(r_1u)^{\frac{D}{2}-1}}\frac{J_{l+\frac{D}{2}-1}(2r_2u)}{(r_2u)^{\frac{D}{2}-1}} = \\
&\quad = \frac{\Gamma\left(l+\frac{\nu}{2}\right)}{2\Gamma\left(\frac{D-\nu}{2}\right)\Gamma\left(l+\frac{D}{2}\right)\,(r_2)^\nu}\left(\frac{r_1}{r_2}\right)^l {}_2F_1\left[\left.\begin{matrix} l+\frac{\nu}{2},\ \frac{\nu-D+2}{2} \\ l+\frac{D}{2}\end{matrix}\right|\left(\frac{r_1}{r_2}\right)^2\right] = \\
&\quad = \frac{2^{\frac{D-\nu-3}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{D-\nu}{2}\right)(r_1r_2)^{\frac{\nu}{2}}}\frac{e^{-i\pi\frac{\nu-D+1}{2}}Q^{\frac{\nu-D+1}{2}}_{l+\frac{D-3}{2}}(z)}{\sqrt{z^2-1}^{\frac{\nu-D+1}{2}}}
\end{aligned} \tag{2.22}$$

$$z = \frac{r_1^2+r_2^2}{2r_1r_2} > 1$$

where in the last expression we expressed the hypergeometric series in terms of the Legendre function of the second kind [11, Ch.3, eq.3.2(45)].

In the paper [5], it was obtained that for the Legendre function of the second kind, the addition theorem of the following form is valid

$$\begin{aligned}
\frac{e^{-i\pi\mu}Q^\mu_v(z)}{\sqrt{z^2-1}^\mu} &= \frac{2^{2v+\frac{3}{2}}\Gamma(v+1)^2}{\sqrt{\pi}\left(\sqrt{z_1^2+1}\sqrt{z_2^2+1}\right)^{v+\mu+1}}\times \\
&\times\sum_{n=0}^{\infty}\frac{(-1)^n n!\,(n+v+1)}{\Gamma(n+2v+2)}C^{v+1}_n\left(\frac{z_1}{\sqrt{z_1^2+1}}\right)C^{v+1}_n\left(\frac{z_2}{\sqrt{z_2^2+1}}\right)\frac{e^{-i\pi\left(\mu-\frac{1}{2}\right)}Q^{\mu-\frac{1}{2}}_{n+v+\frac{1}{2}}(z_3)}{\sqrt{z_3^2-1}^{\mu-\frac{1}{2}}}
\end{aligned}$$

$$z = z_1z_2 + z_3\sqrt{z_1^2+1}\sqrt{z_2^2+1},\ z_3>1$$

given the limit at $z_3\to 1$ of [11, Sec.3.9.2]

$$\lim_{z_3\to 1}\frac{e^{-i\,\pi\,\left(\mu-\frac{1}{2}\right)}Q^{\mu-\frac{1}{2}}_{n+v+\frac{1}{2}}(z_3)}{\sqrt{z_3^2-1}^{\,\mu-\frac{1}{2}}}=\frac{\Gamma(n+v+\mu+1)\Gamma\left(\frac{1}{2}-\mu\right)}{2^{\mu+\frac{1}{2}}\Gamma(n+v-\mu+2)},\ \mu<\frac{1}{2}$$

and also by introducing other variables

$$z_1=\frac{r_1^2-1}{2r_1},\ z_2=\frac{1-r_2^2}{2r_2}$$

$$\text{where}\quad z=z_1z_2+\sqrt{z_1^2+1}\sqrt{z_2^2+1}=\frac{r_1^2+r_2^2}{2r_1r_2}$$

