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THE VALUATION ALGEBRA MOTIVATED BY DISSECTION THEORY

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Abstract A lattice L is said to be lower-finite if the set $[0, a]$ is finite for every element a of L. This article provides a detailed proof that, if M is a subset of a complete lower-finite distributive lattice L that contains the join-irreducible elements of L, and a an element of M which is not join-irreducible, then $\sum_{b \in M \cap [0,a]} \mu_M(b, a)b$ belongs to the submodule $\langle a \wedge b + a \vee b - a - b \mid a, b \in L \rangle$ of the module $\mathbb{Z}L$. This property was initially established by Zaslavsky for finite distributive lattice. It will be seen that this property is the main ingredient to obtain the fundamental theorem of dissection theory of Zaslavsky. This articles ends with a concrete application of that theorem to face counting for submanifold arrangements.

1 Introduction

Recall that a distributive lattice is a partially order set with join and meet operations which distribute over each other. Standard examples are sets whose join and meet are the usual union and intersection. Other examples include the Lindenbaum algebra of logics that support conjunction and disjunction, every Heyting algebra, and Young's lattice formed by all integer partitions ordered by inclusion of their Young diagrams. This article mainly aims to provide a complete proof that, if L is a complete lower-finite distributive lattice, M a subset of L containing its join-irreducible elements, $f : L \to G$ a valuation on L to a module G, and a an element of M which is not join-irreducible, then

$$\sum_{b \in [0,a] \cap M} \mu_M(b, a) f(b) = 0. \tag{1.1}$$

Its proof is carried out in several stages. We first consider the general case of posets in Section 2. A proof of the lemma of Zorn [20] and an introduction to the Möbius algebra $\text{Möb}(L)$ of a lower-finite poset L are namely provided. Although diverse proofs of Zorn's lemma can easily be found in the literature, new ones are still proposed other time like that of Lewin [11]. The proof in Section 2 is inspired by the notes of Debussche [5] in § 2.II. The Möbius algebra was discovered by Solomon [15] who defined it for finite posets. We give a proof of the Möbius inversion formula, and of the fact that $\left\{ \sum_{b \in [0,a]} \mu_L(b, a)b \;\middle|\; a \in L \right\}$ is a complete set of orthogonal idempotents in $\text{Möb}(L)$.

We study the special case of lattices in Section 3. After viewing some essential generalities, we focus on the distributive lattices, and establish diverse properties like the distributivity of a lattice L if and only if, for all $a, b, c \in L$, $c \vee a = c \vee b$ and $c \wedge a = c \wedge b$ imply $a = b$.

Those last properties are necessary to investigate the valuation algebra in Section 4. It is the central part of this article, and principally inspired from the articles of Geissinger [7, 8], and Zaslavsky [19]. In that section is particularly proved that if M is a subset of a complete lower-finite distributive lattice L containing its join-irreducible elements, and a an element of M which is not join-irreducible, then $\sum_{b\in M\cap[0,a]} \mu_M(b,a)b$ belongs to the submodule $\langle a \wedge b + a \vee b - a - b \mid a,b \in L\rangle$ of $\mathbb{Z}L$. That allows to deduce Equation 1.1.

Thereafter, Equation 1.1 is used to deduce the fundamental theorem of dissection theory in Section 5. This latter affirms that, if $\mathscr{A}$ is a subspace arrangement in a simple topological space T, and L a meet-refinement of $L_{\mathscr{A}}$, then $\sum_{C\in C_{\mathscr{A}}} \chi(C) = \sum_{X\in L} \mu_L(X,T)\chi(X)$. In its original form in Theorem 1.2 of his article, Zaslavsky [19] stated this formula for CW complexes. Remark that if all chambers have the same Euler characteristic $c \neq 0$, then they are $\#C_{\mathscr{A}} = \frac{1}{c}\sum_{X\in L} \mu_L(X,T)\chi(X)$ in number. Deshpande [6] showed a similar result in Theorem 4.6 of his article for the special case of a submanifold arrangement with chambers having the same Euler characteristic $(-1)^l$.

We finally compute the f-polynomial of submanifold arrangements from the dissection theorem of Zaslavsky in Section 6. Chamber counting has probably its origin in the article of Steiner [16] who studied the partition of plane using circles and lines, then that of $\mathbb{R}^3$ using planes and spheres. About 150 years later, Alexanderson and Wetzel [1] computed the numbers of the i-dimensional faces for an arbitrary set of planes, and Zaslavsky [18] for hyperplane arrangements in a Euclidean space of any dimension. One of our formulas is a generalization of those results as it considers a submanifold arrangement $\mathscr{A}$ such that $\chi(X) = (-1)^{\dim X}$ for every $X \in L_{\mathscr{A}} \cup F_{\mathscr{A}}$, and states that $\mathrm{f}_{\mathscr{A}}(x) = (-1)^{\mathrm{rk}\,\mathscr{A}}\mathrm{M}_{\mathscr{A}}(-x,-1)$ where $\mathrm{M}_{\mathscr{A}}$ is the Möbius polynomial of $\mathscr{A}$. Moreover, Pakula [12] computed in Corollary 1 of his article the number of chambers of a pseudosphere arrangement with simple complements. Another formula is a generalization of his result considering a submanifold arrangement $\mathscr{A}$ such that

$$\forall C \in F_{\mathscr{A}} : \chi(C) = (-1)^{\dim C} \quad \text{and} \quad \forall X \in L_{\mathscr{A}} : \chi(X) = \begin{cases} 2 & \text{if } \dim X \equiv 0 \mod 2 \\ 0 & \text{otherwise} \end{cases},$$

and states

$$\mathrm{f}_{\mathscr{A}}(x) = (-1)^{n-\mathrm{rk}\,\mathscr{A}}\big(\mathrm{M}_{\mathscr{A}}(x,-1)+\gamma_n\mathrm{M}_{\mathscr{A}}(-x,-1)\big) \text{ with } \gamma_n := \begin{cases} 1 & \text{if } \dim X \equiv 0 \mod 2 \\ -1 & \text{otherwise} \end{cases}.$$

For some related study on lattices, see [2], [10], [13].

2 Poset

We begin with the general case of posets. A proof of the Zorn's lemma is provided in particular, and the Möbius algebra is described. That algebra plays a key role in this article.

Definition 2.1. A **partial order** is a binary relation $\preceq$ over a set L such that, for $a,b,c \in L$,

- $a \preceq a$,
- if $a \preceq b$ and $a \succeq b$, then $a = b$,
- if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

The set L with a partial order is called a **partially ordered set** or **poset**, and two elements $a,b \in L$ are said comparable if $a \preceq b$ or $a \succeq b$.

Definition 2.2. A poset L has an uppest resp. lowest element 1 resp. $0 \in L$ if, for every $a \in L$, one has $a \preceq 1$ resp. $a \succeq 0$. The poset is said to be **complete** if it has an uppest and a lowest element.

2.1 Zorn's Lemma

Definition 2.3. A subset C of a poset P is a **chain** if any two elements in C are comparable.

Denote by $\mathcal{C}_L$ the set formed by the chains of a poset L. A subset S of L has an upper resp. lower bound if there exists u resp. $l \in L$ such that $s \preceq u$ resp. $l \preceq s$ for each $s \in S$. The upper resp. lower bound u resp. l is said strict if u resp. $l \notin S$.

Definition 2.4. A poset L is said to be **inductive** if every chain included in L has an upper bound.

For an inductive poset L, and $C \in \mathcal{C}_L$, let $C_\prec$ be the set formed by the strict upper bound of C, and denote by $\mathcal{E}_L$ the set $\{C \in \mathcal{C}_L \mid C_\prec \neq \emptyset\}$. The axiom of choice allows to deduce the existence of a function $\mathrm{c} : 2^L \setminus \{\emptyset\} \to L$ such that, for every $A \in 2^L \setminus \{\emptyset\}$, we have $\mathrm{c}(A) \in A$. Define the function $\mathrm{m} : \mathcal{E}_L \to L$, for $C \in \mathcal{E}_L$, by $\mathrm{m}(C) := \mathrm{c}(C_\prec)$.

Definition 2.5. Let S, A be subsets of a poset L. The set S is called a **segment** of A if

$$S \subseteq A \quad \text{and} \quad \forall s \in S, \forall a \in A: \ s \succeq a \Rightarrow a \in S.$$

Definition 2.6. An upper resp. lower bound u resp. l of the subset S of a poset L is called a **join** resp. **meet** if $u \preceq a$ resp. $b \preceq l$ for each upper resp. lower bound a resp. b of S.

Definition 2.7. A chain C of an inductive poset L is called a **good set** if, for every segment S of C with $S \neq C$, we have $S_\prec \cap C \neq \emptyset$ and $\mathrm{m}(S)$ is the meet of $S_\prec \cap C$.

For elements a, b of a poset, by $a \prec b$ we mean that $a \preceq b$ and $a \neq b$.

Lemma 2.8. *Let A, B be nonempty good sets of an inductive poset L. Then, either A is a segment of B or vice versa.*

Proof. Note first that $\emptyset$ is a chain of L. As L is inductive, $\emptyset$ has then an upper bound in L which is necessary a strict upper bound, hence $\emptyset \in \mathcal{E}_L$. Moreover, since $\emptyset$ is obviously a segment of both A and B which are good sets, then $\mathrm{m}(\emptyset) \in \emptyset_\prec \cap A \cap B$ and $A \cap B \neq \emptyset$.
For $a \in A \cap B$, the sets $S_{a,A} := \{s \in A \mid s \prec a\}$ and $S_{a,B} := \{s \in B \mid s \prec a\}$ are clearly segments of A and B respectively. Set $C := \{a \in A \cap B \mid S_{a,A} = S_{a,B}\}$, and let $b \in C$, $c \in A$, with $b \succ c$. We have $c \in S_{b,A} = S_{b,B}$, then $c \in B$ which implies $c \in A \cap B$. If $d \in S_{c,A}$, then $d \prec c \prec b$ implies $d \in S_{b,A} = S_{b,B}$, hence $b \in S_{c,B}$ and $S_{c,A} \subseteq S_{c,B}$. Similarly, we have $S_{c,B} \subseteq S_{c,A}$, then $c \in C$. Therefore, C is a segment of A and B.
Suppose now that $C \neq A$ and $C \neq B$. As A, B are good sets, then $\mathrm{m}(C) \in A \cap B$. Remark that $C \sqcup \{\mathrm{m}(C)\} = S_{\mathrm{m}(C),A} = S_{\mathrm{m}(C),B}$, then $\mathrm{m}(C) \in C$ which is absurd. Hence $C = A$ or $C = B$, in other words, A is a segment of B or vice versa. □

Denoting by $\mathcal{G}_L$ the set formed by the good sets of an inductive poset L, set $U_L := \bigcup_{A \in \mathcal{G}_L} A$.

Theorem 2.9. *If L is an inductive poset, then U_L is a good set.*

Proof. For $a, b \in U_L$, there exist good sets S_a, S_b such that $a \in S_a$ and $b \in S_b$. Using Lemma 2.8, we get either $S_a \subseteq S_b$ or $S_b \subseteq S_a$. That means either $a \preceq b$ or $a \succeq b$, and U_L is consequently a chain.
Let $A \in \mathcal{G}_L$, $a \in A$, and $b \in U_L$ with $a \succeq b$. There is $B \in \mathcal{G}_L$ with $b \in B$. From Lemma 2.8,

- if A is a segment of B, then A is a segment and $b \in A$,
- if B is a segment of A, then $B \subseteq A$ and $b \in A$.

