Palestine Journal of Mathematics

Vol 13(4)(2024) , 693–700

A NOTE ON NEAR HARMONIC DIVISOR NUMBER AND ASSOCIATED CONCEPTS

Jaba Kalita and Helen K. Saikia

Communicated by Ayman Badawi

MSC 2020 Classifications: 11A25

Keywords: Harmonic divisor number, perfect number, near perfect number, quasi perfect number.

Corresponding Author: Jaba Kalita

Abstract A positive integer n is called a harmonic divisor number or Ore number if the harmonic mean of all its positive divisors is an integer. In this paper our attempt is to generalize the notion of harmonic divisor number. We define near harmonic divisor number and establish some properties of these numbers. Every prime number is a near harmonic divisor number. If p is a prime number then any integer of the form $2p$ or $3p$ is a near harmonic divisor number for particular values of p. Moreover, any integer of the form p^2 is not a near harmonic divisor number for any prime p.

1 Introduction

A positive integer is known as a harmonic divisor number if the harmonic mean of all of its positive divisors is an integer. Öystein Ore first studied about harmonic divisor numbers in 1948[8]. These numbers are also known as Ore numbers[10]. Since the harmonic mean of all the positive divisors of an integer n is

$$H(n) = \frac{n\tau(n)}{\sigma(n)}$$

where $\tau(n)$ and $\sigma(n)$ denotes the number of divisors and the sum of divisors of n respectively[8]. So a positive integer n is a harmonic divisor number if $\sigma(n)$ divides $n\tau(n)$. A positive integer n is called a perfect number if $\sigma(n) = 2n$. Ore showed that all perfect numbers are harmonic but all harmonic divisor numbers are not perfect. Ore conjectured that 1 is the only one odd harmonic divisor number. If this conjecture is true, then we can say that there doesn't exist any odd perfect number. The connection of harmonic divisor numbers with perfect numbers is the reason that many researchers have interest in it. There are various types of generalized harmonic divisor numbers such as superharmonic numbers[3], unitary harmonic numbers[7] etc. In 2004 T. Goto and S. Shibata found all numbers whose positive divisors have integral harmonic mean up to 300[6]. Many researchers have studied harmonic divisor numbers([5],[4],[11]). Many researchers have been carrying out research work on various aspects of arithmetical functions and related concepts like perfect numbers, Fibonacci sequences and Lucas sequences([12], [1]). In this paper we define a new type of generalized harmonic divisor number which we termed as near harmonic divisor number.

2 Priliminaries:

2.1 Near harmonic divisor number:

A positive integer n is called a near harmonic divisor number if the harmonic mean of all its positive divisors except one of them is an integer. For a positive integer n, the harmonic mean of all the positive divisors except one of them say d is

$$H^*(n) = \frac{n(\tau(n) - 1)}{\sigma(n) - \frac{n}{d}}. \tag{2.1}$$

So we can say that n is a near harmonic divisor number with redundant divisor d if $H^*(n)$ is an integer.
Some examples of near harmonic divisor numbers are
2,3,5,6,7,11,13,15,17,18,19,23,24,28,29,30,31,37,40, 41,43,47,48, 53,59,60,61 etc.

Definition 2.1. A positive integer n is called a near perfect number if it is equal to sum of its proper divisors except one of them which is known as redundant divisor of n. Mathematically a positive integer n is a near perfect number with redundant divisor d if $\sigma(n) = 2n + d$[9].

Definition 2.2. A positive integer n is called a quasi perfect number if $\sigma(n) = 2n + 1$[2].

Theorem 2.1. (Euler). All even perfect numbers have the form $2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are primes.

3 Main Results

In this section we present our main results.

Proposition 3.1. Every prime number is a near harmonic divisor number.

Proof. As every prime number has only two divisors, so clearly we can say that every prime number is a near harmonic divisor number. □

Proposition 3.2. A positive integer of the form $2p$ where p is an odd prime number is a near harmonic divisor number if and only if $p = 3$.

