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Class of Holomorphic Functions Considering Seven-Parameter Mittag-Leffler Function

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Abstract This paper dedicated to introduce new class of operator concurring normalized seven-parameter Mittag-Leffler function of single complex variable and first-order subordination relation. Our new operator defined using the convolution technique for functions belong to the class $\mathcal{A}(p)$. In addition, we apply first-order order differential subordination properties for this class to achieve certain new features for that class.

1 Introduction

Contributions and Methodologies

As one of the vital methods in the geometric function theory, convolution (Hadamard product) used widely to define new operators, as well as performing many differential and integral operators in view of its definition and properties. The convolution formalism play a key role for understanding and discussing numerous geometric properties of operators, see [1, 10, 11, 15, 17, 23].

Let $\mathcal{A}(p)$ be the class of holomorphic functions on the unit disk $\mathbb{D}$, such that [13]

$$f(z) = z^p + \sum_{n=2}^{\infty} \alpha_{n+p-1} z^{n+p-1},\ p \in \mathbb{N}. \tag{1.1}$$

Further, the convolution of two function in the class $\mathcal{A}(p)$ is given as

$$(f * g)(z) = z^p + \sum_{n=2}^{\infty} \alpha_{n+p-1}\, \zeta_{n+p-1} z^{n+p-1} = (g * f)(z)$$

where, $(z) = z^p + \sum_{n=2}^{\infty} \zeta_{n+p-1} z^{n+p-1}$, $p \in \mathbb{N}$, [22].

Further, let $\mathcal{I}$ be the class of holomorphic and convex univalent functions $w(z)$ in $\mathbb{D}$ with $w(0) = 1$. Moreover, $\mathcal{S}^*_\epsilon$ denotes the class of starlike functions of order $\epsilon, \epsilon \leq 1$ such that $f \in \mathcal{S}^*_\epsilon$ if and only if $Re\left(\frac{z\acute{f}(z)}{f(z)}\right) \geq \epsilon$. Also, we mention the relation $\mathcal{S}^*_\epsilon \subset \mathcal{S}^*$ if and only if $0 \leq \epsilon \leq 1$. Further, $\mathcal{R}_\epsilon$ denotes the class of prestarlike functions of order ϵ, a function $f \in \mathcal{A}$ is belonged to $\mathcal{R}_\epsilon$ if and only if $f(z) * \frac{z}{(1-z)^{2(1-\epsilon)}} \in \mathcal{S}^*_\epsilon$, for $\epsilon < 1$. Note that $\mathcal{S}^*_{\frac{1}{2}} = \mathcal{R}_{\frac{1}{2}}$, [21].

Suppose that F and G be holomorphic functions in $\mathbb{D}$, we say that the function F subordinate to G , written as $F \prec G$ if there exist a schwarz function w, such that that $F(z) = G(w(z))$. If G is univalent, then $F \prec G$ if and only if $F(0) = G(0)$ and $F(\mathbb{U}) \subset G(\mathbb{U})$, [18]. In addition, let $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and h be univalent in $\mathbb{D}$. If ρ is holomorphic in $\mathbb{D}$ and satisfies the differential subordination

$$\psi\left(\rho(z), z\acute{\rho}(z); z\right) \prec h(z) \tag{1.2}$$

then ρ is called a solution of (1.2). The univalent function h is called dominant of the solutions of (1.2), if $\rho \prec h$ for all ρ satisfying (1.2). A dominant $\tilde{h}$ that satisfies $\tilde{h} \prec h$ for all dominant h of (1.2) is said to be the best dominant [**?**].

In order to achieve our major outcomes of this paper, we require the following fundamental lemmas: Lemma

Lemma 1.1. *[?] Let F be holomorphic function in $\mathbb{D}$, h be convex $\mathbb{D}$, and $P(z) : \mathbb{D} \to \mathbb{C}$ with $Re[P(z)] > 0$. If*

$$F(z) + P(z).\acute{F}(z) \prec h(z)$$

then $F(z) \prec h(z)$.