In any case, we have $b \in A$, then A is a segment of U_L.
Consider a segment S of U_L such that $S \neq U_L$. Since U_L is a chain, necessarily $U_L \setminus S \subseteq S_\prec$. Let $a \in U_L \setminus S$, and $A \in \mathcal{G}_L$ such that $a \in A$. As A is a segment of U_L, then $S \subsetneq A$ and S is a segment of A. Moreover, $\mathrm{m}(S)$ is the meet of $S_\prec \cap A$. If there exists $b \in S_\prec \cap U_L$ such that $b \prec \mathrm{m}(S)$, we would get $b \in A$, which is absurd. Therefore, $\mathrm{m}(S)$ is the meet of $S_\prec \cap U_L$, and U_L is a good set. □

Definition 2.10. An element a of a poset L is said to be **maximal** if there does not exist an element $b \in L \setminus \{a\}$ such that $b \succ a$.

Corollary 2.11 (Zorn's Lemma). *Every inductive poset L has a maximal element.*

Proof. Recall that, since U_L is a chain, it consequently possesses an upper bound. Suppose $U_{L\prec} \neq \emptyset$, and let $u \in U_{L\prec}$. Then $U_L \sqcup \{u\}$ is a good set which is absurd. Hence, U_L has a unique upper bound, contained in U_L, which is a maximal element of L. □

2.2 Möbius Algebra

For two elements a, b of a poset L such that $a \preceq b$, denote by $[a, b]$ the set $\{c \in L \mid a \preceq c \preceq b\}$.

Definition 2.12. A poset L is **locally finite** if, for all $a, b \in L$ such that $a \preceq b$, $[a, b]$ is finite.

For a locally finite poset L, denote by $\mathsf{Inc}(L)$ the module of the functions $f : L^2 \to \mathbb{Z}$ with the property that, if $x, y \in L$, then $f(x, y) = 0$ if $x \not\preceq y$.

Definition 2.13. The **incidence algebra** $\mathrm{Inc}(L)$ of a locally finite poset L is the module of functions $f : L^2 \to \mathbb{Z}$, having the property $f(a, b) = 0$ if $a \not\preceq b$, with distributive multiplication $h = f \cdot g$ defined, for $f, g \in \mathrm{Inc}(L)$, by

$$h(a,b) := 0 \text{ if } a \not\preceq b \quad \text{and} \quad h(a,b) := \sum_{c\in[a,b]} f(a,c)g(c,b) \text{ otherwise.}$$

Its multiplicative identity is the Kronecker delta $\delta : L^2 \to \mathbb{Z}$ with $\delta(a,b) := \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise} \end{cases}$.

Definition 2.14. For a locally finite poset L, the **zeta function** ζ_L and the **Möbius function** μ_L in the incidence algebra $\mathrm{Inc}(L)$ are defined, for $a, b \in L$ with $a \preceq b$, by

$$\zeta_L(a,b) := 1 \quad \text{and} \quad \mu_L(a,b) := \begin{cases} 1 & \text{if } a = b \\ -\sum\limits_{\substack{c\in[a,b] \\ c\neq b}} \mu_L(a,c) = -\sum\limits_{\substack{c\in[a,b] \\ c\neq a}} \mu_L(c,b) & \text{otherwise} \end{cases}.$$

Lemma 2.15. *For a locally finite poset L, the zeta function is the multiplicative inverse of the Möbius function in the incidence algebra* $\mathrm{Inc}(L)$.

Proof. For $a, b \in L$ with $a \preceq b$, we have $\zeta_L \cdot \mu_L(a,a) = \mu_L \cdot \zeta_L(a,a) = 1 = \delta(a,a)$, but also

$$\zeta_L \cdot \mu_L(a,b) = \sum_{c\in[a,b]} \mu_L(c,b) = 0 = \delta(a,b) \quad \text{and} \quad \mu_L \cdot \zeta_L(a,b) = \sum_{c\in[a,b]} \mu_L(a,c) = 0 = \delta(a,b).$$

□

The proof of the following proposition is inspired from the original proof of Rota [14] in Proposition 2 of his article.

Proposition 2.16 (Möbius Inversion Formula). *Let L be a locally finite poset, $a, b \in L$ with $a \preceq b$, and f, g two functions from L onto a module M over $\mathbb{Z}$. Then,*

$$\forall x \in [a,b] : g(x) = \sum_{c\in[a,x]} f(c) \iff \forall x \in [a,b] : f(x) = \sum_{c\in[a,x]} g(c)\mu_L(c,x).$$

Proof. Assume first that, for every $x \in [a, b]$, $g(x) = \sum_{c\in[a,x]} f(c)$. Using Lemma 2.15, we get

$$\begin{aligned}
\sum_{c\in[a,x]} g(c)\mu_L(c,x) &= \sum_{c\in[a,x]} \sum_{d\in[a,c]} f(d)\mu_L(c,x) = \sum_{c\in[a,x]} \sum_{d\in[a,c]} f(d)\zeta_L(d,c)\mu_L(c,x) \\
&= \sum_{d\in[a,c]} \sum_{c\in[a,x]} f(d)\zeta_L(d,c)\mu_L(c,x) = \sum_{d\in[a,c]} f(d) \sum_{c\in[a,x]} \zeta_L(d,c)\mu_L(c,x) \\
&= \sum_{d\in[a,c]} f(d)\, \zeta_L \cdot \mu_L(d,x) = \sum_{d\in[a,c]} f(d)\delta(d,x) \\
&= f(x).
\end{aligned}$$

Similarly, if $f(x) = \sum_{c\in[a,x]} g(c)\mu_L(c,x)$ for every $x \in [a,b]$, we obtain

$$\sum_{c\in[a,x]} f(c) = \sum_{c\in[a,x]} \sum_{d\in[a,c]} g(d)\mu_L(d,c) = \sum_{c\in[a,x]} \sum_{d\in[a,c]} g(d)\mu_L(d,c)\zeta_L(c,x) = \sum_{d\in[a,c]} g(d)\delta(d,x)$$
$$= g(x).$$

□

Definition 2.17. We say that a poset L is **lower-finite** if the set $\{b \in L \mid b \preceq a\}$ is finite for any $a \in L$.

For a lower-finite poset L and $a \in L$, let $u_L(a)$ be the element $\sum_{\substack{c\in L \\ c\preceq a}} \mu_L(c,a)c$ of $\mathbb{Z}L$.

Definition 2.18. The **Möbius Algebra** $\text{Möb}(L)$ of a lower-finite poset L is the module $\mathbb{Z}L$ with distributive multiplication defined, for $a, b \in L$, by

$$a \cdot b := \sum_{\substack{c\in L \\ c\preceq a,\ c\preceq b}} u_L(c).$$

Remark that the Möbius algebra was initial defined for finite posets by Solomon [15]. For a lower-finite poset L with a lowest element, define the algebra $\mathrm{A}_L := \langle \alpha_a \mid a \in L\rangle$ over $\mathbb{Z}$ with multiplication

$$\alpha_a\alpha_b := \begin{cases} \alpha_a & \text{if } a = b, \\ 0 & \text{otherwise} \end{cases}.$$

To each $a \in L$, associate an element $a' \in \mathrm{A}_L$ by setting $a' := \sum_{b\in[0,a]} \alpha_b$.

Lemma 2.19. *For a lower-finite poset L with a lowest element, the set $\{a' \mid a \in L\}$ forms a basis of the algebra A_L.*

Proof. From the Möbius inversion formula, we get $\alpha_a = \sum_{b\in[0,a]} \mu(b,a)b'$. The set $\{a' \mid a \in L\}$ consequently generates A_L. Suppose that there exists a finite set $I \subseteq L$ and an integer set $\{i_a\}_{a\in I}$ such that $\sum_{a\in I} i_a a' = 0$. If b is a maximal element of I, then $\alpha_b \sum_{a\in I} i_a a' = i_b\alpha_b = 0$, hence $i_b = 0$. Inductively, we deduce that $i_a = 0$ for every $a \in I$. The set $\{a' \mid a \in L\}$ is therefore independent. □

The following results were initially established by Greenl [9] for finite lattice.

Theorem 2.20. *For a lower-finite poset L with a lowest element, the map $\phi : L \to \mathrm{A}_L$, $a \mapsto a'$ extends to an algebra isomorphism from* $\text{Möb}(L)$ *to* A_L.

Proof. The map ϕ clearly becomes a module homomorphism by linear extension, and an isomorphism by Lemma 2.19. Moreover, for $a, b \in L$,

$$\phi(a\cdot b) = \phi\Big(\sum_{c\in[0,a]\cap[0,b]} u_L(c)\Big) = \sum_{c\in[0,a]\cap[0,b]} \phi\Big(\sum_{d\in[0,c]} \mu(d,c)d\Big)$$
$$= \sum_{c\in[0,a]\cap[0,b]} \sum_{d\in[0,c]} \mu(d,c)d' = \sum_{c\in[0,a]\cap[0,b]} \alpha_c,$$

and $\phi(a)\phi(b) = a'b' = \sum_{c\in[0,a]} \alpha_c \times \sum_{d\in[0,b]} \alpha_d = \sum_{c\in[0,a]\cap[0,b]} \alpha_c$. Then $\phi(a\cdot b) = \phi(a)\phi(b)$, and ϕ is consequently an algebra isomorphism. □

Corollary 2.21. *For a lower-finite poset L with a lowest element, the set $\{u_L(a) \mid a \in L\}$ is a complete set of orthogonal idempotents in* $\text{Möb}(L)$.

Proof. Since $\phi\big(u_L(a)\big) = \sum_{b\in[0,a]} \mu_L(b,a)b' = \alpha_a$, then $\{u_L(a) \mid a \in L\}$ is a basis of $\text{Möb}(L)$. Moreover $\phi\big(u_L(a)\cdot u_L(a)\big) = \alpha_a = \phi\big(u_L(a)\big)$, so the $u_L(a)$'s are idempotents.
Finally $\phi\big(u_L(a)\cdot u_L(b)\big) = \alpha_a\alpha_b = 0$ if $a \neq b$, hence the $u_L(a)$'s are orthogonal. □

Corollary 2.22. *Let L be a lower-finite poset with a lowest element, and M a subset of L containing 0. Then, the linear map $\mathrm{j} : \text{Möb}(L) \to \text{Möb}(M)$, which on the basis $\{u_L(a) \mid a \in L\}$ has the values*

$$\mathrm{j}\big(u_L(a)\big) := \begin{cases} u_M(a) & \text{if } a \in M,\\ 0 & \text{otherwise},\end{cases}$$

is an algebra homomorphism.

Proof. Using Corollary 2.21, $\mathrm{j}\big(u_L(a)\cdot u_L(a)\big) = \mathrm{j}\big(u_L(a)\big) = u_M(a) = \mathrm{j}\big(u_L(a)\big)\cdot \mathrm{j}\big(u_L(a)\big)$ if $a \in M$. Otherwise, $\mathrm{j}\big(u_L(a)\cdot u_L(a)\big) = 0 = \mathrm{j}\big(u_L(a)\big)\cdot \mathrm{j}\big(u_L(a)\big)$. For $a, b \in L$ with $a \neq b$, $\mathrm{j}\big(u_L(a)\cdot u_L(b)\big) = 0 = \mathrm{j}\big(u_L(a)\big)\cdot \mathrm{j}\big(u_L(b)\big)$. □

3 Lattice

We study the special but important case of lattices. After viewing some generalities, we focus on distributive ones, and establish diverse properties which are necessary to investigate the valuation algebra in the next section.

Definition 3.1. A poset L is a join-semilattice resp. meet-semilattice if each 2-element subset $\{a, b\} \subseteq L$ has a join resp. meet denoted by $a \vee b$ resp. $a \wedge b$. It is called a **lattice** if L is both a join- and meet-semilattice, moreover $\vee$ and $\wedge$ become binary operations on L.

