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WELL-POSEDNESS AND STABILITY RESULT FOR A THERMOELASTIC LAMINATED BEAM WITH STRUCTURAL DAMPING AND DISTRIBUTED DELAY TERM

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Abstract. This paper deals with a thermoelastic laminated beam along with structural damping and distributed delay term, where both the rotation angle and the transverse displacements are affected by the heat conduction, which is described by Fourier's law. Using the semigroup method, we establish the existence and uniqueness of the solution. Regarding stability results, we demon- strate exponential and polynomial stabilities of the system for the cases of equal and non-equal wave speeds respectively.

1 Introduction

In the current work, we study the following thermoelastic laminated beam, together with structural damping and distributed delay term

$$\begin{cases} \varrho\vartheta_{tt} + G(\phi_x - \vartheta_{xx}) + \gamma\theta_x = 0, \\ \\ I_\varrho(3\psi - \phi)_{tt} - D(3\psi - \phi)_{xx} - G(\phi - \vartheta_x) - \gamma\theta = 0, \\ \\ 3I_\varrho\psi_{tt} - 3D\psi_{xx} + 3G(\phi - \vartheta_x) + 4\delta\psi + 4\beta\psi_t + 4\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\psi_t(x, t - \varsigma)d\varsigma = 0, \\ \\ \varrho_3\theta_t - k\theta_{xx} + \gamma(3\psi - \phi)_t + \gamma\vartheta_{tx} = 0, \end{cases} \tag{1.1}$$

where

$$(x, \varsigma, t) \in (0, 1) \times (\varsigma_1, \varsigma_2) \times (0, +\infty),$$

with the following initial and boundary conditions

$$\begin{cases} \vartheta(x, 0) = \vartheta_0,\ \psi(x, 0) = \psi_0,\ \phi(x, 0) = \phi_0,\ \theta(x, 0) = \theta_0, \quad x \in (0, 1), \\ \vartheta_t(x, 0) = \vartheta_1,\ \psi_t(x, 0) = \psi_1,\ \phi_t(x, 0) = \phi_1, \quad x \in (0, 1), \\ \vartheta_x(1, t) = \vartheta_x(0, t) = \psi(1, t) = \psi(0, t) = 0, \quad t > 0, \\ \phi(0, t) = \phi(1, t) = \theta(0, t) = \theta(1, t) = 0, \quad t > 0, \end{cases} \tag{1.2}$$

here ϑ denotes the transverse displacement, ϕ represents the rotation angle, $\psi$is relative to the amount of slip occurring along the interface and θ is the difference temperature. The coefficients

$\delta,\ \beta,\ \varrho,\ I_\varrho,\ G$, and D are positive constants representing the adhesive stiffness, the adhesive damping parameter, the density, the shear stiffness, the flexural rigidity, and the mass moment of inertia, respectively. We denote by $\varrho_3,\ \ k,\gamma > 0$ the physical parameters from theory of thermoelasticity.
Herein, $\varsigma_1,\ \varsigma_2$ are positive numbers such that $0 \leq \varsigma_1 \leq \varsigma_2$, and μ_2 is an L^∞ function satisfying the following assumption

(A$_1$) The function $\mu_2 : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is bounded, and it fulfills

$$\beta - \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| d\varsigma > 0.$$

The laminated beam model has become quite popular, that scientists and engineers are interested in it. This model is a pertinent study topic, because of the wide industry applicability of such materials. Hansen and Spies in [9], were the first to introduce the following beam with two layers by developing this mathematical model

$$\begin{cases} \rho_1 \vartheta_{tt} + G(\phi - \vartheta_x)_x = 0, \\ \rho_2 (3\psi - \phi)_{tt} - G(\phi - \vartheta_x) - D(3\psi - \phi)_{xx} = 0, \\ \rho_3 \psi_{tt} + G(\phi - \vartheta_x) + \frac{4}{3}\gamma\psi + \frac{4}{3}\beta\psi_t - D\psi_{xx} = 0. \end{cases}$$

Lately, a renewed focus on investigating the asymptotic behavior of the solution of several thermoelastic laminated beams has grown. For more details about this topic the reader may consult [3, 5, 6, 7, 8, 12].

The time delays problems are one of the most significant and active research areas recently. Numerous studies have demonstrated that delay can lead to instability unless certain conditions are incorporated, it also can lead to distinct solutions that differ from those found in prior studies. Therefore, the issue of stability for systems that involve delay is highly crucial. To learn more about this term, we refer the reader to the following papers [1, 2, 4, 13, 14, 15, 16].
In [11], Nicaise and Pignotti made a study on the following wave equation, together with linear frictional damping and internal distributed delay

$$u_{tt} - \Delta u + \mu_1 u_t + a(x) \int_{\varsigma_1}^{\varsigma_2} \mu_2(s) u_t(t-s)\, ds, \qquad \text{in } \Omega \times (0, \infty),$$

assuming that

$$\|a\|_\infty \int_{\varsigma_1}^{\varsigma_2} \mu_2(s)\, ds < \mu_1,$$

the authors managed to prove that the solution is exponentially stable.

Recently, Fayssal [5] considered a thermoelastic laminated beam, along with structural damping and Fourier's law, he established an exponential stability result for the problem with equal wave speeds, i.e.

$$\sqrt{\frac{\varrho}{G}} = \sqrt{\frac{I_\varrho}{D}}. \tag{1.3}$$

The rest of the paper is structured this way, in section 2, we provide some resources required for our research, then highlight our major results. In section 3, we establish the well-posedness of the system, in Section 4, we introduce some fundamental lemmas required in the proof later. In section 5, using the multiplier technique, we prove an exponential stability of the system in case of equal wave speeds, and a polynomial one in the opposite case.

2 Preliminaries and main results

In this section, we state our major results, and provide some practical materials needed in the proof later.

To achieve our goal, we start by introducing

$$\mathscr{Y}(x,p,\varsigma,t)=\psi_t(x,t-\varsigma p),$$

where

$$(x,p,\varsigma,t)\in(0,1)\times(0,1)\times(\varsigma_1,\varsigma_2)\times(0,+\infty),$$

then, the variable $\mathscr{Y}$ surely satisfies

$$\begin{cases}\varsigma\mathscr{Y}_t(x,p,\varsigma,t)+\mathscr{Y}_p(x,p,\varsigma,t)=0,\\ \mathscr{Y}(x,0,\varsigma,t)=\psi_t(x,t).\end{cases}$$

Thereby, system (1.1) can be rewritten as

$$\begin{cases}\varrho\vartheta_{tt}+G(\phi_x-\vartheta_{xx})+\gamma\theta_x=0,\\ I_\varrho(3\psi-\phi)_{tt}-D(3\psi-\phi)_{xx}-G(\phi-\vartheta_x)-\gamma\theta=0,\\ 3I_\varrho\psi_{tt}-3D\psi_{xx}+3G(\phi-\vartheta_x)+4\delta\psi+4\beta\psi_t+4\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathscr{Y}(x,1,\varsigma,t)d\varsigma=0,\\ \varrho_3\theta_t-k\theta_{xx}+\gamma(3\psi-\phi)_t+\gamma\vartheta_{tx}=0,\\ \varsigma\mathscr{Y}_t(x,p,\varsigma,t)+\mathscr{Y}_p(x,p,\varsigma,t)=0,\end{cases}\tag{2.1}$$

certainly, system (2.1) is depending on the initial and boundary conditions below

$$\begin{cases}\vartheta(x,0)=\vartheta_0,\ \phi(x,0)=\phi_0,\ \psi(x,0)=\psi_0,\ \theta(x,0)=\theta_0,\quad x\in(0,1),\\ \vartheta_t(x,0)=\vartheta_1,\ \phi_t(x,0)=\phi_1,\ \psi_t(x,0)=\psi_1,\quad x\in(0,1),\\ \vartheta_x(1,t)=\vartheta_x(0,t)=\phi(1,t)=\phi(0,t)=0,\quad t>0,\\ \psi(0,t)=\psi(1,t)=\theta(0,t)=\theta(1,t)=0,\quad t>0,\\ \mathscr{Y}(x,p,\varsigma,0)=f_0(x,p\varsigma),\quad (x,p,\varsigma)\in(0,1)\times(\varsigma_1,\varsigma_2)\times\mathbb{R}_+.\end{cases}\tag{2.2}$$