Proof. Consider a positive integer $n = 2p$, where p is an odd prime. The positive divisors of n are 1,2,p and $2p$. The harmonic mean of all the positive divisors except the redundant divisor d is

$$H^*(n) = \frac{n(\tau(n) - 1)}{\sigma(n) - \frac{n}{d}}$$

If we take the divisors 1,2,p and $2p$ as redundant divisor of n respectively, then we have the following cases.
Case I: If $d = 1$, then

$$\begin{aligned} H^*(n) &= \frac{3}{\frac{1}{2} + \frac{1}{p} + \frac{1}{2p}} \\ &= \frac{3 \cdot 2p}{p + 3} \\ &= 6 - \frac{18}{p + 3} \end{aligned}$$

For $H^*(n)$ to be an integer, $\frac{18}{p+3}$ must be an integer ,i.e., $p + 3|18$ which is possible only when $p = 3$ as p is an odd prime.

Case II: If $d = 2$, then

$$H^*(n) = \frac{3}{1 + \frac{1}{p} + \frac{1}{2p}}$$
$$= \frac{3 \cdot 2p}{2p + 3}$$
$$= 3 - \frac{9}{2p + 3}$$

For $H^*(n)$ to be an integer, $2p + 3|9$ which is possible only when $p = 3$.
Case III: If $d = p$, then

$$H^*(n) = \frac{3}{1 + \frac{1}{2} + \frac{1}{2p}}$$
$$= \frac{3 \cdot 2p}{3p + 1}$$

which can never be an integer for any value of p.

Case IV: If $d = 2p$, then

$$H^*(n) = \frac{3}{1 + \frac{1}{2} + \frac{1}{p}}$$
$$= \frac{3 \cdot 2p}{3p + 2}$$

which cannot be an integer for any value of p.
Therefore a positive integer of the form $2p$, where p is an odd prime is a near harmonic divisor number if $p = 3$.
Conversely, if $p = 3$, then $n = 2 \cdot 3 = 6$ is obviously a near harmonic divisor number with two redundant divisors 1 and 2.
Hence only positive integer of the form $2p$ (p is an odd prime) which is near harmonic divisor number is 6.

□

Proposition 3.3. A positive integer of the form $3p$ where $p(p \neq 3)$ is a prime number is a near harmonic divisor number if and only if $p = 2$ or $p = 5$.

Proof. We consider a positive integer $n = 3p$, where $p \neq 3$ is a prime. The positive divisors of n are 1,3,p and $3p$. The harmonic mean of all the positive divisors except the redundant divisor d is

$$H^*(n) = \frac{n(\tau(n) - 1)}{\sigma(n) - \frac{n}{d}}$$

Now taking the divisors 1,3,p,$3p$ as redundant divisor of n respectively, then we have the following cases.

Case I: If $d = 1$, then

$$
\begin{aligned}
H^*(n) &= \frac{3}{\frac{1}{3} + \frac{1}{p} + \frac{1}{3p}} \\
&= \frac{3 \cdot 3p}{p+4}
\end{aligned}
$$

For $H^*(n)$ to be an integer, p must be equal to 2 or 5.
Case II: If d=3, then

$$
\begin{aligned}
H^*(n) &= \frac{3}{1 + \frac{1}{p} + \frac{1}{3p}} \\
&= \frac{3 \cdot 3p}{3p+4}
\end{aligned}
$$

It is obvious that $H^*(n)$ can never be an integer for any value of p.
Case III: If $d = p$, then

$$
\begin{aligned}
H^*(n) &= \frac{3}{1 + \frac{1}{3} + \frac{1}{3p}} \\
&= \frac{3 \cdot 3p}{4p+1}
\end{aligned}
$$

which can never be an integer for any value of p.
Case IV: If $d = 3p$, then

$$
\begin{aligned}
H^*(n) &= \frac{3}{1 + \frac{1}{3} + \frac{1}{p}} \\
&= \frac{3 \cdot 3p}{4p+3}
\end{aligned}
$$

which cannot be an integer for any value of p.
Therefore a positive integer of the form $3p$, where $p \neq 3$ is a prime is near harmonic divisor number if $p = 2$ or $p = 5$.
Conversely, if $p = 2$, then $n = 3 \cdot 2 = 6$ is obviously a near harmonic divisor number with two redundant divisors 1 and 2 and for $p = 5$, $n = 15$ which is also a near harmonic divisor number with redundant divisor 1.
Hence only positive integers of the form $3p$ ($p \neq 3$ is a prime) which are near harmonic divisor number are 6 and 15.