Proposition 3.2. *If a lattice L is lower-finite, then it has a lowest element.*

Proof. For any $a \in L$, the principal ideal $\text{id}(a)$ has a lowest element which is $0_a := \bigwedge_{x\in\text{id}(a)} x$. Consider $b \in L \setminus \{a\}$ and the lowest element 0_b of $\text{id}(b)$. The fact $0_a \wedge 0_b \neq 0_a$ would contradict the fact that 0_a is the lowest element of $\text{id}(a)$. Hence, L has a lowest element 0. □

3.1 Generalities on Lattice

Definition 3.3. A **sublattice** of a lattice L is a nonempty subset $M \subseteq L$ such that, for all $a, b \in M$, we have $a \vee b \in M$ and $a \wedge b \in M$.

Definition 3.4. A **lattice homomorphism** is a function $\varphi : L_1 \to L_2$ between two lattices L_1 and L_2 such that, for all $a, b \in L_1$,

$$\varphi(a \vee b) = \varphi(a) \vee \varphi(b) \quad \text{and} \quad \varphi(a \wedge b) = \varphi(a) \wedge \varphi(b).$$

Definition 3.5. An **ideal** of a lattice L is a sublattice $I \subseteq L$ such that, for any $a \in I$ and $b \in L$, we have $a \wedge b \in I$. If in addition $I \neq L$ and, for any $a \wedge b \in I$, either $a \in I$ or $b \in I$, then I is a **prime ideal**.

Definition 3.6. Dually, a **filter** of a lattice L is a sublattice $F \subseteq L$ such that, for any $a \in F$ and $b \in L$, we have $a \vee b \in F$. If in addition $F \neq L$ and, for any $a \vee b \in F$, either $a \in F$ or $b \in F$, then F is a **prime filter**.

Proposition 3.7. *A subset M of a lattice L is a prime ideal if and only if the subset $L \setminus M$ is a prime filter.*

Proof. Assume that M is a prime ideal:

- If $a, b \in L \setminus M$, clearly $a \wedge b \in L \setminus M$ and $a \vee b \in L \setminus M$ since $(a \vee b) \wedge b \in L \setminus M$, then $L \setminus M$ is a sublattice.
- If $a \in M$ and $b \in L \setminus M$, once again $a \vee b \in L \setminus M$ since $(a \vee b) \wedge b \in L \setminus M$, then $L \setminus M$ is a filter.
- If $a \vee b \in L \setminus M$, it is clear that both a, b cannot be all in M, then $L \setminus M$ is prime.

One similarly proves that if M is a prime filter, then $L \setminus M$ is a prime ideal. □

Definition 3.8. Let L be a lattice, and $a \in L$. The **principal ideal** generated by a is the ideal $\mathrm{id}(a) := \{b \in L \mid b \preceq a\}$, dually the **principal filter** generated by a is the filter $\mathrm{fil}(a) := \{b \in L \mid b \succeq a\}$.

Definition 3.9. An element a of a lattice L is **join-irreducible** if, for any subset $S \subseteq L$, $a = \bigvee_{b \in S} b$ implies $a \in S$. Denote by $\mathrm{ji}(L)$ the set formed by the join-irreducible elements of L.

Lemma 3.10. *Let L be a lattice, and $a \in L$. Then, $a \in \mathrm{ji}(L)$ if and only if $a \neq \bigvee_{\substack{b \in L \\ b \prec a}} b$.*

Proof. If $a \in \mathrm{ji}(L)$, as $a \notin \{b \in L \mid b \prec a\}$, then $a \neq \bigvee_{\substack{b \in L \\ b \prec a}} b$.

Assume now that $a \neq \bigvee_{\substack{b \in L \\ b \prec a}} b$, and let $S \subseteq L$ such that $a = \bigvee_{b \in S} b$. Since $b \preceq a$ for every $b \in S$, the only possibility is $a \in S$, and consequently $a \in \mathrm{ji}(L)$. □

The proof of the following proposition is inspired from that of Proposition 2.2 in the article of Bhatta and Ramananda [3].

Proposition 3.11. *Let L be a lower-finite lattice, and $a \in L$. Then, $a = \bigvee_{b \,\in\, \mathrm{id}(a) \cap \mathrm{ji}(L)} b$.*

Proof. It is obvious if $a \in \mathrm{ji}(L)$. Now, assume that $a \in L \setminus \mathrm{ji}(L)$ and $a \neq \bigvee_{b \,\in\, \mathrm{id}(a) \cap \mathrm{ji}(L)} b$. The set $S = \Big\{x \in L \;\Big|\; x \neq \bigvee_{b \,\in\, \mathrm{id}(x) \cap \mathrm{ji}(L)} b\Big\}$ is nonempty and has a minimal element c as L is lower-finite. Since $c \neq \bigvee_{b \,\in\, \mathrm{id}(c) \cap \mathrm{ji}(L)} b$, then $c \notin \mathrm{ji}(L)$, and it follows from Lemma 3.10 that $c = \bigvee_{\substack{b \in L \\ b \prec c}} b$. Clearly, c is an upper bound of the set $X = \bigcup_{\substack{b \in L \\ b \prec c}} \mathrm{id}(b) \cap \mathrm{ji}(L)$. If u is another upper bound of X, then u is an upper bound of $\mathrm{id}(x) \cap \mathrm{ji}(L)$ for every $x \in L$ with $x \prec c$. As c is minimal in S, then $x = \bigvee_{b \,\in\, \mathrm{id}(x) \cap \mathrm{ji}(L)} b$ if $x \prec c$, hence u is an upper bound of $\{b \in L \mid b \prec c\}$ implying $u \succeq c$. Observe that $X = \bigcup_{\substack{b \in L \\ b \prec c}} \big(\mathrm{id}(b) \cap \mathrm{ji}(L)\big) = \mathrm{ji}(L) \cap \bigcup_{\substack{b \in L \\ b \prec c}} \mathrm{id}(b) = \mathrm{ji}(L) \cap \mathrm{id}(c)$. Therefore, c is a minimal upper bound for $\mathrm{id}(c) \cap \mathrm{ji}(L)$ which is a contradiction. □

For two elements a, b of a lattice L such that $a \preceq b$, let $\mathrm{j}_{a,b} : [a \wedge b,\, b] \to [a,\, a \vee b]$ and $\mathrm{m}_{a,b} : [a,\, a \vee b] \to [a \wedge b,\, b]$ be functions respectively defined by

$$\mathrm{j}_{a,b}(x) := a \vee x \quad \text{and} \quad \mathrm{m}_{a,b}(x) := x \wedge b.$$

Definition 3.12. A lattice L is **modular** if, for all $a, b \in L$, $x \in [a \wedge b,\, b]$, and $y \in [a,\, a \vee b]$, we have

$$x = \mathrm{m}_{a,b}\, \mathrm{j}_{a,b}(x) \quad \text{and} \quad y = \mathrm{j}_{a,b}\, \mathrm{m}_{a,b}(y).$$

Proposition 3.13. *A lattice L is modular if and only if, for all $a, b, z \in L$, we have*

$$(a \vee z) \wedge (a \vee b) = a \vee \big(z \wedge (a \vee b)\big) \quad \textit{and} \quad (a \wedge z) \vee (a \wedge b) = a \wedge \big(z \vee (a \wedge b)\big).$$

Proof. Assume first that L is modular. We have $a \preceq (a \vee z) \wedge (a \vee b) \preceq a \vee b$. Letting $u = (a \vee z) \wedge (a \vee b)$, we get

$$u = \mathrm{j}_{a,b}\, \mathrm{m}_{a,b}(u) = a \vee \big((a \vee z) \wedge (a \vee b) \wedge b\big) = a \vee \big((a \vee z) \wedge b\big).$$

Since it is true for all $a, b, z \in L$, interchanging z and b, we obtain $u = a \vee \big(z \wedge (a \vee b)\big)$. Likewise, we have $a \wedge b \preceq (z \wedge b) \vee (a \wedge b) \preceq b$. Letting $v = (z \wedge b) \vee (a \wedge b)$, we get

$$v = \mathrm{m}_{a,b}\, \mathrm{j}_{a,b}(v) = b \wedge \big((z \wedge b) \vee (a \wedge b) \vee a\big) = b \wedge \big((b \wedge z) \vee a\big).$$

Since it is true for all $a, b, z \in L$, interchanging z and a, we obtain $v = b \wedge \big(z \vee (a \wedge b)\big)$.
Assume now that $(a \vee z) \wedge (a \vee b) = a \vee \big(z \wedge (a \vee b)\big)$ and $(a \wedge z) \vee (a \wedge b) = a \wedge \big(z \vee (a \wedge b)\big)$ for all $a, b, z \in L$. If $a \preceq z \preceq a \vee b$, then

$$z = (a \vee b) \wedge z = (a \vee b) \wedge (a \vee z) = a \vee \big(b \wedge (a \vee z)\big).$$

Since $a \vee z = z$, then $z = a \vee (z \wedge b) = \mathrm{j}_{a,b}\, \mathrm{m}_{a,b}(z)$. Likewise, if $a \wedge b \preceq z \preceq b$, then

$$z = (a \wedge b) \vee z = (b \wedge a) \vee (z \wedge b) = b \wedge \big(a \vee (z \wedge b)\big).$$

And since $z \wedge b = z$, then $z = b \vee (a \wedge z) = \mathrm{m}_{a,b}\, \mathrm{j}_{a,b}(z)$. □

3.2 Distributive Lattice

Proposition 3.14. *Let L be a poset, and $a, b, c \in L$. The condition*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \;\; \textit{is equivalent to} \;\; a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Proof. Assume that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Then,

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= \big((a \vee b) \wedge a\big) \vee \big((a \vee b) \wedge c\big)\\ &= a \vee \big((a \vee b) \wedge c\big)\\ &= a \vee (a \wedge c) \vee (b \wedge c)\\ &= a \vee (b \wedge c).\end{aligned}$$

Similarly, if we assume $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, then we obtain

$$(a \wedge b) \vee (a \wedge c) = \big((a \wedge b) \vee a\big) \wedge \big((a \wedge b) \vee c\big) = a \wedge (b \vee c).$$

□

Definition 3.15. A lattice L is **distributive** if, for all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Denote by $\mathcal{I}_L$ the poset formed by the ideals of a lattice L with inclusion as partial order. It is a lattice such that, for $I, J \in \mathcal{I}_L$, $I \vee J := \bigcap_{\substack{K \in \mathcal{I}_L \\ I, J \subseteq K}} K$ and $I \wedge J := \bigcup_{\substack{K \in \mathcal{I}_L \\ K \subseteq I \cap J}} K$.