Now, let

$$\begin{cases}\zeta=3\psi-\phi,\\ \zeta(0,t)=\zeta(1,t)=0,\ \zeta(x,0)=\zeta_0,\ \zeta_t(x,0)=\zeta_1,\ (x,t)\in(0,1)\times\mathbb{R}_+.\end{cases}$$

Hence, (2.1) is equivalent to

$$\begin{cases}\varrho\vartheta_{tt}+G(3\psi-\zeta-\vartheta_x)_x+\gamma\theta_x=0,\\ I_\varrho\zeta_{tt}-D\zeta_{xx}-G(3\psi-\zeta-\vartheta_x)-\gamma\theta=0,\\ 3I_\varrho\psi_{tt}-3D\psi_{xx}+3G(3\psi-\zeta-\vartheta_x)+4\delta\psi+4\beta\psi_t+4\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathscr{Y}(x,1,\varsigma,t)d\varsigma=0,\\ \varrho_3\theta_t-k\theta_{xx}+\gamma\zeta_t+\gamma\vartheta_{tx}=0,\\ \varsigma\mathscr{Y}_t(x,p,\varsigma,t)+\mathscr{Y}_p(x,p,\varsigma,t)=0.\end{cases}\tag{2.3}$$

For the purpose of applying Poincaré's inequality for ϑ, we shall consider minor transformation, using boundary conditions and $(2.1)_1$, we come to the conclusion that

$$\frac{d^2}{dt^2}\int_0^1\vartheta(x,t)dx=0,\qquad\text{for all }\ t\geq0,\tag{2.4}$$

solving this ODE yields

$$\int_0^1 \vartheta(x,t)dx = t\int_0^1 \vartheta_1(x)dx + \int_0^1 \vartheta_0(x)dx, \quad \text{for all } \ t \geq 0.$$

As a consequence, if we denote by

$$\tilde{\vartheta}(x,t) = \vartheta(x,t) - t\int_0^1 \vartheta_1(x)dx - \int_0^1 \vartheta_0(x)dx,$$

we obtain

$$\int_0^1 \tilde{\vartheta}(x,t)dx = 0, \quad \text{for all } \ t \geq 0.$$

Furthermore, one can check that $(\tilde{\vartheta}, \phi, \psi, \theta, \mathscr{Y})$ fulfils system (2.1)–(2.2).
From now on, we will be using $\tilde{\vartheta}$ rather than ϑ, but we will write ϑ for clarity of matters.
At this step, let us introduce the vector function $U = (\vartheta, u, \zeta, \nu, \psi, z, \theta, \mathscr{Y})^T$, with

$$\begin{aligned} u &= \vartheta_t, \\ \nu &= \zeta_t, \\ z &= \psi_t, \end{aligned}$$

then, system (2.3) becomes

$$\begin{cases} \frac{d}{dt}U(t) = \mathfrak{A}U(t), \quad t > 0, \\ U(0) = U_0 = (\vartheta_0, \vartheta_1, \zeta_0, \zeta_1, \psi_0, \psi_1, \theta_0, f_0)^T, \end{cases} \tag{2.5}$$

here, $\mathfrak{A} : D(\mathfrak{A}) \subset \mathcal{H} :\to \mathcal{H}$ stands for a linear operator indicated by

$$\mathfrak{A}U = \begin{pmatrix} u \\ -\frac{1}{\varrho}\left(G(3\psi - \zeta - \vartheta_x)_x + \gamma\theta_x\right) \\ \nu \\ \frac{1}{I_\varrho}\left(D\zeta_{xx} + G(3\psi - \zeta - \vartheta_x) + \gamma\theta\right) \\ z \\ \frac{1}{I_\varrho}\left(D\psi_{xx} - G(3\psi - \zeta - \vartheta_x) - \frac{4}{3}\delta\psi - \frac{4}{3}\beta z - \frac{4}{3}\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathscr{Y}(x,1,\varsigma,t)d\varsigma\right) \\ \frac{1}{\varrho_3}\left(k\theta_{xx} - \gamma\nu - \gamma u_x\right) \\ -\frac{1}{\varsigma}\mathscr{Y}_p \end{pmatrix}.$$

Now, we shall consider the ensuing energy space

$$\mathcal{H} = \mathbb{J}_*^1(0,1) \times \mathbb{L}_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1)$$
$$\times L^2(0,1) \times L^2\left((0,1)\times(0,1)\times(\varsigma_1,\varsigma_2)\right),$$

where

$$\begin{aligned} \mathbb{L}_*^2(0,1) &= \left\{\varphi \in L^2(0,1) : \textstyle\int_0^1 \varphi(x)dx = 0\right\}, \\ \mathbb{J}_*^1(0,1) &= H^1(0,1) \cap \mathbb{L}_*^2(0,1), \\ \mathbb{J}_*^2(0,1) &= \left\{\varphi \in H^2(0,1) : \varphi_x(0) = \varphi_x(1) = 0\right\}. \end{aligned}$$

Then, we introduce

$$\langle U, \bar{U}\rangle_{\mathcal{H}} = \varrho\int_0^1 u\bar{u}\,dx + I_\varrho\int_0^1 \nu\bar{\nu}\,dx + 3I_\varrho\int_0^1 z\bar{z}\,dx + \varrho_3\int_0^1 \theta\bar{\theta}\,dx + D\int_0^1 \zeta_x\bar{\zeta}_x\,dx$$
$$+G\int_0^1 (3\psi - \zeta - \vartheta_x)(3\bar{\psi} - \bar{\zeta} - \bar{\vartheta}_x)dx + 4\delta\int_0^1 \psi\bar{\psi}\,dx + 3D\int_0^1 \psi_x\bar{\psi}_x\,dx$$

$$+4\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2} \varsigma|\mu_2(\varsigma)|\mathscr{Y}\bar{\mathscr{Y}}\ d\varsigma dp dx. \tag{2.6}$$

We deduce that $\mathcal{H}$ along with (2.6) is a Hilbert space, once we do that, we define $D(\mathfrak{A})$ by

$$D(\mathfrak{A}) = \left\{ \begin{array}{c} U \in \mathcal{H}:\ \vartheta \in \mathbb{J}_*^2(0,1)\cap\mathbb{J}_*^1(0,1);\ \zeta,\ \psi,\ \theta \in H^2(0,1)\cap H_0^1(0,1); \\ u \in \mathbb{J}_*^1(0,1);\ \nu,\ z \in H_0^1(0,1); \\ \mathscr{Y},\ \mathscr{Y}_p \in L^2\left((0,1)\times(0,1)\times(\varsigma_1,\varsigma_2)\right),\ \mathscr{Y}(x,0,\varsigma,t) = z. \end{array} \right\}$$

Obviously, $D(\mathfrak{A})$ is dense in $\mathcal{H}$.

Now, we are prepared to state our results.

Theorem 2.1. *For any initial data $U_0 \in D(\mathfrak{A})$, problem (2.2)-(2.3) admits a unique solution*

$$U \in C(\mathbb{R}_+, D(\mathfrak{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

In addition, if $U_0 \in \mathcal{H}$, then

$$U \in C(\mathbb{R}_+, \mathcal{H}).$$

We give the energy functional of the solution of (2.1)-(2.2) by

$$\begin{aligned} \mathscr{E}(t) = \frac{1}{2}\int_0^1 &\left\{\varrho\vartheta_t^2 + G(\phi-\vartheta_x)^2 + I_\varrho(3\psi_t-\phi_t)^2 + D(3\psi_x-\phi_x)^2 + 3I_\varrho\psi_t^2\right. \\ &\left. +3D\psi_x^2 + 4\delta\psi^2 + \varrho_3\theta^2\right\} dx \\ &+2\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2} \varsigma|\mu_2(\varsigma)|\mathscr{Y}^2(x,p,\varsigma,t)\ d\varsigma dp dx. \end{aligned} \tag{2.7}$$

Thereby, the stability results are as follows.

Theorem 2.2 (Exponential stability). *Let $(\vartheta,\phi,\psi,\theta,\mathscr{Y})$ be the solution of (2.1)–(2.2), suppose that (A_1), and (1.3) hold. So, there exist $\alpha_1, \alpha_2 > 0$, such that*

$$\mathscr{E}(t) \le \alpha_1 e^{-\alpha_2 t}, \quad \forall t \ge 0. \tag{2.8}$$

Theorem 2.3 (Polynomial stability). *Let $(\vartheta,\phi,\psi,\theta,\mathscr{Y})$ be the solution of (2.1)–(2.2), suppose that (A_1) holds. So, there exists a positive constant λ such that the energy functional satisfies*

$$\mathscr{E}(t) \le \frac{\lambda}{t}, \quad \textit{for all}\ \ t > 0.$$

3 Well-posedness

In this part, we utilize the semigroup method to prove our well-posedness result.

