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# Uniform approximation by Lupaş operators on non-compact intervals

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**Abstract** In this paper, we study the function's behavior defined on the closed and unbounded interval of a real line. We investigate the particular case of Lupaş operator and prove the result of the uniform convergence of Lupaş operator if the functions have a certain type of behavior. We also derive the result of the rate of convergence with the help of moduli of continuity. The paper also contain the graphical representation of the mathematical results on convergence.

## 1 Introduction

The study of the convergence problem of unbounded functions using linear positive operators is a fast-growing field in approximation theory. Many papers provide the study of the approximation process by positive linear operators in spaces containing unbounded functions. The collection of all such convergence problems and some open problems are listed in the survey paper [1].

Let $\alpha_n : C(I) \to C(I)$ be a sequence of positive linear operators, where $C(I)$ is the space of continuous function defined on a non-compact interval $I$. We aim to study the behaviour of the continuous function defined on a non-compact interval $I$ which can be in uniform approximation process by Lupaş operators and which satisfies the following condition.

$$\sup_{x\in I}|\alpha_n(f,x) - f(x)| = \|\alpha_n f - f\|_\infty \to 0.$$

The problems of the convergence process of unbounded continuous functions by positive linear operators are found in many papers, some of which are [2], [3].

A useful result in this direction is theorem 2 from [4]. In [1], the author has studied the case when the condition of boundedness is dropped from the function. In [5], the author has studied the characterization of a function that is defined on a non-compact interval $I \subseteq \mathbb{R}$ and which is uniformly approximated by different types of positive linear operators.

In this paper, we follow our work from the papers mentioned above. We give the characterization of the function defined on a non-compact interval $A$ which is uniformly approximated by the Lupaş operators. The study of certain operators and their different modifications can be found in ([6], [9], [10], [11], [12], [13], [14], [15]).

## 2 Preliminaries

Let $I \subseteq \mathbb{R}$ be a non-compact interval of $\mathbb{R}$, $J \subseteq \mathbb{R}$ and we define $\Lambda : I \to J$ which is a continuously differentiable one-one map, with the help of this map, we define the space

$$UC_\Lambda = \{f \in C(I) : f \circ \Lambda^{-1} \text{ is uniformly continuous on J}\}.$$

For a fixed map $\Lambda$ and for a function $f \in UC_\Lambda$, the modulus of continuity is defined as

$$w^\Lambda(f,\delta) = \sup_{a,b\in I}\{|f(a) - f(b)| : |\Lambda(a) - \Lambda(b)| \le \delta, \quad \text{for } \delta > 0\},$$

which is the generalization of the usual modulus of continuity. The usual modulus of continuity and the general modulus of continuity is related by the following equation

$$w^{\Lambda}(f,\delta) = w(f \circ \Lambda^{-1}, \delta).$$

**Proposition 2.1.** *For every function function $f \in UC_{\Lambda}$, we have*

$$\lim_{n\to\infty} w^{\Lambda}(f,\delta_n) = 0 \text{ whenever } \delta_n \to 0.$$

*The converse is also true, i.e., if there is a positive sequence $\{\delta_n\}$ converging to $0$, such that $w^{\Lambda}(f,\delta_n) \to 0$, then $f \in UC_{\Lambda}$.*

*Proof.* This property is true because of the property of the usual modulus of continuity that $f$ is uniformly continuous if and only if $w(f,\delta) \to 0$ as $\delta \to 0$. □

**Proposition 2.2.** *For every function $f \in UC_{\Lambda}$, $\forall\, t,\ x \in I$ and $\delta > 0$, we have*

$$|f(t) - f(x)| \leq \left(1 + \frac{|\phi(t) - \phi(x)|}{\delta}\right) w^{\Lambda}(f,\delta).$$

*Proof.* To prove this, we use the following inequality

$$w^{\Lambda}(f,\lambda\delta) = w(f \circ \Lambda^{-1}, \lambda\delta) \leq (1+\lambda) w^{\Lambda}(f,\delta)$$