Theorem 3.16. *Let L be a distributive lattice, I an ideal of L, and F a filter of L such that $I \cap F = \emptyset$. Then, there exists a prime ideal P of L such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Set $\mathcal{X}_{I,F} := \{M \in \mathcal{I}_L \mid I \subseteq M,\ M \cap F = \emptyset\}$. It is a poset with inclusion as partial order, and is nonempty since $I \in \mathcal{X}_{I,F}$. Consider a chain $\mathcal{E} \in \mathcal{C}_{\mathcal{X}_{I,F}}$, and let $E = \bigcup_{C \in \mathcal{E}} E$. If $a, b \in E$, then $a \in A$ and $b \in B$ for some $A, B \in \mathcal{E}$. Since $\mathcal{E}$ is a chain, either $A \subseteq B$ or $A \supseteq B$ hold, so let assume $A \subseteq B$. Then, $a \in B$, and $a \vee b \in B \subseteq E$, as B is an ideal. Moreover, if $c \in L$, then

$a \wedge c \in A \subseteq E$, as A is also an ideal. We deduce that $E \in \mathcal{I}_L$. Besides, $I \subseteq E$ and $E \cap F = \emptyset_{\prec}$ obviously hold. Hence, E is an upper bound of $\mathcal{E}$ in $\mathcal{X}_{I,F}$. Therefore, $\mathcal{X}_{I,F}$ is an inductive poset, and Zorn's lemma allows to state that it has a maximal element P.
Suppose that P is not prime. Then, there exists $a, b \in L$ such that $a, b \notin P$ but $a \wedge b \in P$. The maximality of P yields $\big(P \vee \mathrm{id}(a)\big) \cap F \neq \emptyset$ and $\big(P \vee \mathrm{id}(b)\big) \cap F \neq \emptyset$. Thus, there are $p, q \in P$ such that $p \vee a \in F$, $q \vee b \in F$, and $(p \vee a) \wedge (q \vee b) \in F$ since F is a filter. Expanding by distributivity, we obtain

$$(p \vee a) \wedge (q \vee b) = \big((p \vee a) \wedge q\big) \vee \big((p \vee a) \wedge b\big) = (p \wedge q) \vee (a \wedge q) \vee (p \wedge b) \vee (a \wedge b)$$

which belongs to P. That means $P \cap F \neq \emptyset$ or a contradiction. □

Corollary 3.17. *Let L be a distributive lattice, $I \in \mathcal{I}_L$, and $a \in L$ such that $a \notin I$. Then, there exists a prime ideal P of L such that $I \subseteq P$ and $a \notin P$.*

Proof. Remark that $I \neq \mathrm{fil}(a) = \emptyset$, otherwise, if $b \in I \neq \mathrm{fil}(a)$, then $b \wedge a = a \in I$, which is absurd. Now, for the proof, we apply Theorem 3.16 to I and $F = \mathrm{fil}(a)$. □

Corollary 3.18. *Let L be a distributive lattice, and $a, b \in L$ such that $a \neq b$. Then, L has a prime ideal containing exactly one of a and b.*

Proof. If a and b are not comparable or $b \prec a$, then $a \notin \mathrm{id}(b)$. It remains to apply Corollary 3.17 to $I = \mathrm{id}(b)$. □

Theorem 3.19. *A lattice L is distributive if and only if, for all $a, b, c \in L$, $c \vee a = c \vee b$ and $c \wedge a = c \wedge b$ imply $a = b$.*

Proof. Suppose first that L is distributive and that there exist $a, b, c \in L$ such that $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$. Then,

$$a = a \vee (a \wedge c) = a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge (b \vee c) = b \vee (a \wedge c),$$

which implies $a \preceq b$, and similarly we have $b \preceq a$.
Suppose now that $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$ imply $a = b$. If $x \in [a \wedge b,\, b]$,

- as $x \preceq b \wedge (a \vee x)$ then $a \vee x \preceq a \vee \big(b \wedge (a \vee x)\big)$, as $a \vee x \succeq b \wedge (a \vee x)$ then $a \vee x \succeq a \vee \big(b \wedge (a \vee x)\big)$, hence $a \vee x = a \vee \big(b \wedge (a \vee x)\big)$ on one side,
- on the other side, $a \wedge x = a \wedge b \wedge x = a \wedge b \wedge (a \vee x)$.

By canceling a, we obtain $x = \mathrm{m}_{a,b}\, \mathrm{j}_{a,b}(x)$. If $y \in [a,\, a \vee b]$, as $y \succeq a \vee (b \wedge y)$ then $b \wedge y \succeq b \wedge \big(a \vee (b \wedge y)\big)$, as $b \wedge y \preceq a \vee (b \wedge y)$ then $b \wedge y \preceq b \wedge \big(a \vee (b \wedge y)\big)$, hence $b \wedge y = b \wedge \big(a \vee (b \wedge y)\big)$ on one side, and $b \vee y = a \vee b \vee y = a \vee b \vee (b \wedge y)$ on the other side. By canceling b, we obtain $y = \mathrm{j}_{a,b}\, \mathrm{m}_{a,b}(y)$. Therefore, L is modular.
Let $a^* = a \wedge (b \vee c)$, $b^* = b \wedge (c \vee a)$, and $c^* = c \wedge (a \vee b)$. Then, $a^* \wedge b^* = a \wedge (c \vee a) \wedge b \wedge (b \vee c) = a \wedge b$, $a^* \wedge c^* = a \wedge c$, and $b^* \wedge c^* = b \wedge c$. Set $d = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$. Using twice Proposition 3.13, we get

$$\begin{aligned} a^* \vee b^* &= a^* \vee \big(b \wedge (a \vee c)\big) = (a^* \vee b) \wedge (a \vee c) \\ &= \Big(\big((b \vee c) \wedge a\big) \vee b\Big) \wedge (a \vee c) = (b \vee c) \wedge (a \vee b) \wedge (a \vee c) \\ &= d. \end{aligned}$$

By symmetry, we also have $a^* \vee c^* = b^* \vee c^* = d$. Hence,

- $c^* \vee a^* \vee (b \wedge c) = c^* \vee b^* \vee (a \wedge c) = d$,
- and $c^* \wedge \big(a^* \vee (b \wedge c)\big) = (c^* \wedge a^*) \vee (b \wedge c) = (c^* \wedge b^*) \vee (a \wedge c) = c^* \wedge \big(b^* \vee (a \wedge c)\big)$.

By canceling c^*, we obtain $a^* \vee (b \wedge c) = b^* \vee (a \wedge c)$, whence

$$a^* \vee (b \wedge c) = a^* \vee (b \wedge c) \vee b^* \vee (a \wedge c) = a^* \vee b^* = d.$$

It follows that $(a \vee b) \wedge c = c^* = c^* \wedge d = c^* \wedge \big(a^* \vee (b \wedge c)\big) = (a \wedge c) \vee (b \wedge c)$, hence L is consequently distributive. □

4 Valuation on Lattice

This section is the central part of this article. After defining the valuation algebra and showing some important properties, we prove that if M is a subset of a complete lower-finite distributive lattice L containing its join-irreducible elements, and a an element of M which is not join-irreducible, then $\sum_{b\in M\cap[0,a]} \mu_M(b,a)b$ belongs to the submodule $\langle a \wedge b + a \vee b - a - b \mid a, b \in L\rangle$ of $\mathbb{Z}L$. It would not have been possible to write the first two subsections without the articles of Geissinger [7, 8], and the third without that of Zaslavsky [19].

Definition 4.1. A **valuation** on a lattice L is a function f from L to a module G such that, for all $a, b \in L$,

$$f(a \wedge b) + f(a \vee b) = f(a) + f(b).$$

4.1 Valuation Module

Definition 4.2. The **valuation module** of a lattice L is the module $\mathrm{Val}(L) := \mathbb{Z}L/\mathrm{N}(L)$, where $\mathrm{N}(L)$ is the submodule $\langle a \wedge b + a \vee b - a - b \mid a, b \in L\rangle$ of the module $\mathbb{Z}L$.

Proposition 4.3. *Let* $\mathrm{i} : L \to \mathrm{Val}(L)$ *be the natural induced map for a lattice* L*. Then,* i *is a valuation, and, for every valuation* $f : L \to G$*, there exists a unique module homomorphism* $h : \mathrm{Val}(L) \to G$ *such that the following diagram is commutative*

$$\begin{array}{ccc} L & \xrightarrow{\ \mathrm{i}\ } & \mathrm{Val}(L) \\ & \searrow^{f} & \downarrow h \\ & & G \end{array}$$

Proof. It is clear that i is a valuation as $\mathrm{i}(a\wedge b)+\mathrm{i}(a\vee b)-\mathrm{i}(a)-\mathrm{i}(b) = a\wedge b+a\vee b-a-b = 0$. Besides, we get the homomorphism h by setting

$$\forall a \in L : \ h(a) := f(a) \quad \text{and} \quad \forall x, y \in \mathrm{Val}(L) : \ h(x + y) = h(x) + h(y).$$

□

Proposition 4.4. *For lattices* L_1, L_2 *with natural induced maps* $\mathrm{i}_1, \mathrm{i}_2$ *respectively, a lattice homomorphism* $\varphi : L_1 \to L_2$ *induces a unique module homomorphism* $\psi : \mathrm{Val}(L_1) \to \mathrm{Val}(L_2)$ *such that, for every* $a \in L_1$*,* $\psi\mathrm{i}_1(a) = \mathrm{i}_2\varphi(a)$*.*

Proof. We obtain the homomorphism ψ by setting

$$\forall a \in L_1 : \ \psi(a) := \varphi(a) \quad \text{and} \quad \forall x, y \in \mathrm{Val}(L_1) : \ \psi(x + y) = \psi(x) + \psi(y).$$

□

Proposition 4.5. *For any prime ideal or prime filter* M *of a lattice* L *with natural induced map* i*, each element of* $\mathrm{i}(M)$ *is linearly independent of those in* $\mathrm{i}(L \setminus M)$ *and vise versa.*

Proof. Assume that M is a prime ideal, and consider the indicator function $1_M : L \to \mathbb{Z}$ defined as $1_M(a) := \begin{cases} 1 & \text{if } a \in M \\ 0 & \text{otherwise} \end{cases}$. For $a, b \in L$,

- if $a, b \in M$, we clearly have $1_M(a \wedge b) + 1_M(a \vee b) = 1_M(a) + 1_M(b) = 2$,
- if $a \in M$ and $b \notin M$, since $(a\vee b)\wedge b = b \notin M$, then $a\vee b \notin M$ and $1_M(a\wedge b)+1_M(a\vee b) = 1_M(a) + 1_M(b) = 1$,
- if $a, b \notin M$, then $a \wedge b \notin M$, the fact $(a \vee b) \wedge b = b \notin M$ implies $a \vee b \notin M$, and $1_M(a \wedge b) + 1_M(a \vee b) = 1_M(a) + 1_M(b) = 0$.

Therefore, 1_M is a valuation on L. One similarly proves that if M is prime filter, then 1_M is also a valuation on L. We know from Proposition 4.3 that there exists a unique homomorphism $h : \mathrm{Val}(L) \to \mathbb{Z}$ such that the diagram
$$\begin{array}{ccc} L & \xrightarrow{\ \mathrm{i}\ } & \mathrm{Val}(L) \\ & {\scriptstyle 1_M} \searrow & \downarrow{\scriptstyle h} \\ & & \mathbb{Z} \end{array}$$
is commutative. As $h\mathrm{i}(a) = 1$, for every $a \in M$, and $\big\langle \mathrm{i}(b) \bigm| b \in L \setminus M \big\rangle \subseteq \ker h$, each element of $\mathrm{i}(M)$ is then linearly independent of those in $\mathrm{i}(L \setminus M)$. Likewise, Proposition 3.7 allows to state that $1_{L\setminus M}$ is a valuation, then one also proves that each element of $\mathrm{i}(L \setminus M)$ is linearly independent of those in $\mathrm{i}(M)$. □

Proposition 4.6. *The natural induced map* $\mathrm{i} : L \to \mathrm{Val}(L)$ *of a lattice* L *is an injection if and only if* L *is distributive.*

Proof. If L is distributive, we know from Corollary 3.18 that any two different elements $a, b \in L$ can be separated by a prime ideal, hence Proposition 4.5 allows to deduce that $\mathrm{i}(a)$ and $\mathrm{i}(b)$ are independent in $\mathrm{Val}(L)$.
If L is not distributive, then, by Theorem 3.19, it contains distinct elements a, b, c with $c \vee a = c \vee b$ and $c \wedge a = c \wedge b$. Hence, $\mathrm{i}(a) + \mathrm{i}(c) = \mathrm{i}(c \vee a) + \mathrm{i}(c \wedge a) = \mathrm{i}(c \vee b) + \mathrm{i}(c \wedge b) = \mathrm{i}(b) + \mathrm{i}(c)$, and $\mathrm{i}(a) = \mathrm{i}(b)$. □

Proposition 4.7. *Let* L *be a distributive lattice, and* $a_1, \dots, a_n, b \in L$ *with* $b \notin \big[\bigwedge_{i\in[n]} a_i, \bigvee_{i\in[n]} a_i \big]$. *Then,* b *is linearly independent of* $\{a_1, \dots, a_n\}$ *in* $\mathrm{Val}(L)$.