□

Proposition 3.4. Any positive integer of the form p^2 , where p is a prime number cannot be a near harmonic divisor number.

Proof. Let $n = p^2$ be a positive integer, where p is a prime.
Positive divisors of n are 1, p and p^2.

The harmonic mean of all the positive divisors except the redundant divisor d is

$$H^*(n) = \frac{n(\tau(n)-1)}{\sigma(n)-\frac{n}{d}}$$

If we take 1,p,p^2 as redundant divisor of n respectively, then we have the following cases.

Case I: If $d = 1$, then

$$\begin{aligned} H^*(n) &= \frac{2}{\frac{1}{p}+\frac{1}{p^2}} \\ &= \frac{2p^2}{p+1} \end{aligned}$$

Clearly $(p+1) \nmid 2p^2$. So $H^*(n)$ is not an integer for any value of p. Thus n is not a near harmonic divisor number with redundant divisor $d = 1$.

Case II: If $d = p$, then

$$\begin{aligned} H^*(n) &= \frac{2}{1+\frac{1}{p^2}} \\ &= \frac{2p^2}{p^2+1} \end{aligned}$$

which can never be an integer for any value of p. Thus n is not a near harmonic divisor number with redundant divisor $d = p$.
Case III: If $d = p^2$, then

$$\begin{aligned} H^*(n) &= \frac{2}{1+\frac{1}{p}} \\ &= \frac{2p}{p+1} \end{aligned}$$

Clearly, $p + 1 \nmid 2p$. So n is not a near harmonic divisor number with redundant divisor $d = p^2$. Therefore no positive integer of the form p^2 where p is a prime is near harmonic divisor number. □

Proposition 3.5. Every even perfect number is a near harmonic divisor number with redundant divisor 1.

Proof. Let n be an even perfect number. Then by Theorem 2.1, n must be of the form $2^{k-1}(2^k - 1)$, where $2^k - 1$ is a Mersenne prime.

Let $n = 2^{k-1}(2^k - 1)$. Then, $\tau(n) = 2k$ and $\sigma(n) = 2n = 2^k(2^k - 1)$.
Thus from Equation (2.1) the harmonic mean of all the divisors of n except the redundant divisor 1 is

$$\begin{aligned} H^*(n) &= \frac{n(\tau(n)-1)}{\sigma(n)-\frac{n}{1}} \\ &= \frac{(2k-1)2^{k-1}(2^k-1)}{2^k(2^k-1)-2^{k-1}(2^k-1)} \\ &= 2k-1 \end{aligned}$$

Which is clearly an integer.
Hence every even perfect number is a near harmonic divisor number with redundant divisor 1.

□

Proposition 3.6. If n is a near perfect number with redundant divisor d and there are odd number of divisors of n, then n is a near harmonic divisor number with redundant divisor $\frac{n}{d}$.

Proof. Let n be a near perfect number with redundant divisor d. Then by definition of near perfect number,

$$\sigma(n) = 2n + d$$

From Equation (2.1), the harmonic mean of all the divisors of n except the redundant divisor $\frac{n}{d}$ is

$$\begin{aligned} H^*(n) &= \frac{n(\tau(n)-1)}{\sigma(n)-\frac{n}{\frac{n}{d}}} \\ &= \frac{n(\tau(n)-1)}{2n+d-d} \\ &= \frac{\tau(n)-1}{2} \end{aligned}$$

Since n has odd number of divisors, so $\tau(n)$ is odd and $\tau(n)-1$ is even and divisible by 2.
Thus $H^*(n)$ is an integer.
Hence n is a near harmonic divisor number with redundant divisor $\frac{n}{d}$. □

Proposition 3.7. If there exist a divisor d of a positive integer n such that $\sigma(n)-d|n$, where $\sigma(n)$ denotes the sum of the divisors of n, then n is a near harmonic divisor number with redundant divisor $\frac{n}{d}$.