Proof of Theorem 2.1. Let's prove the dissipativity of $\mathfrak{A}$. By (2.6) and for any $U \in D(\mathfrak{A})$, we find

$$\begin{aligned} \langle\mathfrak{A}U,U\rangle_{\mathcal{H}} = &-4\beta\int_0^1 z^2\ dx - k\int_0^1 \theta_x^2\ dx - 4\int_0^1\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| z\mathscr{Y}(x,1,\varsigma,t)\ d\varsigma dx \\ &-4\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathscr{Y}_p\mathscr{Y}\ d\varsigma dp dx. \end{aligned}$$

One can notice that

$$\begin{aligned} -4\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathscr{Y}_p\mathscr{Y}\ d\varsigma dp dx &= -2\int_0^1\int_{\varsigma_1}^{\varsigma_2}\int_0^1 |\mu_2(\varsigma)|\partial_p\mathscr{Y}^2\ dp d\varsigma dx \\ &= -2\int_0^1\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathscr{Y}^2(x,1,\varsigma,t)\ d\varsigma dx \\ &\quad + 2\int_0^1\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathscr{Y}^2(x,0,\varsigma,t)\ d\varsigma dx. \end{aligned} \tag{3.1}$$

Applying Youg's inequality, we obtain

$$-4\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|z\mathscr{Y}(x,1,\varsigma,t)\,d\varsigma dx \le 2\left(\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\,d\varsigma\right)\int_0^1 z^2\,dx + 2\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathscr{Y}^2(x,1,\varsigma,t)\,d\varsigma dx,$$

therefore, by (A_1) and given $\mathscr{Y}(x,0,\varsigma,t)=z(x,t)$, we end up with

$$\langle\mathfrak{A}U,U\rangle_{\mathcal{H}} = -4\left(\beta-\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\,d\varsigma\right)\int_0^1 z^2\,dx - k\int_0^1\theta_x^2\,dx \le 0.$$

Thereby, $\mathfrak{A}$ is dissipative.
Thereafter, we establish the surjectivity of $(I-\mathfrak{A})$, that is, we show that

$$\forall H=(h_1,h_2,h_3,h_4,h_5,h_6,h_7,h_8)^T\in\mathcal{H},\ \exists U\in D(\mathfrak{A}):$$
$$(I-\mathfrak{A})U=H. \tag{3.2}$$

We have

$$\begin{cases}\vartheta-u=h_1,\\ \varrho u+G(3\psi-\zeta-\vartheta_x)_x+\gamma\theta_x=\varrho h_2,\\ \zeta-\nu=h_3,\\ I_\varrho\nu-D\zeta_{xx}-G(3\psi-\zeta-\vartheta_x)-\gamma\theta=I_\varrho h_4,\\ \psi-z=h_5,\\ 3I_\varrho z-3D\psi_{xx}+3G(3\psi-\zeta-\vartheta_x)+4\delta\psi+4\beta z+4\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathscr{Y}(x,1,\varsigma,t)d\varsigma=3I_\varrho h_6,\\ \varrho_3\theta-k\theta_{xx}+\gamma\nu+\gamma u_x=\varrho_3 h_7,\\ \varsigma\mathscr{Y}(x,p,\varsigma,t)+\mathscr{Y}_p(x,p,\varsigma,t)=\varsigma h_8.\end{cases} \tag{3.3}$$

Solving (3.3)$_8$, and using $\mathscr{Y}(x,0,\varsigma,t)=z(x,t)$, we find

$$\mathscr{Y}(x,p,\varsigma,t)=ze^{-\varsigma p}+\varsigma e^{-\varsigma p}\int_0^p e^{\varsigma\sigma}h_8(x,\sigma,\varsigma,t)\,d\sigma.$$

Hence,

$$\mathscr{Y}(x,1,\varsigma,t)=ze^{-\varsigma}+\varsigma e^{-\varsigma}\int_0^1 e^{\varsigma\sigma}h_8(x,\sigma,\varsigma,t)\,d\sigma. \tag{3.4}$$

Inserting

$$\begin{cases}u=\vartheta-h_1,\\ \nu=\zeta-h_3,\\ z=\psi-h_5,\end{cases}$$

and (3.4) into (3.3)$_2$, (3.3)$_4$, (3.3)$_6$, and (3.3)$_7$, we get

$$\begin{cases}\varrho\vartheta+G(3\psi-\zeta-\vartheta_x)_x+\gamma\theta_x=\varrho(h_1+h_2),\\ I_\varrho\zeta-D\zeta_{xx}-G(3\psi-\zeta-\vartheta_x)-\gamma\theta=I_\varrho(h_3+h_4),\\ \mu_1\psi-3D\psi_{xx}+3G(3\psi-\zeta-\vartheta_x)=\tilde{\mu}_1h_5+3I_\varrho h_6-4\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\varsigma e^{-\varsigma}\int_0^1 e^{\varsigma\sigma}h_8\,d\sigma d\varsigma,\\ \varrho_3\theta-k\theta_{xx}+\gamma\zeta+\gamma\vartheta_x=\gamma h_{1x}+\gamma h_3+\varrho_3h_7,\end{cases} \tag{3.5}$$

where,

$$\begin{cases}\mu_1=3I_\varrho+4\delta+4\beta+4\int_{\varsigma_1}^{\varsigma_2}e^{-\varsigma}|\mu_2(\varsigma)|\,d\varsigma,\\ \tilde{\mu}_1=3I_\varrho+4\beta+4\int_{\varsigma_1}^{\varsigma_2}e^{-\varsigma}|\mu_2(\varsigma)|\,d\varsigma.\end{cases}$$

We take the following variational formulation, to solve (3.5)

$$\mathscr{Q}((\vartheta,\zeta,\psi,\theta),(\bar{\vartheta},\bar{\zeta},\bar{\psi},\bar{\theta}))=L(\bar{\vartheta},\bar{\zeta},\bar{\psi},\bar{\theta})),\quad\forall(\bar{\vartheta},\bar{\zeta},\bar{\psi},\bar{\theta}))\in X, \tag{3.6}$$

with, $X = \mathbb{J}^1_*(0,1) \times \left(H^1_0(0,1)\right)^3$ is a Hilbert space endowed with the following norm

$$\|(\vartheta,\zeta,\psi,\theta)\|^2_X = \|3\psi-\zeta-\vartheta_x\|^2_2 + \|\vartheta\|^2_2 + \|\zeta_x\|^2_2 + \|\psi_x\|^2_2 + \|\theta\|^2_2 + \|\theta_x\|^2_2.$$

As a part of this step, we provide definitions for both the bilinear form $\mathscr{Q} : X \times X \to \mathbb{R}$, and the linear form $L : X \to \mathbb{R}$, as follows

$$\begin{aligned}
\mathscr{Q}((\vartheta,\zeta,\psi,\theta),(\bar\vartheta,\bar\zeta,\bar\psi,\bar\theta)) =& \varrho\int_0^1 \vartheta\bar\vartheta\,dx + I_\varrho\int_0^1 \zeta\bar\zeta\,dx + \mu_1\int_0^1 \psi\bar\psi\,dx \\
&+\varrho_3\int_0^1 \theta\bar\theta\,dx + D\int_0^1 \zeta_x\bar\zeta_x\,dx + 3D\int_0^1 \psi_x\bar\psi_x\,dx \\
&+G\int_0^1 (3\psi-\zeta-\vartheta_x)(3\bar\psi-\bar\zeta-\bar\vartheta_x)\,dx + k\int_0^1 \theta_x\bar\theta_x\,dx \\
&+\gamma\int_0^1 (\theta_x\bar\vartheta+\vartheta_x\bar\theta)\,dx + \gamma\int_0^1 (\zeta\bar\theta-\bar\zeta\theta)\,dx,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
L(\bar\vartheta,\bar\zeta,\bar\psi,\bar\theta) =& \varrho\int_0^1 \bar\vartheta(h_1+h_2)\,dx + I_\varrho\int_0^1 \bar\zeta(h_3+h_4)\,dx + \int_0^1 \bar\theta(\gamma h_{1x}+\gamma h_3+\varrho_3 h_7)\,dx \\
&+\int_0^1 \bar\psi\left[\bar\mu_1 h_5 + 3I_\varrho h_6 - 4\int_{\varsigma_1}^{\varsigma_2} \varsigma|\mu_2(\varsigma)|e^{-\varsigma}\int_0^1 e^{\varsigma\sigma}h_8\,d\sigma d\varsigma\right]dx.
\end{aligned}$$