Proof. If $b \notin \mathrm{id}\Big(\bigvee_{i\in[n]} a_i \Big)$, then there exists a prime ideal P such that $\{a_1, \dots, a_n\} \subseteq P$ and $b \notin P$ by Corollary 3.17, and b is linearly independent of $\{a_1, \dots, a_n\}$ by Proposition 4.5.
If $b \in \mathrm{id}\Big(\bigvee_{i\in[n]} a_i \Big)$, then $b \notin \mathrm{fil}\Big(\bigwedge_{i\in[n]} a_i \Big)$, otherwise $b \in \big[\bigwedge_{i\in[n]} a_i, \bigvee_{i\in[n]} a_i \big]$ which is a contradiction. Hence, $\mathrm{id}(b) \cap \mathrm{fil}\Big(\bigwedge_{i\in[n]} a_i \Big) = \emptyset$, and there exists a prime ideal P such that $\mathrm{id}(b) \subseteq P$ and $P \cap \mathrm{fil}\Big(\bigwedge_{i\in[n]} a_i \Big) = \emptyset$ by Theorem 3.16. As $\{a_1, \dots, a_n\} \subseteq \mathrm{fil}\Big(\bigwedge_{i\in[n]} a_i \Big)$, we once again obtain the independence of b by Proposition 4.5. □

As the lattice L with either the operation $\vee$ or $\wedge$ form a semigroup, the module $\mathbb{Z}L$ may consequently be considered as an algebra with either $\vee$ or $\wedge$ as multiplication. Besides, if L is distributive, Proposition 4.6 allows to identify L with $\mathrm{i}(L)$.

Proposition 4.8. *If* L *is a distributive lattice, then* $\mathrm{N}(L)$ *is an ideal of the algebra* $\mathbb{Z}L$ *for both* $\vee$ *and* $\wedge$ *as multiplication.*

Proof. For $a, b, c \in L$, we have
$$\begin{aligned}(a \wedge b + a \vee b - a - b) \wedge c &= (a \wedge b) \wedge c + (a \vee b) \wedge c - a \wedge c - b \wedge c \\ &= (a \wedge c) \wedge (b \wedge c) + (a \wedge c) \vee (b \wedge c) - a \wedge c - b \wedge c\end{aligned}$$
which belongs to $\mathrm{N}(L)$. Then, by linearly extension, we get $(a \wedge b + a \vee b - a - b) \wedge t \in \mathrm{N}(L)$ for any $t \in \mathbb{Z}L$. Similarly, we have
$$(a \wedge b + a \vee b - a - b) \vee c = (a \vee c) \wedge (b \vee c) + (a \vee c) \vee (b \vee c) - a \vee c - b \vee c \in \mathrm{N}(L).$$
□

4.2 Valuation Algebra

If the lattice L is distributive, Proposition 4.8 allows to state that the valuation module $\mathrm{Val}(L)$ becomes a commutative algebra for either $\vee$ or $\wedge$ as multiplication.

Definition 4.9. The **valuation algebra** is the algebra $\big(\mathrm{Val}(L), \vee\big)$ or $\big(\mathrm{Val}(L), \wedge\big)$ for a distributive lattice L.

Lemma 4.10. *Let L be a complete distributive lattice, and define the map $\tau : \mathrm{Val}(L) \to \mathrm{Val}(L)$ by $\tau(x) := 1 + 0 - x$. Then, for $a, b \in L$, we have $\tau(a \vee b) = \tau(a) \wedge \tau(b)$.*

Proof. We have $1 + 0 - a \vee b = 1 + 0 + a \wedge b - a - b = (1 + 0 - a) \wedge (1 + 0 - b)$. □

Proposition 4.11. *Let L be a complete distributive lattice, $n \in \mathbb{N}^*$, and $a_1, \dots, a_n \in L$. Then, we have $1 - \bigvee_{i\in[n]} a_i = \bigwedge_{i\in[n]} (1 - a_i)$, that is*

$$\bigvee_{i\in[n]} a_i = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I\subseteq[n] \\ \#I=k}} \bigwedge_{i\in I} a_i.$$

Proof. Using Lemma 4.10 and $0 \wedge (1 - a_i) = 0$, we obtain

$$\tau\Big(\bigvee_{i\in[n]} a_i\Big) = 0 + 1 - \bigvee_{i\in[n]} a_i = \bigwedge_{i\in[n]} \tau(a_i) = \bigwedge_{i\in[n]} (0 + 1 - a_i) = 0 + \bigwedge_{i\in[n]} (1 - a_i).$$

Then $1 - \bigvee_{i\in[n]} a_i = \bigwedge_{i\in[n]} \tau(a_i) = \bigwedge_{i\in[n]} (1 - a_i) = 1 + \sum_{k=1}^{n} (-1)^{k} \sum_{\substack{I\subseteq[n] \\ \#I=k}} \bigwedge_{i\in I} a_i.$ □

Corollary 4.12. *Let L be a complete distributive lattice, $n \in \mathbb{N}^*$, $a_1, \dots, a_n \in L$, and f a valuation on L. Then,*

$$f\Big(\bigvee_{i\in[n]} a_i\Big) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I\subseteq[n] \\ \#I=k}} f\Big(\bigwedge_{i\in I} a_i\Big).$$

Proof. If f is a valuation to module G, we know from Proposition 4.3 that a unique module homomorphism $h : \mathrm{Val}(L) \to G$ such that $hi = f$ exists. Then, using Proposition 4.11, we obtain

$$f\Big(\bigvee_{i\in[n]} a_i\Big) = h\Big(\bigvee_{i\in[n]} a_i\Big) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I\subseteq[n] \\ \#I=k}} h\Big(\bigwedge_{i\in I} a_i\Big) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I\subseteq[n] \\ \#I=k}} f\Big(\bigwedge_{i\in I} a_i\Big).$$

□

Theorem 4.13. *Let L be a complete lower-finite distributive lattice. Then, $\mathrm{Val}(L)$ is equal to $\mathbb{Z}\mathrm{ji}(L)$ as modules.*

Proof. We obviously have $0 \in \mathrm{ji}(L)$. Let $a \in L$, and assume that every $b \in L$ such that $a \succ b$ is a linear combination in $\mathrm{Val}(L)$ of a finite number of elements in $\mathrm{ji}(L)$. We know from Proposition 3.11 that there exists a subset $\{b_1, \dots, b_n\}$ of $\mathrm{ji}(L)$ such that $a = \bigvee_{i\in[n]} b_i$. Using Proposition 4.11, we get $a = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I\subseteq[n] \\ \#I=k}} \bigwedge_{i\in I} b_i$ with $a \succ \bigwedge_{i\in I} b_i$ for each $I \subseteq [n]$. Thus $\mathrm{ji}(L)$ generates $\mathrm{Val}(L)$.

Assume now that every subset with cardinality $n - 1$ in $\mathrm{ji}(L)$ is independent, and consider a subset of n elements $\{a_1, \dots, a_n\} \subseteq \mathrm{ji}(L)$. We can suppose that a_n is a maximal element in that set. Since $a_n \neq \bigvee_{i\in[n-1]} a_i$, then $a_n \notin \Big[\bigwedge_{i\in[n-1]} a_i, \bigvee_{i\in[n-1]} a_i\Big]$. We deduce from Proposition 4.7 that $\{a_1, \dots, a_n\}$ is independent. Hence $\mathrm{ji}(L)$ is an independent set in $\mathrm{Val}(L)$. □

Corollary 4.14. *If L is a complete lower-finite distributive lattice, then every valuation of L is determined by its values on* $\mathrm{ji}(L)$ *which can be assigned arbitrarily.*

Proof. If f is a valuation to a module G, we know from Proposition 4.3 that a unique module homomorphism $h : \mathrm{Val}(L) \to G$ such that $hi = f$ exists. We know from Theorem 4.13 that, if $a \in L$, there exist subsets $\{\lambda_1, \dots, \lambda_n\} \subseteq \mathbb{Z}$ and $\{a_1, \dots, a_n\} \subseteq \mathrm{ji}(L)$ such that $a = \sum_{i \in [n]} \lambda_i a_i$.

Then, $f(a) = h(a) = h\Big(\sum_{i \in [n]} \lambda_i a_i\Big) = \sum_{i \in [n]} \lambda_i h(a_i) = \sum_{i \in [n]} \lambda_i f(a_i)$. □

For a poset L, and $a, b \in L$, we write $a \lessdot b$ if $a \prec b$ and $\{c \in L \mid a \prec c \prec b\} = \emptyset$.

Proposition 4.15. *Let L be a distributive lattice, and $a \in \mathrm{ji}(L)$ such that a is not minimal. Then, there exists a unique element $a^* \in L$ such that $a^* \lessdot a$.*

Proof. Suppose that there exist two different elements $b, c \in L$ such that $b \lessdot a$ and $c \lessdot a$. Then, $b \vee c \succeq b$, $b \vee c \succeq c$, and $b \vee c \notin \{b, c\}$. The only possibility is $b \vee c = a$ which contradicts the join-irreducibility of a. □

Let L be a distributive lattice having a lowest element $\mathsf{0}$. Define

$$e_{\mathsf{0}} := \mathsf{0} \in \mathrm{Val}(L) \quad \text{and} \quad e_a := a - a^* \in \mathrm{Val}(L) \text{ for each } a \in \mathrm{ji}(L) \setminus \{\mathsf{0}\}.$$

Theorem 4.16. *Let L be a complete lower-finite distributive lattice. Then, $\big\{e_a \mid a \in \mathrm{ji}(L)\big\}$ is an orthogonal idempotent basis of* $\mathrm{Val}(L)$.