by $v=l+\frac{D-3}{2},\ \mu=\frac{\nu-D+1}{2}$, we get the following expansion

$$\frac{e^{-i\,\pi\,\left(\frac{\nu-D+1}{2}\right)}Q^{\frac{\nu-D+1}{2}}_{l+\frac{D-3}{2}}(z)}{\sqrt{z^2-1}^{\,\frac{\nu-D+1}{2}}}=\frac{2^{4l+\frac{3M+\nu-5}{2}}\Gamma\left(\frac{D-\nu}{2}\right)\Gamma\left(l+\frac{D-1}{2}\right)^2(r_1r_2)^{l+\frac{\nu}{2}}}{\sqrt{\pi}\left((r_1^2+1)(r_2^2+1)\right)^{l+\frac{\nu}{2}}}\times$$
$$\times\sum_{n=l}^{\infty}\frac{(n-l)!\,\left(n+\frac{D-1}{2}\right)\Gamma\left(n+\frac{\nu}{2}\right)}{(n+l+D-2)!\,\Gamma\left(n+D-\frac{\nu}{2}\right)}C^{l+\frac{D-1}{2}}_{n-l}\left(\frac{r_1^2-1}{r_1^2+1}\right)C^{l+\frac{D-1}{2}}_{n-l}\left(\frac{r_2^2-1}{r_2^2+1}\right)$$
$$\nu<D$$

Thus, introducing functions of the form (2.6), we write (2.22) as

$$\int_0^\infty du\,u^{\nu-1}\frac{J_{l+\frac{D}{2}-1}(2r_1u)}{(r_1u)^{\frac{D}{2}-1}}\frac{J_{l+\frac{D}{2}-1}(2r_2u)}{(r_2u)^{\frac{D}{2}-1}}=$$
$$=\frac{1}{4}\left((r_1^2+1)(r_2^2+1)\right)^{\frac{D-\nu}{2}}\sum_{n=l}^{\infty}\frac{\Gamma\left(n+\frac{\nu}{2}\right)}{\Gamma\left(n+D-\frac{\nu}{2}\right)}\eta^{(D)}_{n,l}(r_1)\eta^{(D)}_{n,l}(r_2)$$

given this expression, we get for (2.21) in the form

$$\int_0^\infty du\,u^{\nu-1}\xi^{(\nu,D)}_{n_1,l}(u)\xi^{(\nu,D)}_{n_2,l}(u)=(-1)^l\,\pi^D\sum_{n=l}^{\infty}\frac{\Gamma\left(n+\frac{\nu}{2}\right)}{\Gamma\left(n+D-\frac{\nu}{2}\right)}\times$$
$$\times\int_E dr_1\,\rho(r_1)\phi_{n_1}(r_1)(r_1^2+1)^{\frac{D-\nu}{2}}\eta^{(D)}_{n,l}(r_1)\int_E dr_2\,\rho(r_2)\phi_{n_2}(r_2)(r_2^2+1)^{\frac{D-\nu}{2}}\eta^{(D)}_{n,l}(r_2)\quad(2.23)$$

Obviously, for

$$\phi_n(r)=(r^2+1)^{\frac{D-\nu}{2}}\eta^{(D)}_{n,l}(r),\quad\rho(r)=\frac{r^{D-1}}{(r^2+1)^{D-\nu}}\quad(2.24)$$

provided (2.7) and the domain $E>0$, the function $\xi^{(\nu,D)}_{n,l}(u)$ will form an orthogonal system (2.4). Comparing (2.12a) with (2.12c) and also with (2.18)-(2.20), introducing the functions (2.2) and (2.5) and changing the order of summing the series, we get (2.1). □

3 The function $\xi^{(\nu,D)}_{n,l}(u)$ and its other representations.

From (2.19) and (2.24) we have that

$$\xi^{(\nu,D)}_{n,l}(u)=2\,i^l\,\pi^{\frac{D}{2}}\int_0^\infty dr\,r^{D-1}\frac{\eta^{(D)}_{n,l}(r)}{(r^2+1)^{\frac{D-\nu}{2}}}\frac{J_{l+\frac{D}{2}-1}(2ru)}{(ru)^{\frac{D}{2}-1}}\quad(3.1)$$

or in the form, using the expression (2.6) for $\eta_{n,l}^{(D)}(r)$ as the hypergeometric Gauss function for the Gegenbauer function [11, Sec. 3.15]

$$\xi_{n,l}^{(\nu,D)}(u) = \frac{i^l\, 4\,\pi^{\frac{D}{2}}}{\Gamma\left(l+\frac{D}{2}\right)}\sqrt{\frac{\left(n+\frac{D-1}{2}\right)(n+l+D-2)!}{(n-l)!}}\times$$
$$\times\, u^{l+D-\nu}\int_0^\infty dt\, \frac{t^{l+\frac{D}{2}}J_{l+\frac{D}{2}-1}(2t)}{(t^2+u^2)^{l+D-\frac{\nu}{2}}}\; {}_2F_1\left[\left.\begin{matrix}-n+l\,,\; n+l+D-1\\ l+\dfrac{D}{2}\end{matrix}\,\right|\, \frac{u^2}{t^2+u^2}\right] \tag{3.2}$$