Proof. Let n be a positive integer. From Equation (2.1), the harmonic mean of the divisors of n except the redundant divisor $\frac{n}{d}$ is

$$\begin{aligned} H^*(n) &= \frac{n(\tau(n)-1)}{\sigma(n)-\frac{n}{\frac{n}{d}}} \\ &= \frac{n(\tau(n)-1)}{\sigma(n)-d} \end{aligned}$$

Since $\sigma(n)-d|n$, so clearly $H^*(n)$ is an integer.
Therefore n is a near harmonic divisor number with redundant divisor $\frac{n}{d}$.

□

Proposition 3.8. If any quasi-perfect number n exists which have odd number of positive divisors, then n is a near harmonic divisor number with redundant divisor n.

Proof. Let n be a quasi-perfect number. So,

$$\sigma(n) = 2n + 1$$

Now from Equation (2.1), the harmonic mean of the divisors of n except the redundant divisor n is

$$H^*(n) = \frac{n(\tau(n)-1)}{\sigma(n)-\frac{n}{n}}$$
$$= \frac{n(\tau(n)-1)}{2n+1-1}$$
$$= \frac{\tau(n)-1}{2}$$

which is an integer as $\tau(n)$ is odd.
Hence a quasi- perfect number with odd number of positive divisors is a near harmonic divisor number. □

Proposition 3.9. If (m, n) is an amicable pair and $m|n(\tau(n)-1)$, then n is a near harmonic divisor number with redundant divisor 1.

Proof. Here (m,n) is an amicable pair.
Therefore

$$\sigma(m) - m = n$$

and

$$\sigma(n) - n = m.$$

Now the harmonic mean of the divisors of n except the redundant divisor 1 is

$$H^*(n) = \frac{n(\tau(n)-1)}{\sigma(n)-n}$$
$$= \frac{n(\tau(n)-1)}{m}$$

As $m|n(\tau(n)-1)$, so $H^*(n)$ is an integer.
Hence n is a near harmonic divisor number. □

4 Conclusion

In this paper we have defined near harmonic divisor number and establish several properties of these numbers. In our future work our objective will be to characterize near harmonic divisor number.

References

[1] Bitim, B. & Topal, N. Binomial sum formulas from the exponential generating functions of (P, q)- fibonacci and (p, q)- lucas quaternions. *Palestine Journal Of Mathematics*. **10**, 279-289 (2021).

[2] Cohen, G. On odd perfect numbers (ii), multiperfect numbers and quasiperfect numbers. *Journal Of The Australian Mathematical Society*. **29**, 369-384 (1980).

[3] Cohen, G. Superharmonic numbers. *Mathematics Of Computation*. **78**, 421-429 (2009).

[4] Cohen, G. & Sorli, R. Odd harmonic numbers exceed 10^{24}. *Mathematics Of Computation*. **79**, 2451-2460 (2010).

[5] Garcia, M. On numbers with integral harmonic mean. *The American Mathematical Monthly*. **61**, 89-96 (1954).

[6] Goto, T. & Shibata, S. All numbers whose positive divisors have integral harmonic mean up to 300. *Mathematics Of Computation*. **73**, 475-491 (2004).

[7] Hagis, P. & Lord, G. Unitary harmonic numbers. *Proceedings Of The American Mathematical Society*. **51**, 1-7 (1975).

[8] Ore, O. On the averages of the divisors of a number. *The American Mathematical Monthly*. **55**, 615-619 (1948).

[9] Pollack, P. & Shevelev, V. On perfect and near-perfect numbers. *Journal Of Number Theory*. **132**, 3037-3046 (2012).

[10] Pomerance, C. On a problem of Ore: Harmonic numbers (unpublished typescript); see Abstract 709-A5. *Notices Amer. Math. Soc*. **20** (1973).

[11] Rajkhowa, P. & Bharali, H. Some aspects of Harmonic Numbers which divide the sum of its Positive divisors. *IOSR Journal of Mathematics*.**7**, 39-45 (2013).

[12] Verma, K. ON AN ARITHMETICAL FUNCTIONS INVOLVING GENERAL EXPONENTIAL. *Palestine Journal Of Mathematics*. **11**, 496-504 (2022).

Author information

Jaba Kalita, Department of Mathematics, Gauhati University, India.
E-mail: jabakalita95@gmail.com

Helen K. Saikia, Department of Mathematics, Gauhati University, India.
E-mail: hsaikia@yahoo.com

Received: 2023-07-22
Accepted: 2024-01-27