Proof. For $a, b \in \mathrm{ji}(L) \setminus \{\mathsf{0}\}$ with $a \neq b$, we have $e_{\mathsf{0}} \wedge e_{\mathsf{0}} = e_{\mathsf{0}}$ and $e_a \wedge e_{\mathsf{0}} = a \wedge \mathsf{0} - a^* \wedge \mathsf{0} = \mathsf{0}$, $e_a \wedge e_a = a \wedge a - a \wedge a^* - a^* \wedge a + a^* \wedge a^* = a - a^* - a^* + a^* = e_a$, and

$$\begin{aligned} e_a \wedge e_b &= a \wedge b - a \wedge b^* - a^* \wedge b + a^* \wedge b^* \\ &= \begin{cases} b - b^* - b + b^* & \text{if } a^* = b \\ a^* \wedge b^* - a^* \wedge b^* - a^* \wedge b^* + a^* \wedge b^* & \text{otherwise} \end{cases} \\ &= \mathsf{0}. \end{aligned}$$

Then, $\big\{e_a \mid a \in \mathrm{ji}(L)\big\}$ is orthogonal idempotent. Assume now that every subset with cardinality $n-1$ in $\big\{e_a \mid a \in \mathrm{ji}(L)\big\}$ is independent, and consider a subset of n elements $\{e_{a_1}, \dots, e_{a_n}\}$. We can suppose that a_n is a maximal element in the set $\{a_1, \dots, a_n\}$. Since $a_n \neq \bigvee_{i \in [n-1]} a_i \vee \bigvee_{i \in [n]} a_i^*$,

then $a_n \notin \Big[\bigwedge_{i \in [n-1]} a_i \wedge \bigwedge_{i \in [n]} a_i^*, \bigvee_{i \in [n-1]} a_i \vee \bigvee_{i \in [n]} a_i^*\Big]$. We deduce from Proposition 4.7 that a_n is independent of $\{a_1, \dots, a_{n-1}, a_1^*, \dots, a_n^*\}$. Hence e_{a_n} is independent of $\{e_{a_1}, \dots, e_{a_{n-1}}\}$, and $\{e_{a_1}, \dots, e_{a_n}\}$ is consequently an independent set in $\mathrm{Val}(L)$. Finally, since there is a natural bijection $a \mapsto e_a$ between $\mathrm{ji}(L)$ and $\big\{e_a \mid a \in \mathrm{ji}(L)\big\}$, by Theorem 4.13 the latter is also a basis of $\mathrm{Val}(L)$. □

4.3 Identities on Valuation Algebra

Theorem 4.17. *Let L be a complete lower-finite distributive lattice. Then,*

$$\forall x \in L : \ x = \sum_{\substack{a, b \in \mathrm{ji}(L) \\ b \preceq a \preceq x}} \mu_{\mathrm{ji}(L)}(b, a) b.$$

Proof. If $a \in \mathrm{ji}(L)$, then $a = e_a + a^*$, particularly $\mathsf{0} = e_{\mathsf{0}}$. Now, consider any $x \in L \setminus \mathrm{ji}(L)$, and assume that, for every $b \in L$ such that $b \prec x$, we have $b = \sum_{\substack{d \in \mathrm{ji}(L) \\ d \preceq b}} e_d$. There exist $b, c \in L \setminus \{x\}$

such that $x = b \vee c$. Note that $b \wedge c = \sum_{\substack{d \in \mathrm{ji}(L) \\ d \preceq b \wedge c}} e_d$ as the e_d's are orthogonal idempotent. Hence, $b \vee c = b + c - b \wedge c = \sum_{d \in \mathrm{ji}(L) \cap (\mathrm{id}(b) \cup \mathrm{id}(c))} e_d$. Besides, remark that, for any $y \in \mathrm{ji}(L) \cap \mathrm{id}(x)$, there exist $b, c \in L \setminus \{x\}$ such that $y \preceq b$ and $b \vee c = x$. Therefore, $x = \sum_{\substack{d \in \mathrm{ji}(L) \\ d \preceq x}} e_d$.

Let b be the natural bijection $a \mapsto e_a$ between $\mathrm{ji}(L)$ and $\{e_a \mid a \in \mathrm{ji}(L)\}$. For $a \in \mathrm{ji}(L)$, we have $\mathrm{i}(a) = \sum_{d \in \mathrm{ji}(L) \cap [0,a]} \mathrm{b}(d)$. Then, using the Möbius inversion formula, we obtain

$$\mathrm{b}(a) = \sum_{d \in \mathrm{ji}(L) \cap [0,a]} \mu_{\mathrm{ji}(L)}(d,a)\mathrm{i}(d) \quad \text{or} \quad e_a = \sum_{\substack{d \in \mathrm{ji}(L) \\ d \preceq a}} \mu_{\mathrm{ji}(L)}(d,a)d.$$

We obtain the result by combining $x = \sum_{\substack{a \in \mathrm{ji}(L) \\ a \preceq x}} e_a$ with $e_a = \sum_{\substack{d \in \mathrm{ji}(L) \\ d \preceq a}} \mu_{\mathrm{ji}(L)}(d,a)d$. □

Lemma 4.18. *If L is a lower-finite distributive lattice, then $(\mathbb{Z}L, \wedge)$ is naturally isomorphic to the Möbius algebra $(\mathrm{Möb}(L), \cdot)$.*

Proof. For $a \in L$, we have $u_L(a) = \sum_{c \in [0,a]} \mu_L(c,a)c$. The Möbius inversion formula consequently allows to state that $a = \sum_{c \in [0,a]} u_L(c)$. Then, for $a, b \in L$, we have

$$a \cdot b = \sum_{c \in [0,a] \cap [0,b]} u_L(c) = \sum_{c \in [0,a \wedge b]} u_L(c) = a \wedge b.$$

□

Lemma 4.19. *If L is a complete lower-finite distributive lattice, then $(\mathrm{Möb}(L)/\mathrm{N}(L), \cdot)$ is isomorphic to the Möbius algebra $\big(\mathrm{Möb}(\mathrm{ji}(L)), \cdot\big)$.*

Proof. By Lemma 4.18, we get $\mathrm{Möb}(L)/\mathrm{N}(L) \simeq \mathbb{Z}L/\mathrm{N}(L) \simeq \mathrm{Val}(L)$. We know from Theorem 4.13 that $\mathrm{Val}(L)$ is isomorphic to $\mathbb{Z}\mathrm{ji}(L)$ as modules. Now, as algebras, $(\mathrm{Val}(L), \wedge)$ is naturally isomorphic to $\big(\mathrm{Möb}(\mathrm{ji}(L)), \cdot\big)$ since, for $a, b \in \mathrm{ji}(L)$, Theorem 4.17 allows to state that

$$a \cdot b = \sum_{c \in [0,a] \cap [0,b] \cap \mathrm{ji}(L)} u_{\mathrm{ji}(L)}(c) = \sum_{c \in [0,a \wedge b] \cap \mathrm{ji}(L)} u_{\mathrm{ji}(L)}(c) = a \wedge b.$$

□

The following theorem is the main result of this article. Zaslavsky [19] originally proved it in Theorem 2.1 of his article for every finite distributive lattice. This latter is obviously complete, lower-finite, and contains its join-irreducible elements.

Theorem 4.20. *Let L be a complete lower-finite distributive lattice, and M a subset of L such that $\mathrm{ji}(L) \subseteq M$. If $a \in M \setminus \mathrm{ji}(L)$, then*

$$u_M(a) \in \mathrm{N}(L).$$

Proof. Consider the linear maps $\mathrm{j} : \mathrm{Möb}(L) \to \mathrm{Möb}\big(\mathrm{ji}(L)\big)$, $\mathrm{j}_1 : \mathrm{Möb}(L) \to \mathrm{Möb}(M)$, and $\mathrm{j}_2 : \mathrm{Möb}(M) \to \mathrm{Möb}\big(\mathrm{ji}(L)\big)$ which on the basis $\{u_L(a) \mid a \in L\}$, and $\{u_M(a) \mid a \in M\}$

respectively have the values

$$\mathrm{j}\big(u_L(a)\big) := \begin{cases} u_{\mathrm{ji}(L)}(a) & \text{if } a \in \mathrm{ji}(L), \\ 0 & \text{otherwise} \end{cases}, \quad \mathrm{j}_1\big(u_L(a)\big) := \begin{cases} u_M(a) & \text{if } a \in M, \\ 0 & \text{otherwise} \end{cases},$$

$$\text{and } \mathrm{j}_2\big(u_M(a)\big) := \begin{cases} u_{\mathrm{ji}(L)}(a) & \text{if } a \in \mathrm{ji}(L), \\ 0 & \text{otherwise} \end{cases}.$$

Then, j, j_1, and j_2 are algebra homomorphisms by Corollary 2.22. Moreover, as the diagram

$$\begin{array}{ccc} \mathrm{M\ddot{o}b}(L) & \xrightarrow{\ \mathrm{j}_1\ } & \mathrm{M\ddot{o}b}(M) \\ & \searrow^{\mathrm{j}} & \downarrow{\scriptstyle \mathrm{j}_2} \\ & & \mathrm{M\ddot{o}b}\big(\mathrm{ji}(L)\big) \end{array}$$

is commutative, then $u_M(a) \in \ker \mathrm{j}_2 \subseteq \ker \mathrm{j}$ if $a \in M \setminus \mathrm{ji}(L)$.
Finally, since $\mathrm{M\ddot{o}b}\big(\mathrm{ji}(L)\big) \simeq \mathrm{M\ddot{o}b}(L)/\ker \mathrm{j}$ like proved in II-Theorem 6.12 of the book of Burris and Sankappanavar [4], we obtain $\ker \mathrm{j} = \mathrm{N}(L)$ using Lemma 4.19, and then $u_M(a) \in \mathrm{N}(L)$ if $a \in M \setminus \mathrm{ji}(L)$. □

Corollary 4.21. *Let L be a complete lower-finite distributive lattice, M a subset of L such that $\mathrm{ji}(L) \subseteq M$, and $f : L \to G$ a valuation on L. If $a \in M \setminus \mathrm{ji}(L)$, then*

$$\sum_{b \in [0,a] \cap M} \mu_M(b,a) f(b) = 0.$$

Proof. Let $h : \mathrm{Val}(L) \to G$ be the module homomorphism associated to f as in Proposition 4.3. We already know from Lemma 4.18 that $\mathrm{Val}(L) \simeq \mathrm{M\ddot{o}b}(L)/\mathrm{N}(L)$. By Theorem 4.20, we then obtain

$$\sum_{b \in [0,a] \cap M} \mu_M(b,a) b = 0$$

$$h\Big(\sum_{b \in [0,a] \cap M} \mu_M(b,a) b\Big) = h(0)$$

$$\sum_{b \in [0,a] \cap M} \mu_M(b,a) h(b) = 0$$

$$\sum_{b \in [0,a] \cap M} \mu_M(b,a) f(b) = 0.$$

□

5 Dissection Theory

We use Corollary 4.21 to prove the fundamental theorem of dissection theory in this section. Denote by $H_n(T)$ the n^{th} singular homology group of a topological space T for $n \in \mathbb{N}$.

Definition 5.1. A topological space T is **simple** if the groups $H_n(T)$ have finite ranks, only finitely many of them are nontrivial, and rank $H_0(T) = 1$.

Definition 5.2. Let us call **subspace arrangement** a finite set of simple subspaces in in a simple topological space T.

For a subspace arrangement $\mathscr{A}$ in T, let $L_{\mathscr{A}} := \Big\{ \bigcap_{H \in \mathscr{B}} H \in 2^T \setminus \{\emptyset\} \ \Big|\ \mathscr{B} \subseteq \mathscr{A} \Big\}$ be the poset with partial order $\preceq$ defined, for $A, B \in L_{\mathscr{A}}$, by $A \preceq B$ if and only if $A \subseteq B$.

Definition 5.3. Let $\mathscr{A}$ be a subspace arrangement in a simple topological space T. A **meet-refinement** of $L_{\mathscr{A}}$ is a finite poset $L \subseteq 2^T \setminus \{\emptyset\}$ with the same partial order as that defined for $L_{\mathscr{A}}$ such that $\bigcup_{X \in L} X = \bigcup_{H \in \mathscr{A}} H$ and

- any element in $L_{\mathscr{A}}$ is a union of elements in L,
- any nonempty intersection of elements in L is also a union of elements in L.

Denote by $\mathrm{C}(X)$ the set formed by the connected components of a topological space X, and let $\mathscr{A}$ be a subspace arrangement of T. The set $L^c_{\mathscr{A}} := L_{\mathscr{A}} \sqcup \left\{ \mathrm{C}\Big(\bigcap_{H \in \mathscr{B}} H\Big) \,\middle|\, \mathscr{B} \subseteq \mathscr{A},\ \bigcap_{H \in \mathscr{B}} H \neq \emptyset \right\}$ is for instance a meet-refinement of $L_{\mathscr{A}}$.