Let $t, x \in I$, then by above inequality

$$|f(t) - f(x)| \leq w^{\Lambda}(f, |\Lambda(t) - \Lambda(x)|) \leq \left(1 + \frac{|\Lambda(t) - \Lambda(x)|}{\delta}\right) w^{\Lambda}(f,\delta).$$

□

**Theorem 2.3.** *[5] Let $\alpha_n : C(I) \to C(I)$ be a sequence of positive linear operators preserving constant functions. Then*

*(i) If $\sup_{a\in I} \alpha_n(|\Lambda(t) - \Lambda(a)|, a) = k_n \to 0$ and $f \in UC_{\Lambda}$, then following conditions hold*

$$\|\alpha_n f - f\|_{\infty} \to 0 \quad \text{and} \quad \|\alpha_n f - f\|_{\infty} \leq 2 \cdot w^{\Lambda}(f, k_n).$$

*(ii) If $\|\alpha_n f - f\|_{\infty} \to 0$ and $\alpha_n f \in UC_{\Lambda}$, then $f \in UC_{\Lambda}$.*

*Proof.* For $f \in UC_{\Lambda}$, $t, x \in I$ and $\delta > 0$, we have

$$|f(t) - f(x)| \leq \left(1 + \frac{|\Lambda(t) - \Lambda(x)|}{\delta}\right) w^{\Lambda}(f,\delta)$$

Applying the positive operators $\alpha_n$ to the above inequality, we obtain

$$|\alpha_n(f,x) - f(x)| \leq \left(1 + \frac{\alpha_n|\Lambda(t) - \Lambda(x)|}{\delta}\right) w^{\Lambda}(f,\delta_n).$$

By choosing $\delta_n = k_n$, we obtain to the following inequality

$$\|\alpha_n f - f\|_{\infty} \leq 2 \cdot w^{\Lambda}(f,\delta_n).$$

Now using the fact from proposition 2.1, we can prove $(i)$. To prove $(ii)$, we use the semi-norm property of $w^{\Lambda}$ and one can obtain the following result

$$w^{\Lambda}(f,\delta_n) \leq w^{\Lambda}(f - \alpha_n f, \delta_n) + w^{\Lambda}(\alpha_n f, \delta_n) \leq 2\|f - \alpha_n f\|_{\infty} + w^{\Lambda}(\alpha_n f, \delta_n).$$

Using the fact from proposition 2.1 , we obtain $w^{\Lambda}(\alpha_n f, \delta_n) \to 0$, for a sequence $\delta_n \to 0$, which leads us to $f \in UC_{\Lambda}$. □

## 3 Method of constructing function $\Lambda$

For the sequence of linear positive linear operators $\alpha_n$ that maps the continuous functions into differentiable functions, the following method helps us to find a function $\Lambda$, so that $\alpha_n f \circ \Lambda^{-1}$ is uniformly continuous.
By mean value theorem, we have for some $c$ between $x$ and $t$, the following equation holds

$$\Lambda'(c)[\alpha_n f(x) - \alpha_n f(t)] = (\alpha_n f)'(c)[\Lambda(x) - \Lambda(t)]$$

So,

$$w^{\Lambda}(\alpha_n f, \delta_n) = \sup_{x,t\in I} \{|\alpha_n f(x) - \alpha_n f(t)\} : \{\Lambda(x) - \Lambda(t)| \leq \delta_n\} \leq \delta_n \cdot \sup_{c\in I} \left| \frac{(\alpha_n f)'(c)}{\Lambda'(c)} \right| .$$

From the above equation, in order to have $w^{\Lambda}(\alpha_n f, \delta_n) \to 0$ as $\delta_n \to 0$, then we must have $\left\| \frac{(\alpha_n f)^{'}}{\Lambda^{'}} \right\|_{\infty} < \infty$. So, $\Lambda$ should be chosen such that $\left\| \frac{(\alpha_n f)'}{\Lambda'} \right\|_{\infty} < \infty$.