We can easily prove the continuity of $\mathscr{Q}$ and L. Moreover, from (3.7) together with integration by parts, we conclude

$$\begin{aligned}
\mathscr{Q}((\vartheta,\zeta,\psi,\theta),(\vartheta,\zeta,\psi,\theta)) =& \varrho\int_0^1 \vartheta^2\,dx + I_\varrho\int_0^1 \zeta^2\,dx + \mu_1\int_0^1 \psi^2\,dx + \varrho_3\int_0^1 \theta^2\,dx + D\int_0^1 \zeta_x^2\,dx \\
&+3D\int_0^1 \psi_x^2\,dx + G\int_0^1 (3\psi-\zeta-\vartheta_x)^2\,dx + k\int_0^1 \theta_x^2\,dx \\
&\geq M\|(\vartheta,\zeta,\psi,\theta)\|^2_X,\ M>0.
\end{aligned}$$

From which, we conclude the coercivity of $\mathscr{Q}$. It follows from the Lax-Milgram lemma that (3.5) admits a unique solution satisfying

$$\vartheta \in \mathbb{J}^1_*(0,1),$$

and

$$\zeta,\ \psi,\ \theta \in H^1_0(0,1).$$

If we substitute ϑ, ζ, and ψ into $(3.3)_1$, $(3.3)_3$ and $(3.3)_5$, we find

$$u \in \mathbb{J}^1_*(0,1),$$

and

$$\nu,\ z \in H^1_0(0,1).$$

Besides, taking $(\bar\zeta,\bar\psi,\bar\theta) \equiv (0,0,0) \in \left(H^1_0(0,1)\right)^3$, relation (3.6) becomes

$$\varrho\int_0^1 \vartheta\bar\vartheta\,dx - G\int_0^1 (3\psi-\zeta-\vartheta_x)\bar\vartheta_x\,dx + \gamma\int_0^1 \bar\vartheta\theta_x\,dx = \varrho\int_0^1 \bar\vartheta(h_1+h_2)\,dx,$$

which indicates that

$$-G\int_0^1 \bar\vartheta_x\vartheta_x\,dx = \int_0^1 \bar\vartheta\left(\varrho\vartheta + 3G\psi_x - G\zeta_x + \gamma\theta_x - \varrho(h_1+h_2)\right)dx, \tag{3.8}$$

for any $\bar{\vartheta} \in \mathbb{J}_*^1(0,1)$. Next, considering $\bar{\varphi} \in H_0^1(0,1)$, with

$$\bar{\vartheta}(x) = \bar{\varphi}(x) - \int_0^1 \bar{\varphi}(x)\, dx,$$

it follows that, $\bar{\vartheta} \in \mathbb{J}_*^1(0,1)$, then, replacing in (3.8), we arrive at

$$-G \int_0^1 \bar{\varphi}_x \vartheta_x \, dx = \int_0^1 \bar{\varphi} \left(\varrho\vartheta + 3G\psi_x - G\zeta_x + \gamma\theta_x - \varrho(h_1 + h_2)\right) dx.$$

Thus, $\vartheta \in H^2(0,1)$, and we have

$$G\vartheta_{xx} = \varrho\vartheta + 3G\psi_x - G\zeta_x + \gamma\theta_x - \varrho(h_1 + h_2).$$

Now, for $\Phi \in H^1(0,1)$

$$G \int_0^1 \vartheta_{xx}\Phi \, dx = \int_0^1 \Phi \left(\varrho\vartheta + 3G\psi_x - G\zeta_x + \gamma\theta_x - \varrho(h_1 + h_2)\right) dx, \quad \forall \Phi \in H^1(0,1).$$

Thereby,

$$G\vartheta_x \Phi\,|_0^1 - G \int_0^1 \vartheta_x \Phi_x \, dx = \int_0^1 \Phi \left(\varrho\vartheta + 3G\psi_x - G\zeta_x + \gamma\theta_x \right.$$
$$\left. - \varrho(h_1 + h_2)\right) dx.$$

Because $\mathbb{J}_*^1(0,1) \subset H^1(0,1)$, we see that

$$G\bar{\vartheta}\vartheta_x|_0^1 - G \int_0^1 \bar{\vartheta}_x \vartheta_x \, dx = \int_0^1 \left[\varrho\vartheta + 3G\psi_x - G\zeta_x + \gamma\theta_x - \varrho(h_1 + h_2)\right] \bar{\vartheta}\, dx,$$

for all $\bar{\vartheta} \in \mathbb{J}_*^1(0,1)$. From (3.8), we get

$$\bar{\vartheta}(1)\vartheta_x(1) - \bar{\vartheta}(0)\vartheta_x(0) = 0, \quad \bar{\vartheta} \in \mathbb{J}_*^1(0,1).$$

Therefore, we can write $\vartheta_x(0) = \vartheta_x(1) = 0$. Hence $\vartheta \in \mathbb{J}_*^2(0,1)$.
Likewise, it is simple to show that

$$(\zeta, \psi, \theta) \in \left(H^2(0,1) \cap H_0^1(0,1)\right)^3.$$

The standard elliptic regularity guarantees the existence of a unique $U \in D(\mathfrak{A})$ which fulfils (3.2). Thereby, $\mathfrak{A}$ is surjective.
As a consequence, $\mathfrak{A}$ is a maximal dissipative operator. Then, the well-posedness result follows using Lumer–Philips theorem [10]. □

4 Technical lemmas

The main purpose of this section is to establish the essential practical lemmas required to prove our stability results. To attain this goal, we apply a specific approach known as the multiplier technique, which enables us to prove the stability results of problem (2.1). Nevertheless, this method necessitates creating an appropriate Lyapunov functional equivalent to the energy, and we will clarify on this in the next section. To simplify matters, we will employ $\chi_* > 0$ to represent a generic constant.

Lemma 4.1. *Let $(\vartheta, \phi, \psi, \theta, \mathscr{Y})$ be the solution of (2.1)–(2.2), then, the energy functional satisfies*

$$\frac{d}{dt}\mathscr{E}(t) \leq -k \int_0^1 \theta_x^2 dx - \varpi \int_0^1 \psi_t^2 dx, \quad \textit{where } \varpi, t > 0. \tag{4.1}$$

Proof. As a start, we multiply (2.1)$_1$, (2.1)$_2$, (2.1)$_3$, and (2.1)$_4$ by ϑ_t, $(3\psi_t - \phi_t)$, ψ_t, and θ respectively, we then integrate over $(0,1)$, and use both integration by parts and boundary conditions (2.2) to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 \{\varrho\vartheta_t^2 + G(\phi-\vartheta_x)^2 + I_\varrho(3\psi_t-\phi_t)^2 + D(3\psi_x-\phi_x)^2 + 3I_\varrho\psi_t^2\\ +3D\psi_x^2 + 4\delta\psi^2 + \varrho_3\theta^2\}\,dx + 4\beta\int_0^1 \psi_t^2dx + k\int_0^1 \theta_x^2dx\\ +4\int_0^1\int_{\varsigma_1}^{\varsigma_2} \psi_t|\mu_2(\varsigma)|\mathscr{Y}(x,1,\varsigma,t)\,d\varsigma dx = 0.\end{aligned} \tag{4.2}$$

Applying Young's inequality, we find

$$\begin{aligned}\int_0^1\int_{\varsigma_1}^{\varsigma_2} \psi_t|\mu_2(\varsigma)|\mathscr{Y}(x,1,\varsigma,t)\,d\varsigma dx \le \frac{1}{2}\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathscr{Y}^2(x,1,\varsigma,t)\,d\varsigma dx\\ +\frac{1}{2}\left(\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|d\varsigma\right)\int_0^1 \psi_t^2\,dx.\end{aligned} \tag{4.3}$$

Next, we multiply (2.1)$_5$ by $\mathscr{Y}|\mu_2(\varsigma)|$ and integrate the result over $(0,1)\times(0,1)\times(\varsigma_1,\varsigma_2)$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2}\varsigma|\mu_2(\varsigma)|\mathscr{Y}^2(x,p,\varsigma,t)\,d\varsigma dp dx &= -\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathscr{Y}_p\mathscr{Y}(x,p,\varsigma,t)\,d\varsigma dp dx\\ &= -\frac{1}{2}\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\partial_p\mathscr{Y}^2(x,p,\varsigma,t)\,d\varsigma dp dx\\ &= \frac{1}{2}\left(\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|d\varsigma\right)\int_0^1\psi_t^2\,dx\\ &\quad -\frac{1}{2}\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathscr{Y}^2(x,1,\varsigma,t)\,d\varsigma dx.\end{aligned} \tag{4.4}$$