Definition 5.4. Let $\mathscr{A}$ be a subspace arrangement in a simple topological space T, and denote by $C_{\mathscr{A}}$ the set formed by the connected components of $T \setminus \bigcup_{H \in \mathscr{A}} H$. An element of $C_{\mathscr{A}}$ is called **chamber**.

Consider a subspace arrangement $\mathscr{A}$, and a meet-refinement L of $L_{\mathscr{A}}$. Let $\mathrm{D}(L)$ be the finite distributive lattice of sets generated by $L \sqcup C_{\mathscr{A}}$ through unions and intersections, that is

$$\mathrm{D}(L) := \left\{ \bigcup_{A \in M} A \sqcup \bigcup_{X \in D} X \,\middle|\, M \subseteq L,\ D \subseteq C_{\mathscr{A}} \right\}.$$

In that case, for $A, B \in \mathrm{D}(L)$, we have $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

Lemma 5.5. *Let $\mathscr{A}$ be a subspace arrangement in a simple topological space T, and L a meet-refinement of $L_{\mathscr{A}}$. Then,* $\mathrm{ji}\big(\mathrm{D}(L)\big) \subseteq \{\emptyset\} \sqcup L \sqcup C_{\mathscr{A}}$.

Proof. Every element of $\mathrm{D}(L) \setminus (\{\emptyset\} \sqcup L \sqcup C_{\mathscr{A}})$ is the union of at least two elements of $L \sqcup C_{\mathscr{A}}$. Then, none of them can be join-irreducible. ◻

Theorem 5.6. *Let $\mathscr{A}$ be a subspace arrangement in a simple topological space T, L a meet-refinement of $L_{\mathscr{A}}$, and f a valuation on* $\mathrm{D}(L)$*. Then,*

$$\sum_{C \in C_{\mathscr{A}}} f(C) = \sum_{X \in L \sqcup \{\emptyset\}} \mu_{L \sqcup \{\emptyset\}}(X, T) f(X).$$

Proof. Note first that $T \in L$ but $T \notin \mathrm{ji}\big(\mathrm{D}(L)\big)$ as $T = \bigcup_{H \in \mathscr{A}} H \sqcup \bigcup_{C \in C_{\mathscr{A}}} C$. From Corollary 4.21 and Lemma 5.5, we get

$$\sum_{A \in \{\emptyset\} \sqcup L \sqcup C_{\mathscr{A}}} \mu_{\{\emptyset\} \sqcup L \sqcup C_{\mathscr{A}}}(A, T) f(A) = 0.$$

The result is finally obtained after taking into account the following remarks:

- if $C \in C_{\mathscr{A}}$, then $\mu_{\{\emptyset\} \sqcup L \sqcup C_{\mathscr{A}}}(C, T) = -\mu_{\{\emptyset\} \sqcup L \sqcup C_{\mathscr{A}}}(C, C) = -1$,
- if $X \in \{\emptyset\} \sqcup L$, then $[X, T] \cap C_{\mathscr{A}} = \emptyset$, hence $\mu_{\{\emptyset\} \sqcup L \sqcup C_{\mathscr{A}}}(X, T) = \mu_{\{\emptyset\} \sqcup L}(X, T)$.

◻

Definition 5.7. The **Euler characteristic** of a topological space T is

$$\chi(T) := \sum_{n \in \mathbb{N}} (-1)^n \operatorname{rank} H_n(T).$$

We can now state the fundamental theorem of dissection theory.

Corollary 5.8 (Fundamental Theorem of Dissection Theory). *Let $\mathscr{A}$ be a subspace arrangement in a simple topological space T, and L a meet-refinement of $L_{\mathscr{A}}$. Then,*

$$\sum_{C \in C_{\mathscr{A}}} \chi(C) = \sum_{X \in L} \mu_L(X, T) \chi(X).$$

Proof. It is known that $\chi(A) + \chi(B) = \chi(A \cup B) + \chi(A \cap B)$, for $A, B \subseteq T$, like stated for example at the end of § 12.4 in the book of tom Dieck [17]. The Euler characteristic is then a valuation on $\mathrm{D}(L)$. Moreover, $\chi(\emptyset) = 0$ by definition. We consequently obtain the result by using Theorem 5.6 with χ as a valuation. □

Example 5.9. Consider the arrangement $\mathscr{A}$ of parametric 1-spheres $H_1 : \begin{cases} x = \cos(\frac{\pi}{4}) \\ y = \sin(\frac{\pi}{4})\cos(t), \\ z = \sin(\frac{\pi}{4})\sin(t) \end{cases}$

$H_2 : \begin{cases} x = -\cos(\frac{\pi}{8}) \\ y = \sin(\frac{\pi}{8})\cos(t), \\ z = \sin(\frac{\pi}{8})\sin(t) \end{cases}$ $H_3 : \begin{cases} x = \cos(\frac{\pi}{6})\sin(t) \\ y = \cos(\frac{\pi}{6})\cos(t), \\ z = \sin(\frac{\pi}{6}) \end{cases}$ $H_4 : \begin{cases} x = \cos(\frac{\pi}{3})\sin(t) \\ y = \cos(\frac{\pi}{3})\cos(t), \\ z = -\sin(\frac{\pi}{3}) \end{cases}$ where $t \in [0, 2\pi]$, in $\mathbb{S}^2$ represented on Figure 1. On one side, $C_{\mathscr{A}}$ has 6 chambers having Euler characteristic 1, and 1 with Euler characteristic 0, then $\sum_{C \in C_{\mathscr{A}}} \chi(C) = 6$. On the other side,

$$\begin{aligned} \sum_{X \in L_{\mathscr{A}}} \mu_{L_{\mathscr{A}}}(X, \mathbb{S}^2)\chi(X) &= \mu_{L_{\mathscr{A}}}(\mathbb{S}^2, \mathbb{S}^2)\chi(\mathbb{S}^2) + \sum_{i \in [4]} \mu_{L_{\mathscr{A}}}(H_i, \mathbb{S}^2)\chi(H_i) \\ &\quad + \mu_{L_{\mathscr{A}}}(H_1 \cap H_3, \mathbb{S}^2)\chi(H_1 \cap H_3) + \mu_{L_{\mathscr{A}}}(H_2 \cap H_3, \mathbb{S}^2)\chi(H_2 \cap H_3) \\ &= 1 \times 2 + 4 \times (-1) \times 0 + 1 \times 2 + 1 \times 2 \\ &= 6. \end{aligned}$$

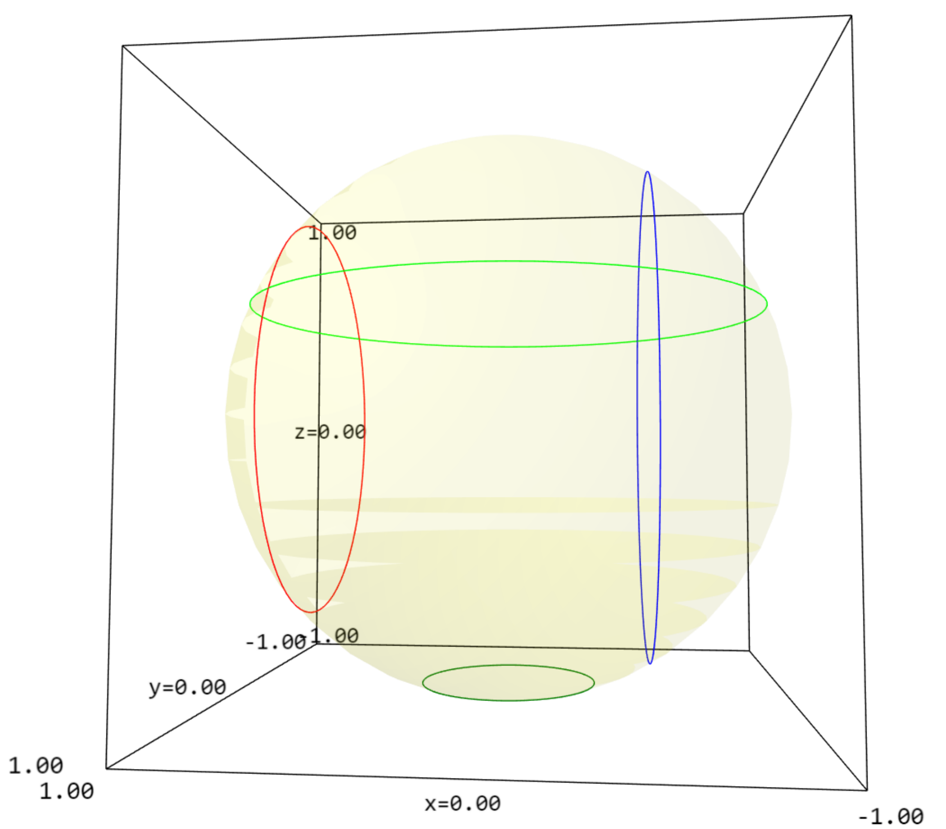

Figure 1. 1-Sphere Arrangement of Example 5.9

Corollary 5.10. *Let $\mathscr{A}$ be a subspace arrangement in a simple topological space T, and L a meet-refinement of $L_{\mathscr{A}}$. Suppose that every chamber of $\mathscr{A}$ has the same Euler characteristic $c \neq 0$. Then,*

$$\#C_{\mathscr{A}} = \frac{1}{c} \sum_{X \in L} \mu_L(X, T)\chi(X).$$

Proof. It is obviously a consequence of the fundamental theorem of dissection theory where $\chi(C) = c$ for $C \in C_{\mathscr{A}}$. □

6 Face Counting for Submanifold Arrangement

We use the fundamental theorem of dissection theory to compute the f-polynomial of submanifold arrangements having specific face properties.

Definition 6.1. Let $\mathscr{A}$ be a subspace arrangement in a simple topological space T, and $X \in L_{\mathscr{A}}$. The **induced subspace arrangement** on X is the subspace arrangement in X defined by

$$\mathscr{A}_X := \big\{ H \cap X \;\big|\; H \in \mathscr{A},\, H \cap X \notin \{\emptyset, X\} \big\}.$$

Let $F_{\mathscr{A}} := \bigsqcup_{X \in L_{\mathscr{A}}} C_{\mathscr{A}_X}$, and call an element of $F_{\mathscr{A}}$ a **face** of $\mathscr{A}$.

Definition 6.2. A n**-dimensional manifold** or n**-manifold** is a topological space with the property that each point has a neighborhood that is homeomorphic to $\mathbb{R}^n$, and a **submanifold** of a n-manifold T is a k-manifold included in T where $k \in [0, n]$. Moreover, we say that a manifold is simple if it is simple as a topological space.

Definition 6.3. Let us call **submanifold arrangement** in a simple n-manifold T a finite set $\mathscr{A}$ of simple submanifolds in T such that every element of $L_{\mathscr{A}} \cup F_{\mathscr{A}}$ is a submanifold.

Example 6.4. Consider the arrangement $\mathscr{A}$ of 1-manifolds $H_1 : y = 6\sin(x)$, $H_2 : y = x + \cos(x)$, $H_3 : \dfrac{x^2}{64} + \dfrac{y^2}{25} = 1$ in $\mathbb{R}^2$ represented on Figure 2. We see that

$$\begin{aligned}
\sum_{X \in L_{\mathscr{A}}} \mu_{L_{\mathscr{A}}}(X, \mathbb{R}^2)\chi(X) &= \mu_{L_{\mathscr{A}}}(\mathbb{R}^2, \mathbb{R}^2)\chi(\mathbb{R}^2) + \mu_{L_{\mathscr{A}}}(H_1, \mathbb{R}^2)\chi(H_1) + \mu_{L_{\mathscr{A}}}(H_2, \mathbb{R}^2)\chi(H_2) \\
&\quad + \mu_{L_{\mathscr{A}}}(H_3, \mathbb{R}^2)\chi(H_3) + \mu_{L_{\mathscr{A}}}(H_1 \cap H_2, \mathbb{R}^2)\chi(H_1 \cap H_2) \\
&\quad + \mu_{L_{\mathscr{A}}}(H_1 \cap H_3, \mathbb{R}^2)\chi(H_1 \cap H_3) + \mu_{L_{\mathscr{A}}}(H_2 \cap H_3, \mathbb{R}^2)\chi(H_2 \cap H_3) \\
&= 1 \times 1 + (-1) \times (-1) + (-1) \times (-1) + (-1) \times 0 + 1 \times 3 + 1 \times 10 + 1 \times 2 \\
&= 18
\end{aligned}$$

is the number of chamber in $C_{\mathscr{A}}$.