Unlike $\eta_{n,l}^{(D)}(r)$(2.6), the function $\xi_{n,l}^{(\nu,D)}(u)$ is not expressed elementary in the general case. Thus, revealing the hypergeometric series in (3.2) and using integrals from the Bessel function [10, Ch.2, eq.2.12.4(28)]

$$\int_0^\infty dx\, \frac{x^{v+1}J_v(cx)}{(x^2+z^2)^\rho} = \frac{c^{\rho-1}z^{v-\rho+1}}{2^{\rho-1}\Gamma(\rho)}K_{v-\rho+1}(cz), \quad -1<v<2\rho-\frac{1}{2}$$

we can represent (3.2) as a finite series (here, by definition, the condition $K_{-\mu}(z) = K_\mu(z)$ holds for the MacDonald function)

$$\xi_{n,l}^{(\nu,D)}(u) = \frac{i^l\, 4\,\pi^{\frac{D}{2}}}{\Gamma\left(l+\frac{D}{2}\right)\Gamma\left(l+D-\dfrac{\nu}{2}\right)}\sqrt{\frac{\left(n+\frac{D-1}{2}\right)(n+l+D-2)!}{(n-l)!}}\times$$
$$\times\sum_{m=0}^{n-l}\frac{(-n+l)_m(n+l+D-1)_m}{\left(l+\frac{D}{2}\right)_m\left(l+D-\frac{\nu}{2}\right)_m m!}(u)^{l+m+\frac{D-\nu}{2}}K_{\frac{D-\nu}{2}+m}(2u)$$

Using an integral of the form [12, Ch.2, eq.2.3.16(1)] ($u^2>0$)

$$\int_0^\infty dt\, e^{-t-\frac{u^2}{t}}t^{\frac{D-\nu}{2}+m-1} = 2(u)^{\frac{D-\nu}{2}+m}K_{\frac{D-\nu}{2}+m}(2u)$$

and comparing this expression with the previous one, we get (2.3). Similarly to the expression (3.2), can also get a general expression

$$\xi_{n,l}^{(\nu,D)}(u) = \frac{i^l\, 4\,\pi^{\frac{D}{2}}\,\Gamma\left(\frac{D-\nu}{2}+\lambda+1\right)}{\Gamma\left(l+\frac{D}{2}\right)\Gamma\left(l+D-\frac{\nu}{2}\right)}\sqrt{\frac{\left(n+\frac{D-1}{2}\right)(n+l+D-2)!}{(n-l)!}}\times$$
$$\times\, u^{l+D-\nu}\int_0^\infty dt\, \frac{t^{\lambda+1}J_\lambda(2t)}{(t^2+u^2)^{\frac{D-\nu}{2}+\lambda+1}}\; {}_3F_2\left[\left.\begin{matrix}-n+l\,,\; n+l+D-1\,,\; \dfrac{D-\nu}{2}+\lambda+1\\ l+\dfrac{D}{2}\,,\; l+D-\dfrac{\nu}{2}\end{matrix}\,\right|\, \frac{u^2}{t^2+u^2}\right]$$
$$\lambda>-1,\;\; \lambda+D-\nu+\frac{3}{2}>0$$

4 Conclusion

The paper presents the expansion of $|\mathbf{r}_2+\ldots+\mathbf{r}_N|^{-\nu}$ from N vectors $\mathbf{r}_k\in\mathbf{R}^D$ in D dimensional space in the form of (2.1) by functions (2.5), which represents the product of an orthogonal radial function, and an angular hyperspherical function on the unit sphere $\mathbf{S}^{D-1}$. Or an equivalent decomposition of (2.9) by hyperspherical functions on the unit sphere $\mathbf{S}^D$. The choice of such functions in the expansion is because, as can be seen from (2.23) and (2.24), the function (2.3) formed an orthogonal system. This is convenient because for N=2 the expansions (2.1) and (2.9) take a simple form as in the Example 2, in which there are no complex integral coefficients $G_{\mathbf{n},\mathbf{l},\mathbf{m_k}}$.

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