Combining (4.2), (4.3), (4.4), and (A$_1$), we obtain

$$\frac{d}{dt}\mathscr{E}(t) \le -k\int_0^1\theta_x^2dx - 4\left(\beta - \int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|d\varsigma\right)\int_0^1\psi_t^2dx \le 0.$$

□

Lemma 4.2. *Consider the functional*

$$\mathscr{I}_1(t) := \varrho\int_0^1(\phi-\vartheta_x)\int_0^x\vartheta_t(y)dy\,dx - \frac{\varrho_3\varrho}{\gamma}\int_0^1\theta\int_0^x\vartheta_t(y)dydx, \tag{4.5}$$

then, it satisfies, for any $\epsilon_1 > 0$,

$$\begin{aligned}\mathscr{I}_1'(t) \le \frac{-G}{2}\int_0^1(\phi-\vartheta_x)^2dx + \epsilon_1\int_0^1\vartheta_t^2dx + \frac{\chi_*}{\epsilon_1}\int_0^1\psi_t^2dx\\ +\chi_*\left(1+\frac{1}{\epsilon_1}\right)\int_0^1\theta_x^2dx.\end{aligned} \tag{4.6}$$

Proof. We proceed by differentiating $\mathscr{I}_1$, we find

$$\begin{aligned}\mathscr{I}_1'(t) = \varrho\int_0^1(\phi_t-\vartheta_{tx})\int_0^x\vartheta_t(y)dy\,dx + \varrho\int_0^1(\phi-\vartheta_x)\int_0^x\vartheta_{tt}(y)dydx\\ -\frac{\varrho_3\varrho}{\gamma}\int_0^1\theta_t\int_0^x\vartheta_t(y)dydx - \frac{\varrho_3\varrho}{\gamma}\int_0^1\theta\int_0^x\vartheta_{tt}(y)dydx.\end{aligned} \tag{4.7}$$

Then, making use of equations $(2.1)_1$, $(2.1)_4$ and boundary conditions (2.2), we obtain

$$\begin{cases} \varrho \int_0^x \vartheta_{tt}(y)dy = -(G(\phi - \vartheta_x) + \gamma\theta), \\ \varrho_3\theta_t = k\theta_{xx} - 3\gamma\psi_t + \gamma(\phi - \vartheta_x)_t, \end{cases} \tag{4.8}$$

next, we substitute (4.8) into (4.7), integrate by parts and use $\int_0^1 \vartheta(x,t)dx = 0$, to end up with

$$\begin{aligned} \mathscr{I}_1'(t) = {} & \varrho_3 \int_0^1 \theta^2 dx + \frac{\varrho k}{\gamma} \int_0^1 \vartheta_t \theta_x dx - G \int_0^1 (\phi - \vartheta_x)^2 dx \\ & + 3\varrho \int_0^1 \psi_t \int_0^x \vartheta_t(y)dydx + \left[\frac{\varrho_3 G}{\gamma} - \gamma\right] \int_0^1 (\phi - \vartheta_x)\theta dx. \end{aligned} \tag{4.9}$$

Now, Young's, Cauchy-Schwarz and Poincaré's inequalities, give us for $\epsilon_1 > 0$,

$$\frac{\varrho k}{\gamma} \int_0^1 \vartheta_t \theta_x dx \le \frac{\chi_*}{\epsilon_1} \int_0^1 \theta_x^2 dx + \frac{\epsilon_1}{2} \int_0^1 \vartheta_t^2 dx, \tag{4.10}$$

$$\left[\frac{\varrho_3 G}{\gamma} - \gamma\right] \int_0^1 (\phi - \vartheta_x)\theta \, dx \le \chi_* \int_0^1 \theta_x^2 dx + \frac{G}{2} \int_0^1 (\phi - \vartheta_x)^2 dx, \tag{4.11}$$

and

$$3\varrho \int_0^1 \psi_t \int_0^x \vartheta_t(y)dydx \le \frac{\chi_*}{\epsilon_1} \int_0^1 \psi_t^2 dx + \frac{\epsilon_1}{2} \int_0^1 \vartheta_t^2 dx. \tag{4.12}$$

Finally, we decisively obtain estimate (4.6), once inserting (4.10)–(4.12) into (4.9). □

Lemma 4.3. *Consider the functional*

$$\mathscr{I}_2(t) = 3I_\varrho \int_0^1 \psi_t \psi \, dx + 2\beta \int_0^1 \psi^2 \, dx,$$

then, it satisfies

$$\begin{aligned} \mathscr{I}_2'(t) \le {} & -2\delta \int_0^1 \psi^2 \, dx - 3D \int_0^1 \psi_x^2 \, dx + 3I_\varrho \int_0^1 \psi_t^2 \, dx + \chi_* \int_0^1 (\phi - \vartheta_x)^2 \, dx \\ & + \chi_* \int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathscr{Y}^2(x,1,p,t) \, d\varsigma dx. \end{aligned} \tag{4.13}$$

Proof. Simple calculations using $(2.1)_3$ and integration by parts, indicate that

$$\begin{aligned} \mathscr{I}_2'(t) = {} & 3I_\varrho \int_0^1 \psi_t^2 \, dx - 3D \int_0^1 \psi_x^2 \, dx - 4\delta \int_0^1 \psi^2 \, dx - 3G \int_0^1 \psi(\phi - \vartheta_x) \, dx \\ & -4 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \psi |\mu_2(\varsigma)| \mathscr{Y}(x,1,p,t) \, d\varsigma dx. \end{aligned}$$

Thereby, using Young's inequality, we obtain estimate (4.13). □

Lemma 4.4. *Consider the functional*

$$\mathscr{I}_3(t) := -\frac{I_\varrho}{D} \int_0^1 (3\psi_t - \phi_t)(\vartheta_x + 3\psi - \phi)dx - \frac{\varrho}{G} \int_0^1 \vartheta_t (3\psi_x - \phi_x) \, dx, \tag{4.14}$$

then, it satisfies for any $\epsilon_2 > 0$,

$$\begin{aligned} \mathscr{I}_3'(t) \le {} & -\frac{I_\varrho}{D} \int_0^1 (3\psi_t - \phi_t)^2 dx + \chi_* \int_0^1 (\phi - \vartheta_x)^2 dx + \epsilon_2 \int_0^1 (3\psi_x - \phi_x)^2 dx \\ & + \chi_* \left(1 + \frac{1}{\epsilon_2}\right) \int_0^1 (\theta_x^2 + \psi_x^2) dx. \end{aligned} \tag{4.15}$$

Proof. We differentiate $\mathscr{I}_3$, use equations (2.1)$_{1,2}$, and integrate by parts, to obtain

$$\begin{aligned}\mathscr{I}_3'(t) &= \left(\frac{I_\varrho}{D}-\frac{\varrho}{G}\right)\int_0^1 \vartheta_t(3\psi_t-\phi_t)_x\,dx - \frac{I_\varrho}{D}\int_0^1(3\psi_t-\phi_t)^2dx \\ &-\frac{I_\varrho}{D}\int_0^1(3\psi-\phi)_{tt}(3\psi+\vartheta_x-\phi)dx - \frac{\varrho}{G}\int_0^1\vartheta_{tt}(3\psi_x-\phi_x)dx \\ &= \left(\frac{I_\varrho}{D}-\frac{\varrho}{G}\right)\int_0^1 \vartheta_t(3\psi_t-\phi_t)_x\,dx - \frac{I_\varrho}{D}\int_0^1(3\psi_t-\phi_t)^2dx \\ &+\frac{1}{D}\int_0^1(\phi-\vartheta_x)\left\{\gamma\theta+D(3\psi-\phi)_{xx}+G(\phi-\vartheta_x)\right\}dx \\ &+\frac{1}{G}\int_0^1(3\psi_x-\phi_x)\left\{\gamma\theta_x+G(\phi_x-\vartheta_{xx})\right\}dx \\ &-\frac{3}{D}\int_0^1\psi\left\{\gamma\theta+G(\phi-\vartheta_x)+D(3\psi-\phi)_{xx}\right\}dx.\end{aligned}$$