Definition 6.5. Let $\mathscr{A}$ be a submanifold arrangement in a simple n-manifold T, and x a variable. For $k \in [0, n]$, denote by $\mathrm{f}_k(\mathscr{A})$ the number of k-dimensional faces of $\mathscr{A}$. The **f-polynomial** of $\mathscr{A}$ is

$$\mathrm{f}_{\mathscr{A}}(x) := \sum_{k \in [0,n]} \mathrm{f}_k(\mathscr{A}) x^{n-k}.$$

Proposition 6.6. *Let $\mathscr{A}$ be a submanifold arrangement in a simple n-manifold T. Suppose that*

$$\begin{aligned}
&\forall k \in [0, n],\, \forall X \in L_{\mathscr{A}},\, \dim X = k : \; \chi(X) = l_k, \\
&\forall k \in [0, n],\, \forall C \in F_{\mathscr{A}},\, \dim C = k : \; \chi(C) = c_k \neq 0.
\end{aligned}$$

Then,

$$\mathrm{f}_{\mathscr{A}}(x) = \sum_{i \in [0,n]} \sum_{\substack{Y \in L_{\mathscr{A}} \\ \dim Y = i}} \sum_{k \in [0,i]} \sum_{\substack{X \in L_{\mathscr{A}_Y} \\ \dim X = k}} \frac{l_k}{c_i} \mu_{L_{\mathscr{A}}}(X, Y) x^{n-k}.$$

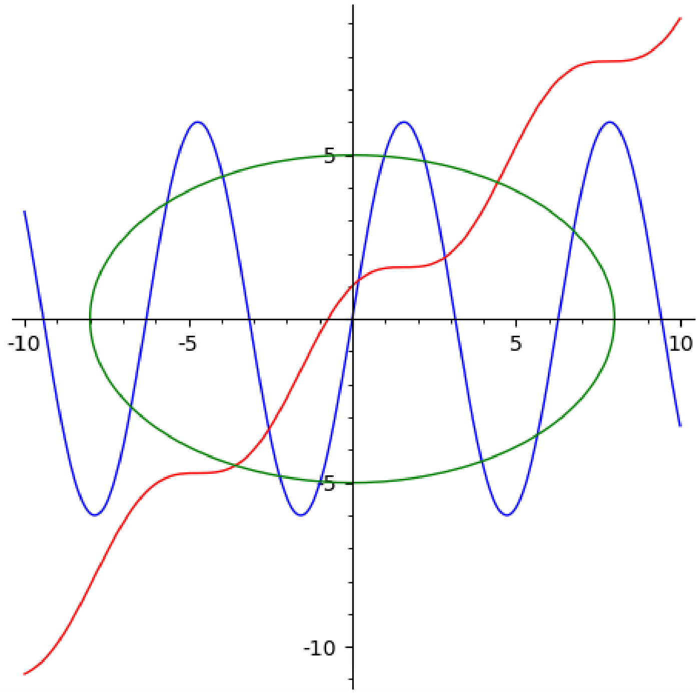

Figure 2. Submanifold Arrangement of Example 6.4

Proof. Using the fundamental theorem of dissection theory, we get

$$\begin{aligned} \mathrm{f}_i(\mathscr{A}) &= \sum_{\substack{Y \in L_{\mathscr{A}} \\ \dim Y = i}} \# C_{L_{\mathscr{A}_Y}} \\ &= \frac{1}{c_i} \sum_{\substack{Y \in L_{\mathscr{A}} \\ \dim Y = i}} \sum_{X \in L_{\mathscr{A}_Y}} \mu_{L_{\mathscr{A}_Y}}(X, Y)\chi(X) \\ &= \sum_{\substack{Y \in L_{\mathscr{A}} \\ \dim Y = i}} \sum_{k \in [0,i]} \sum_{\substack{X \in L_{\mathscr{A}_Y} \\ \dim X = k}} \frac{l_k}{c_i} \mu_{L_{\mathscr{A}_Y}}(X, Y) \\ &= \sum_{\substack{Y \in L_{\mathscr{A}} \\ \dim Y = i}} \sum_{k \in [0,i]} \sum_{\substack{X \in L_{\mathscr{A}_Y} \\ \dim X = k}} \frac{l_k}{c_i} \mu_{L_{\mathscr{A}}}(X, Y). \end{aligned}$$

□

Definition 6.7. Let $\mathscr{A}$ be a submanifold arrangement in a simple n-manifold T. The **rank** of $X \in L_{\mathscr{A}}$ is $\mathrm{rk}\, X := n - \dim X$, and that of $\mathscr{A}$ is $\mathrm{rk}\, \mathscr{A} := \max\{\mathrm{rk}\, X \mid X \in L_{\mathscr{A}}\}$.

Definition 6.8. Let $\mathscr{A}$ be a submanifold arrangement in a simple n-manifold T, and x, y two variables. The **Möbius Polynomial** of $\mathscr{A}$ is

$$\mathrm{M}_{\mathscr{A}}(x, y) := \sum_{X, Y \in L_{\mathscr{A}}} \mu_{L_{\mathscr{A}}}(X, Y) x^{\mathrm{rk}\, X} y^{\mathrm{rk}\, \mathscr{A} - \mathrm{rk}\, Y}.$$

Corollary 6.9. *Let $\mathscr{A}$ be a submanifold arrangement in a simple n-manifold T. Suppose that $\chi(X) = (-1)^{\dim X}$ for every $X \in L_{\mathscr{A}} \cup F_{\mathscr{A}}$. Then,*

$$\mathrm{f}_{\mathscr{A}}(x) = (-1)^{\mathrm{rk}\, \mathscr{A}} \mathrm{M}_{\mathscr{A}}(-x, -1).$$

Proof. From Proposition 6.6, we obtain

$$\begin{aligned}
\mathrm{f}_{\mathscr{A}}(x) &= \sum_{i\in[0,n]} \sum_{\substack{Y\in L_{\mathscr{A}} \\ \dim Y=i}} \sum_{k\in[0,i]} \sum_{\substack{X\in L_{\mathscr{A}_Y} \\ \dim X=k}} (-1)^{k-i}\mu_{L_{\mathscr{A}}}(X,Y)x^{n-k} \\
&= \sum_{Y\in L_{\mathscr{A}}} \sum_{X\in L_{\mathscr{A}_Y}} (-1)^{\dim X-\dim Y}\mu_{L_{\mathscr{A}}}(X,Y)x^{n-\dim X} \\
&= \sum_{Y\in L_{\mathscr{A}}} \sum_{X\in L_{\mathscr{A}_Y}} (-1)^{n-\dim Y}\mu_{L_{\mathscr{A}}}(X,Y)(-1)^{\dim X-n}x^{n-\dim X} \\
&= \sum_{Y\in L_{\mathscr{A}}} \sum_{X\in L_{\mathscr{A}_Y}} (-1)^{\operatorname{rk} Y}\mu_{L_{\mathscr{A}}}(X,Y)(-x)^{\operatorname{rk} X} \\
&= (-1)^{\operatorname{rk}\mathscr{A}} \sum_{Y\in L_{\mathscr{A}}} \sum_{X\in L_{\mathscr{A}_Y}} \mu_{L_{\mathscr{A}}}(X,Y)(-x)^{\operatorname{rk} X}(-1)^{\operatorname{rk} Y-\operatorname{rk}\mathscr{A}} \\
&= (-1)^{\operatorname{rk}\mathscr{A}}\mathrm{M}_{\mathscr{A}}(-x,-1).
\end{aligned}$$

□

Corollary 6.10. *Let $\mathscr{A}$ be a submanifold arrangement in a simple n-manifold T. Suppose*

$$\forall C\in F_{\mathscr{A}}:\ \chi(C)=(-1)^{\dim C} \quad \textit{and} \quad \forall X\in L_{\mathscr{A}}:\ \chi(X)=\begin{cases}2 & \textit{if } \dim X\equiv 0 \mod 2\\ 0 & \textit{otherwise}\end{cases}.$$

Moreover, define $\gamma_n := \begin{cases}1 & \textit{if } \dim X\equiv 0 \mod 2\\ -1 & \textit{otherwise}\end{cases}$. *Then,*

$$\mathrm{f}_{\mathscr{A}}(x)=(-1)^{n-\operatorname{rk}\mathscr{A}}\big(\mathrm{M}_{\mathscr{A}}(x,-1)+\gamma_n\mathrm{M}_{\mathscr{A}}(-x,-1)\big).$$

Proof. From Proposition 6.6, we obtain

$$\begin{aligned}
\mathrm{f}_{\mathscr{A}}(x) &= \sum_{i\in[0,n]} \sum_{\substack{Y\in L_{\mathscr{A}} \\ \dim Y=i}} \sum_{k\in[0,i]} \sum_{\substack{X\in L_{\mathscr{A}_Y} \\ \dim X=k}} (-1)^{-i}\chi(X)\mu_{L_{\mathscr{A}}}(X,Y)x^{n-k} \\
&= \sum_{Y\in L_{\mathscr{A}}} \sum_{X\in L_{\mathscr{A}_Y}} (-1)^{-\dim Y}\chi(X)\mu_{L_{\mathscr{A}}}(X,Y)x^{n-\dim X} \\
&= (-1)^{n} \sum_{Y\in L_{\mathscr{A}}} \sum_{X\in L_{\mathscr{A}_Y}} \chi(X)\mu_{L_{\mathscr{A}}}(X,Y)x^{\operatorname{rk} X}(-1)^{\operatorname{rk} Y} \\
&= (-1)^{n-\operatorname{rk}\mathscr{A}} \sum_{Y\in L_{\mathscr{A}}} \sum_{X\in L_{\mathscr{A}_Y}} \chi(X)\mu_{L_{\mathscr{A}}}(X,Y)x^{\operatorname{rk} X}(-1)^{\operatorname{rk}\mathscr{A}-\operatorname{rk} Y} \\
&= (-1)^{n-\operatorname{rk}\mathscr{A}} \sum_{Y\in L_{\mathscr{A}}} \sum_{X\in L_{\mathscr{A}_Y}} \mu_{L_{\mathscr{A}}}(X,Y)x^{\operatorname{rk} X}(-1)^{\operatorname{rk}\mathscr{A}-\operatorname{rk} Y} \\
&\quad + (-1)^{n-\operatorname{rk}\mathscr{A}}\gamma_n \sum_{Y\in L_{\mathscr{A}}} \sum_{X\in L_{\mathscr{A}_Y}} \mu_{L_{\mathscr{A}}}(X,Y)(-x)^{\operatorname{rk} X}(-1)^{\operatorname{rk}\mathscr{A}-\operatorname{rk} Y} \\
&= (-1)^{n-\operatorname{rk}\mathscr{A}}\mathrm{M}_{\mathscr{A}}(x,-1)+(-1)^{n-\operatorname{rk}\mathscr{A}}\gamma_n\mathrm{M}_{\mathscr{A}}(-x,-1).
\end{aligned}$$

□

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