## 4 Main Result

**Theorem 4.1.** *For the Lupaş operators $L_n : C[0,\infty) \to C[0,\infty)$ defined by*

$$L_n(f,x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right),$$

*we have $L_n(f)$ converges to $f$ if $f(x^2)$ is uniformly continuous on $[0,\infty)$. Also, $f(x^2)$ is uniformly continuous provided $L_n(f)$ converges to $f$ and $f$ is bounded and continuous on non-compact interval $[0,\infty)$. Additionally, we have the following estimation*

$$\|L_n f - f\|_{\infty} \leq 2 \cdot w\left(f(t^2), \sqrt{\frac{2}{n}}\right), \quad n \geq 1.$$

*Proof.* For the Lupaş operators, we know from [7] that, $L_n(1,x) = 1, L_n(t,x) = x, L_n(t^2,x) = x^2 + \frac{2x}{n}$. Using Cauchy-Schwarz inequality for positive linear operators, we have the following inequality

$$\begin{aligned} L_n(|t-x|,x) &\leq \sqrt{L_n((t-x)^2,x)} \\ &= \sqrt{L_n((t^2+x^2-2tx),x)} \\ &= \sqrt{\frac{2x}{n}}. \end{aligned}$$

By using properties of digamma function for bounded function $f \in C[0,\infty)$ and computing the derivatives, we get the following outcomes

$$\begin{aligned}
(L_n f)'(x) &= \frac{d}{dx}\left[2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{k!2^k} f\left(\frac{k}{n}\right)\right] \\
&= -nlog2\sum_{k=0}^{\infty}\frac{(nx)_k}{k!2^k}2^{-nx} f\left(\frac{k}{n}\right) + \frac{1}{x}\sum_{k=0}^{\infty}\frac{k(nx)_k}{k!2^k}2^{-nx} f\left(\frac{k}{n}\right) \\
&\leq n\sum_{k=0}^{\infty}\frac{(nx)_k}{k!2^k}2^{-nx} f\left(\frac{k}{n}\right) + \frac{1}{x}\sum_{k=0}^{\infty}\frac{k(nx)_k}{k!2^k}2^{-nx} f\left(\frac{k}{n}\right) \\
&= \left|\frac{n}{x}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)\frac{2^{-nx}(nx)_k}{k!2^k}\right| \\
&\leq \frac{n}{x}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\left|\frac{k}{n}-x\right|\frac{2^{-nx}(nx)_k}{k!2^k} \\
&\leq \frac{n}{x}\|f\|_\infty L_n(\mid t-x\mid, x) \\
&\leq \frac{n}{x}\|f\|_\infty\sqrt{\frac{2x}{n}} \\
&= \sqrt{\frac{2n}{x}}\ \|f\|_\infty
\end{aligned}$$

We want to choose the function $\Lambda$ such that $\Lambda'(x) = \dfrac{1}{\sqrt{x}}$. So, we take $\Lambda(x) = 2\sqrt{x}$ and hence $\Lambda^{-1}(x) = \dfrac{x^2}{4}$. By using section 3, we obtain

$$\begin{aligned}
w^{\Lambda}(L_n f, \delta_n) &\leq \delta_n \cdot \sup\left|\frac{(L_n f)'(x)}{\Lambda'(x)}\right| \\
&\leq \frac{1}{n}\cdot\sqrt{2n}\cdot\|f\|_\infty \\
&= \frac{\sqrt{2}\|f\|_\infty}{\sqrt{n}}
\end{aligned}$$

which proves that $L_n f \circ \Lambda^{-1}$ is uniformly continuous. Using theorem 2.3, we deduce that $f \circ \Lambda^{-1}(x) = f\left(\dfrac{x^2}{4}\right)$ and hence $f(x^2)$ is uniformly continuous if $L_n f$ converges uniformly to $f$ on $[0, \infty)$. For the other part of the theorem, we use

$$\begin{aligned}
|\Lambda(x) - \Lambda(t)| &= \left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{t}}\right| \\
&\leq \frac{|x-t|}{\sqrt{x}}, \qquad \text{for } x \geq 0, t \geq 0,
\end{aligned}$$