Again, we integrate by parts and use boundary conditions, to get

$$\begin{aligned}\mathscr{I}_3'(t) &= \left(\frac{I_\varrho}{D}-\frac{\varrho}{G}\right)\int_0^1 \vartheta_t(3\psi_t-\phi_t)_x\,dx - \frac{I_\varrho}{D}\int_0^1(3\psi_t-\phi_t)^2dx \\ &+\frac{G}{D}\int_0^1(\phi-\vartheta_x)^2dx+\frac{\gamma}{G}\int_0^1(3\psi_x-\phi_x)\theta_x\,dx+3\int_0^1(3\psi_x-\phi_x)\psi_x\,dx \\ &+\frac{\gamma}{D}\int_0^1(\phi-\vartheta_x)\theta dx-\frac{3G}{D}\int_0^1(\phi-\vartheta_x)\psi dx-\frac{3\gamma}{D}\int_0^1\psi\theta dx.\end{aligned} \tag{4.16}$$

Now, we make use of both Young's and Poincaré's inequalities, to find

$$\frac{\gamma}{G}\int_0^1(3\psi-\phi)_x\theta_x\,dx \le \frac{\chi_*}{\epsilon_2}\int_0^1\theta_x^2dx+\frac{\epsilon_2}{2}\int_0^1(3\psi_x-\phi_x)^2dx, \tag{4.17}$$

$$3\int_0^1(3\psi-\phi)_x\psi_x\,dx \le \frac{\chi_*}{\epsilon_2}\int_0^1\psi_x^2dx+\frac{\epsilon_2}{2}\int_0^1(3\psi_x-\phi_x)^2dx, \tag{4.18}$$

$$\frac{\gamma}{D}\int_0^1(\phi-\vartheta_x)\theta dx \le \chi_*\int_0^1(\phi-\vartheta_x)^2dx+\chi_*\int_0^1\theta_x^2dx, \tag{4.19}$$

$$-\frac{3G}{D}\int_0^1(\phi-\vartheta_x)\psi dx \le \chi_*\int_0^1(\phi-\vartheta_x)^2dx+\chi_*\int_0^1\psi_x^2dx, \tag{4.20}$$

$$-\frac{3\gamma}{D}\int_0^1\psi\theta dx \le \chi_*\int_0^1\psi_x^2dx+\chi_*\int_0^1\theta_x^2dx. \tag{4.21}$$

Estimate (4.15) is established by replacing (4.17)–(4.21) into (4.16) and maintaining hypothesis (1.3). □

Lemma 4.5. *Consider the functional*

$$\mathscr{I}_4(t) := -\varrho\int_0^1\vartheta\vartheta_t\,dx, \tag{4.22}$$

then, it satisfies, for any $\epsilon_3>0$,

$$\begin{aligned}\mathscr{I}_4'(t) &\le -\varrho\int_0^1\vartheta_t^2dx+2\epsilon_3\int_0^1(3\psi_x-\phi_x)^2dx+18\epsilon_3\int_0^1\psi_x^2dx \\ &+\chi_*\left(1+\frac{1}{\epsilon_3}\right)\int_0^1((\phi-\vartheta_x)^2+\theta_x^2)dx.\end{aligned} \tag{4.23}$$

Proof. By differentiating the functional $\mathscr{I}_4$, utilizing $(2.1)_1$ along with integration by parts, we find

$$\begin{aligned}\mathscr{I}_4'(t) =& -\varrho\int_0^1 \vartheta_t^2 dx - \gamma\int_0^1 \vartheta_x\theta dx - G\int_0^1 (\phi-\vartheta_x)\vartheta_x\, dx \\ =& -\varrho\int_0^1 \vartheta_t^2 dx + G\int_0^1 (\phi-\vartheta_x)^2 dx - G\int_0^1 (\phi-\vartheta_x)\phi\, dx \\ &+\gamma\int_0^1 (\phi-\vartheta_x)\theta\, dx - \gamma\int_0^1 \phi\theta\, dx.\end{aligned} \tag{4.24}$$

We then exploit Young's and Poincaré's inequalities, to obtain

$$-G\int_0^1 (\phi-\vartheta_x)\phi\, dx \le \frac{\chi_*}{\epsilon_3}\int_0^1 (\phi-\vartheta_x)^2 dx + \frac{\epsilon_3}{2}\int_0^1 \phi_x^2 dx. \tag{4.25}$$

$$\gamma\int_0^1 (\phi-\vartheta_x)\theta\, dx \le \chi_*\int_0^1 (\phi-\vartheta_x)^2 dx + \chi_*\int_0^1 \theta_x^2 dx. \tag{4.26}$$

$$-\gamma\int_0^1 \phi\theta\, dx \le \frac{\chi_*}{\epsilon_3}\int_0^1 \theta_x^2 dx + \frac{\epsilon_3}{2}\int_0^1 \phi_x^2 dx. \tag{4.27}$$

Then, substituting (4.25), (4.26), and (4.27) into (4.24), we get

$$\begin{aligned}\mathscr{I}_4'(t) \le& -\varrho\int_0^1 \vartheta_t^2 dx + \epsilon_3\int_0^1 \phi_x^2 dx + \chi_*\left(1+\frac{1}{\epsilon_3}\right)\int_0^1 (\phi-\vartheta_x)^2 dx \\ &+ \chi_*\left(1+\frac{1}{\epsilon_3}\right)\int_0^1 \theta_x^2 dx.\end{aligned}$$

Notice that

$$\int_0^1 \phi_x^2 dx = \int_0^1 (\phi_x - 3\psi_x + 3\psi_x)^2 dx \le 2\int_0^1 (3\psi_x-\phi_x)^2 dx + 18\int_0^1 \psi_x^2 dx.$$

Hence, estimate (4.23) is proved. □

Lemma 4.6. *Consider the functional*

$$\mathscr{I}_5(t) := I_\varrho\int_0^1 (3\psi-\phi)(3\psi-\phi)_t dx, \tag{4.28}$$

then, it satisfies the estimate

$$\begin{aligned}\mathscr{I}_5'(t) \le& -\frac{D}{2}\int_0^1 (3\psi_x-\phi_x)^2 dx + \chi_*\int_0^1 \theta_x^2 dx + I_\varrho\int_0^1 (3\psi_t-\phi_t)^2 dx \\ &+\chi_*\int_0^1 (\phi-\vartheta_x)^2 dx.\end{aligned} \tag{4.29}$$

Proof. Easy calculations, by $(2.1)_2$ pursued by integration by parts, yield

$$\begin{aligned}\mathscr{I}_5'(t) =& -D\int_0^1 (3\psi_x-\phi_x)^2 dx + I_\varrho\int_0^1 (3\psi_t-\phi_t)^2 dx + \gamma\int_0^1 (3\psi-\phi)\theta\, dx \\ &+G\int_0^1 (3\psi-\phi)(\phi-\vartheta_x) dx.\end{aligned} \tag{4.30}$$

Employing Young's and Poincaré's inequalities, we find

$$\gamma\int_0^1 (3\psi-\phi)\theta\, dx \le \chi_*\int_0^1 \theta_x^2 dx + \frac{D}{4}\int_0^1 (3\psi_x-\phi_x)^2 dx, \tag{4.31}$$

$$G\int_0^1 (3\psi-\phi)(\phi-\vartheta_x) dx \le \chi_*\int_0^1 (\phi-\vartheta_x)^2 dx + \frac{D}{4}\int_0^1 (3\psi_x-\phi_x)^2 dx. \tag{4.32}$$

Consequently, if we insert (4.31) and (4.32) into (4.30), we obtain (4.29).

□

Lemma 4.7. *Consider the functional*

$$\mathscr{I}_6(t) := \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma p} |\mu_2(\varsigma)| \mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx, \tag{4.33}$$

then, it satisfies

$$\begin{aligned} \mathscr{I}_6'(t) \leq \beta \int_0^1 \psi_t^2\, dx - \varpi_1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathscr{Y}^2(x,1,\varsigma,t)\, d\varsigma dx \\ - \varpi_1 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma |\mu_2(\varsigma)| \mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx, \end{aligned} \tag{4.34}$$

where ϖ_1 is a positive constant.