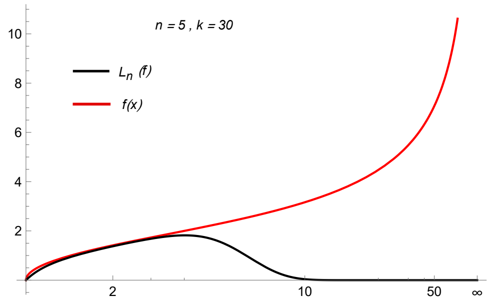
to obtain $k_n$. We get,

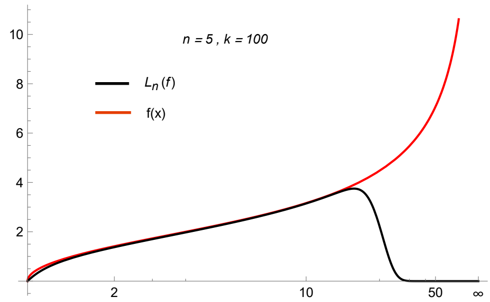
$$k_n = \sup_{x>0} L_n(|\Lambda(x) - \Lambda(t)|, x) \leq \sup_{x>0}\frac{L_n(|t-x|, x)}{\sqrt{x}} \leq \sqrt{\frac{2}{n}}.$$

By Theorem 2.3, we obtain that $L_n f \to f$ on $[0, \infty)$, if $f(x^2)$ is uniformly continuous on $[0, \infty)$. □

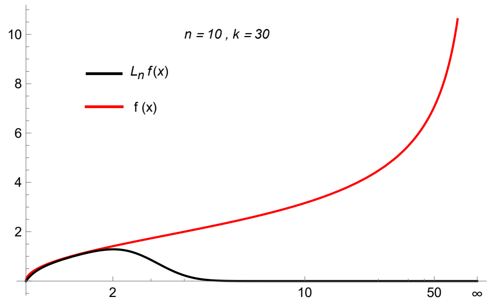
## 5 Graphical Representation

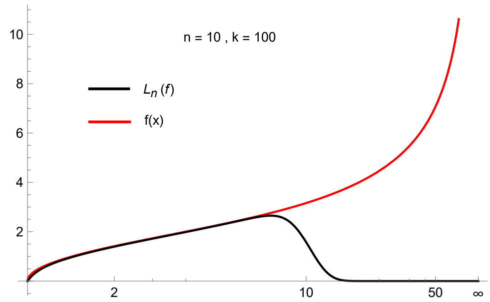
In [8] and [9], authors have explained the study through a graphical presentation. Inspired by that, we have represented our work in a graphical method to observe the convergence of operators and the function with the listed characterization. We take $f(x) = \sqrt{x}$, for $x \in [0, \infty)$. It is clear that $f(x^2)$ is uniformly continuous.




Above two graphs are shown for a fixed value of $n$, i.e., 5. For $n = 5$, different values of $k$ are to be considered. It can be seen that as $k$ approaches infinity, graph of the the operator for that fixed value of n, approaches the graph of the function.
Following two graphs are for $n = 10$. Different values of $k$ are taken for the keen observation. It can be seen that as $k \to \infty$, both curves of the function and operator overlap. Also, it is to be observed that the error of the graph of function and operator is lesser for $n = 10$ than $n = 5$. So, it can be observed that as $n$ and $k$ approach $\infty$, curves of the function and of the operator at that function overlap.

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