Proof. Taking the derivative of $\mathscr{I}_6$, using $(2.1)_5$, and $\mathscr{Y}(x,0,t) = \psi_t$, we achieve what follows

$$\begin{aligned} \mathscr{I}_6'(t) &= -2 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} e^{-\varsigma p} |\mu_2(\varsigma)| \mathscr{Y}_p \mathscr{Y}(x,p,\varsigma,t)\, d\varsigma dp dx \\ &= - \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma p} |\mu_2(\varsigma)| \mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx \\ &\quad - \int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \left\{ e^{-\varsigma} \mathscr{Y}^2(x,1,\varsigma,t) - \psi_t^2(x,t) \right\} d\varsigma dx. \end{aligned}$$

From $e^{-\varsigma} \leq e^{-\varsigma p} \leq 1$, where $0 < p < 1$, we arrive at

$$\begin{aligned} \mathscr{I}_6'(t) \leq & - \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma} |\mu_2(\varsigma)| \mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx \\ & + \left(\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\, d\varsigma \right) \int_0^1 \psi_t^2(x,t)\, dx \\ & - \int_0^1 \int_{\varsigma_1}^{\varsigma_2} e^{-\varsigma} |\mu_2(\varsigma)| \mathscr{Y}^2(x,1,\varsigma,t) d\varsigma dx. \end{aligned}$$

Since $-e^{-\varsigma}$ is an increasing function, then

$$-e^{-\varsigma} \leq -e^{-\varsigma_2}, \quad \text{forall } \varsigma \in [\varsigma_1, \varsigma_2].$$

Hence, if we denote $\varpi_1 = e^{-\varsigma_2}$ and use (A_1), we easily prove (4.34). □

5 Stability results

Let us here prove our stability results by using the lemmas already mentioned in section 4.

5.1 Exponential stability

Here, we establish our exponential stability result.

Proof of Theorem 2.2. We proceed by introducing a Lyapunov functional

$$\mathscr{L}(t) = N\mathscr{E}(t) + \sum_{j=1}^{6} N_j \mathscr{I}_j(t), \tag{5.1}$$

where constants $N, N_j > 0,\ j = 1, \cdots, 6$, will be fixed later.
From (5.1), we are in liberty to write

$$|\mathscr{L}(t) - N\mathscr{E}(t)| \leq \varrho N_1 \int_0^1 \left|(\phi - \vartheta_x)\int_0^x \vartheta_t(y)dy\right| dx + \frac{\varrho\varrho_3}{\gamma} N_1 \int_0^1 \left|\theta \int_0^x \vartheta_t(y)dy\right| dx$$
$$+ 3I_\varrho N_2 \int_0^1 |\psi_t \psi|\, dx + 2\beta N_2 \int_0^1 \psi^2 dx + \frac{\varrho N_3}{G}\int_0^1 |(3\psi_x - \phi_x)\vartheta_t|\, dx$$
$$+ \frac{I_\varrho N_3}{D}\int_0^1 |(\phi - \vartheta_x)(3\psi_t - \phi_t)|\, dx + \frac{3I_\varrho N_3}{D}\int_0^1 |(3\psi_t - \phi_t)\psi|\, dx$$
$$+ \varrho N_4 \int_0^1 |\vartheta\vartheta_t|\, dx + N_5 \int_0^1 |(3\psi - \phi)(3\psi_t - \phi_t)|\, dx$$
$$+ N_6 \int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma p}|\mu_2(\varsigma)|\mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx.$$

Thanks to Young's, Cauchy-Schwarz and Poincaré's inequalities, we get

$$|\mathscr{L}(t) - N\mathscr{E}(t)| \leq \eta_1 \int_0^1 \left\{\vartheta_t^2 + (\phi - \vartheta_x)^2 + (3\psi_t - \phi_t)^2 + (3\psi_x - \phi_x)^2 + \psi_t^2 + \psi_x^2 + \psi^2 + \theta^2\right\} dx$$
$$+ \eta_1 \int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2} \varsigma|\mu_2(\varsigma)|\mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx, \qquad \text{where } \eta_1 > 0.$$

Therefore, using the definition of the energy (2.7), we come to

$$|\mathscr{L}(t) - N\mathscr{E}(t)| \leq \eta_2 \mathscr{E}(t), \qquad \text{where } \eta_2 > 0,$$

i.e.

$$(N - \eta_2)\mathscr{E}(t) \leq \mathscr{L}(t) \leq (N + \eta_2)\mathscr{E}(t). \tag{5.2}$$

Now, differentiating the Lyapunov functional $\mathscr{L}(t)$, employing (4.1), (4.6), (4.13), (4.15), (4.23), (4.29), and (4.34), and fixing

$$N_4 = N_5 = 1,\ \epsilon_1 = \frac{\varrho}{4N_3},\ \epsilon_2 = \frac{D}{4N_3},$$

we find

$$\frac{d}{dt}\mathscr{L}(t) \leq -\left(\frac{G}{2}N_1 - \chi_*(N_2 + N_3) - \chi_*\left(1 + \frac{1}{\epsilon_3}\right) - \chi_*\right)\int_0^1 (\phi - \vartheta_x)^2 dx$$
$$- \left(\frac{I_\varrho}{D}N_3 - I_\varrho\right)\int_0^1 (3\psi_t - \phi_t)^2 dx - \left(\frac{D}{4} - 2\epsilon_3\right)\int_0^1 (3\psi_x - \phi_x)^2 dx$$
$$- 2\delta N_2 \int_0^1 \psi^2 dx - (3DN_2 - (1 + N_3)\chi_* N_3 - 18\epsilon_3)\int_0^1 \psi_x^2 dx$$
$$- \left(kN - (1 + N_1)\chi_* N_1 - (1 + N_3)\chi_* N_3 - \chi_* - \chi_*\left(1 + \frac{1}{\epsilon_3}\right)\right)\int_0^1 \theta_x^2 dx \tag{5.3}$$
$$- \left(\varpi N - \chi_* N_1^2 - 3I_\varrho N_2 - \beta N_6\right)\int_0^1 \psi_t^2 dx - \frac{\varrho}{2}\int_0^1 \vartheta_t^2 dx$$
$$- \varpi_1 N_6 \int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2} \varsigma|\mu_2(\varsigma)|\mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx$$
$$- (\varpi_1 N_6 - \chi_* N_2)\int_0^1\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathscr{Y}^2(x,1,\varsigma,t)\, d\varsigma dx.$$

Next, we choose our coefficients in (5.3), in a way that, they all become negative. We start by selecting N_3 big enough so that

$$\frac{I_\varrho}{D}N_3 - I_\varrho > 0,$$

then, we take N_2 fairly wide, such that

$$3DN_2 - (1+N_3)\chi_* N_3 > 0.$$

After fixing the above constants, we can pick ϵ_3 sufficiently small, so that

$$\epsilon_3 < \min\left\{\frac{D}{8}, \frac{3DN_2 - (1+N_3)\chi_* N_3}{18}\right\},$$

now, we select N_6 huge enough such that

$$\varpi_1 N_6 - \chi_* N_2 > 0.$$

Next, we fix N_1 sufficiently large such that

$$\frac{G}{2}N_1 - \chi_*(N_2+N_3) - \chi_*\left(1+\frac{1}{\epsilon_3}\right) - \chi_* > 0.$$

Lastly, we take N fairly huge, in a way that, we have (5.2) and

$$\begin{cases} \varpi N - \chi_* N_1^2 - 3I_\varrho N_2 - \beta N_6 > 0, \\ kN - (1+N_1)\chi_* N_1 - (1+N_3)\chi_* N_3 - \chi_* - \chi_*\left(1+\frac{1}{\epsilon_3}\right) > 0. \end{cases}$$

Hence, we conclude that

$$\begin{aligned} \frac{d}{dt}\mathscr{L}(t) \leq -\eta_3 \int_0^1 \{\vartheta_t^2 + (3\psi_t - \phi_t)^2 + (\phi - \vartheta_x)^2 + \psi^2 + \psi_t^2 + \psi_x^2 + (3\psi_x - \phi_x)^2 + \theta_x^2\}\, dx \\ - \eta_3 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma |\mu_2(\varsigma)| \mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx, \quad \text{where } \eta_3 > 0. \end{aligned} \tag{5.4}$$

Now, taking advantage of both (2.7) and Poincaré's inequality, we get

$$\begin{aligned} \mathscr{E}(t) \leq \eta_4 \int_0^1 \{\vartheta_t^2 + (3\psi_t - \phi_t)^2 + (\phi - \vartheta_x)^2 + \psi^2 + \psi_t^2 + \psi_x^2 + (3\psi_x - \phi_x)^2 + \theta_x^2\}\, dx \\ + \eta_4 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma |\mu_2(\varsigma)| \mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx, \quad \text{where } \eta_4 > 0, \end{aligned}$$

From which

$$\begin{aligned} - \int_0^1 \{\vartheta_t^2 + (3\psi_t - \phi_t)^2 + (\phi - \vartheta_x)^2 + \psi^2 + \psi_t^2 + \psi_x^2 + (3\psi_x - \phi_x)^2 + \theta_x^2\}\, dx \\ - \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma |\mu_2(\varsigma)| \mathscr{Y}^2(x,p,\varsigma,t)\, d\varsigma dp dx \\ \leq -\eta_5 \mathscr{E}(t), \quad \text{where } \eta_5 > 0. \end{aligned} \tag{5.5}$$

Thereby, if we merge (5.4) and (5.5), we arrive at

$$\frac{d}{dt}\mathscr{L}(t) \leq -\eta_6 \mathscr{E}(t), \quad \text{where } \eta_6 > 0, \tag{5.6}$$

next, using the fact that (5.2) is valid, we get

$$\frac{d}{dt}\mathscr{L}(t) \leq -\eta_7 \mathscr{L}(t), \quad \text{where } \eta_7 = \frac{\eta_6}{N+\eta_2} > 0. \tag{5.7}$$

Finally, by a simple integration of (5.7), and using (5.2), we obtain estimate (2.8). □

5.2 Polynomial stability

In this subsection, we establish our polynomial stability result for (2.1) provided that

$$\frac{\varrho}{G} \neq \frac{I_\varrho}{D}. \tag{5.8}$$

Proof of Theorem 2.3. Let us start by introducing the second order energy functional

$$\mathscr{E}_2(t) := \mathscr{E}(\vartheta_t, \phi_t, \psi_t, \theta_t, \mathscr{Y}_t). \tag{5.9}$$

Proceeding exactly as we did in Lemma 4.1, we easily show that

$$\frac{d}{dt}\mathscr{E}_2(t) \leq -k\int_0^1 \theta_{tx}^2 dx - \varpi \int_0^1 \psi_{tt}^2 dx. \tag{5.10}$$

In case of different wave speeds, we shall restimate the derivative of $\mathscr{I}_3(t)$, from lemma 4.4, we have

$$\begin{aligned}\frac{d}{dt}\mathscr{I}_3(t) \leq\ & \epsilon_2 \int_0^1 (3\psi_x - \phi_x)^2\, dx + \chi_* \int_0^1 (\phi - \vartheta_x)^2\, dx \\ & - \frac{I_\varrho}{D}\int_0^1 (3\psi_t - \phi_t)^2 dx + \chi_*\left(1 + \frac{1}{\epsilon_2}\right)\int_0^1 \theta_x^2\, dx \\ & + \chi_*\left(1 + \frac{1}{\epsilon_2}\right)\int_0^1 \psi_x^2\, dx + \left(\frac{\varrho}{G} - \frac{I_\varrho}{D}\right)\int_0^1 \vartheta_{tx}(3\psi_t - \phi_t)\, dx.\end{aligned} \tag{5.11}$$

By using the fourth equation in (2.1), and the integration by parts, we arrive at

$$\begin{aligned}\int_0^1 \vartheta_{tx}(3\psi_t - \phi_t)\, dx =& -\int_0^1 (3\psi_t - \phi_t)^2\, dx - \frac{\varrho_3}{\gamma}\int_0^1 \theta_t(3\psi_t - \phi_t)\, dx \\ & - \frac{k}{\gamma}\int_0^1 \theta_x(3\psi_t - \phi_t)_x\, dx \\ =& -\int_0^1 (3\psi_t - \phi_t)^2\, dx - \frac{\varrho_3}{\gamma}\int_0^1 \theta_t(3\psi_t - \phi_t)\, dx \\ & - \frac{d}{dt}\frac{k}{\gamma}\int_0^1 \theta_x(3\psi_x - \phi_x)\, dx + \frac{k}{\gamma}\int_0^1 \theta_{tx}(3\psi_x - \phi_x)\, dx,\end{aligned}$$

which, together with estimate (5.11), gives us

$$\begin{aligned}\frac{d}{dt}\left\{\mathscr{I}_3(t) + \frac{k}{\gamma}\int_0^1 \theta_x(3\psi_x - \phi_x)\, dx\right\} \leq\ & \epsilon_2\int_0^1 (3\psi_x - \phi_x)^2\, dx + \chi_*\int_0^1 (\phi - \vartheta_x)^2\, dx \\ & - \frac{\varrho}{G}\int_0^1 (3\psi_t - \phi_t)^2\, dx + \chi_*\left(1 + \frac{1}{\epsilon_2}\right)\int_0^1 (\theta_x^2 + \psi_x^2)\, dx \\ & + \frac{k}{\gamma}\left(\frac{\varrho}{G} - \frac{I_\varrho}{D}\right)\int_0^1 \theta_{tx}(3\psi_x - \phi_x)\, dx \\ & - \frac{\varrho_3}{\gamma}\left(\frac{\varrho}{G} - \frac{I_\varrho}{D}\right)\int_0^1 \theta_t(3\psi_t - \phi_t)\, dx.\end{aligned}$$

Applying Young's and Poincaré's inequalities, the above estimate takes the following form

$$\begin{aligned}\frac{d}{dt}\left\{\mathscr{I}_3(t) + \frac{k}{\gamma}\int_0^1 \theta_x(3\psi_x - \phi_x)\, dx\right\} \leq\ & 2\epsilon_2\int_0^1 (3\psi_x - \phi_x)^2\, dx + \chi_*\int_0^1 (\phi - \vartheta_x)^2\, dx \\ & + \chi_*\left(1 + \frac{1}{\epsilon_2}\right)\int_0^1 (\theta_{tx}^2 + \theta_x^2 + \psi_x^2)\, dx \\ & - \frac{\varrho}{2G}\int_0^1 (3\psi_t - \phi_t)^2\, dx.\end{aligned} \tag{5.12}$$

At this point, we introduce a Lyapunov functional

$$\begin{aligned}\mathscr{R}(t) := N(\mathscr{E}(t)+\mathscr{E}_2(t)) + \sum_{j=1,\ j\neq 3}^{6} N_j \mathscr{F}_j(t)\\ + N_3\left[\mathscr{I}_3(t) + \frac{k}{\gamma}\int_0^1 \theta_x(3\psi_x-\phi_x)\,dx\right],\end{aligned} \tag{5.13}$$

where constants $N, N_j > 0,\ j = 1,\cdots,6$, will be fixed later.
Now, we take the derivative of $\mathscr{R}(t)$, make use of (4.1), (4.13), (4.23), (4.29), (4.34), (5.10), (5.12), and set

$$\epsilon_2 = \frac{D}{8N_3}.$$

If we maintain the same selection of $N_1,\ N_2,\ N_4,\ N_5,\ N_6,\ \epsilon_1,\ \epsilon_3$ as in the previous subsection, and choose $N_3,\ N$ sufficiently large so that

$$\frac{\varrho}{2G}N_3 - I_\varrho > 0,$$

and

$$\begin{cases}\varpi N - \chi_* N_1^2 - 3I_\varrho N_2 - \beta N_6 > 0,\\ kN - \chi_*(1+N_1)N_1 - \chi_*(1+N_3)N_3 - \chi_* - \chi_*\left(1+\frac{1}{\epsilon_3}\right) > 0,\\ kN - \chi_*(1+N_3)N_3 > 0,\end{cases}$$

we arrive at

$$\frac{d}{dt}\mathscr{R}(t) \le -\gamma_1\mathscr{E}(t), \qquad \text{where } \gamma_1 > 0. \tag{5.14}$$

Therefore, integrating (5.14) over $(0,t)$, and keeping in mind that $\mathscr{E}'(t) \le 0$, we reach the following result

$$\begin{aligned} t\mathscr{E}(t) &\le \int_0^t \mathscr{E}(\sigma)d\sigma\\ &\le \frac{1}{\gamma_1}(\mathscr{R}(0)-\mathscr{R}(t))\\ &\le \frac{1}{\gamma_1}\mathscr{R}(0)\\ &\le \frac{1}{\gamma_1}(\mathscr{E}(0)+\mathscr{E}_2(t)), \quad \forall t>0.\end{aligned}$$

Hence,

$$\mathscr{E}(t) \le \frac{\lambda}{t}, \quad \forall t > 0,$$

where $\lambda = \frac{\mathscr{E}(0)+\mathscr{E}_2(t)}{\gamma_1}$. This proof is then completed. □

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