Lemma 1.2. *[21] Let $\epsilon < 1, f_1 \in \mathcal{R}_\epsilon$ and $f_2 \in \mathcal{S}^*_\epsilon$. Then for any holomorphic function T in $\mathbb{D}$*

$$\frac{f_1 * (f_2 T)}{f_1 * f_2}(\mathbb{D}) \subset \bar{co}(T(\mathbb{D})),$$

where $\bar{co}(T(\mathbb{D}))$ is the closed convex hull of $T(\mathbb{D})$.

Lemma 1.3. *[12] Let $B(z) = 1 + \sigma_1 z + \sigma_2 z^2 + ...$ be holomorphic function with positive real part in $\mathbb{D}$ and $\kappa \in \mathbb{C}$, then $|\sigma_2 - \kappa\sigma_1^2| \leq 2\max\{1, |2\kappa - 1|\}$.*

Lemma 1.4. *[6] Let $P(z)$ be convex univalent in $\mathbb{D}$, and $T(z)$ be holomorphic in $\mathbb{D}$ with $P(0) = T(0) = 1$, then for all $\eta \neq 0$ with $Re(\eta) \geq 0$ we have $T(z) \prec P(z)$ where*

$$T(z) = \eta z^{-\eta}\int_0^z r^{\eta-1}T(r)dr.$$

Inception and Motivation

As one of the well-known entire functions, Mittag-Leffler function, was introduced in 1903 the mathematician Gosta Mittag-Leffler as

$$E_\tau(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\tau n + 1)} \tag{1.3}$$

where, z $\in \mathbb{C}$ and $Re(\tau) > 0$ [14]. Then, many mathematicians interested to generalize that functions for more parameters and variables with different significant technique, see [5, 8, 9, 24, 25, 16].

Here, it is notable to review the seven-parameter Mittag-Leffler function of single complex variable proposed by Rasheed and Majeed [20],

$$E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,\Gamma(\tau_1 n + \lambda_1)\Gamma(\tau_2 n + \lambda_2)n!}\,z^n, \tag{1.4}$$

where $\min\{Re(a), Re(b), Re(c), Re(\tau_1), Re(\tau_2), Re(\lambda_1), Re(\lambda_2)\} > 0$. It was raised as entire function of estimated order, type and another numerous holomorphic properties.

Accordingly, indicate the normalized seven-parameter Mittag-Leffler function that recently presented by Rasheed and Majeed [19],

$$\begin{aligned}\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) &= \Gamma(\lambda_1)\Gamma(\lambda_2)zE^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z),\\ &= z + \sum_{n=2}^{\infty}\frac{(a)_{n-1}\,(b)_{n-1}\Gamma(\lambda_1)\Gamma(\lambda_2)}{(c)_{n-1}\,\Gamma(n)\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)}z^n, \end{aligned} \tag{1.5}$$

where $z \in \mathbb{C}$ and $min\{Re(a), Re(b), Re(c), Re(\tau_1), Re(\tau_2), Re(\lambda_1), Re(\lambda_2)\} > 0$.

Our motivation for this paper is to propose the function

$$\begin{aligned} {}_p\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) &= z^{p-1}\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) \\ &= z^p + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}\Gamma(\lambda_1)\Gamma(\lambda_2)}{(c)_{n-1}\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)(n-1)!} z^{p+n-1}, \end{aligned} \tag{1.6}$$

where $p \in \mathbb{N}$. Further, let $f \in \mathcal{A}(p)$, by virtue of the convolution method we suggest the new operator $\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) : \mathcal{A}(p) \to \mathcal{A}(p)$ such that

$$\begin{aligned} (\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z) &= {}_p\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) * f(z) \\ &= z^p + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}\Gamma(\lambda_1)\Gamma(\lambda_2)}{(c)_{n-1}\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)(n-1)!} \alpha_{n+p-1}\, z^{p+n-1}. \end{aligned} \tag{1.7}$$

Note that, if we set $p = 1$ we observe that the generalized Bernardi operator [2], Carlson-Shiffer operator [3], Hohlove linear operator [7] and Choi-Saigo-Srivastava operator [4], are respective special cases in their convolution forms as

$$\begin{aligned} (\mathcal{J}^{1,1+b,2+b}_{0,\lambda_1,0,\lambda_2} f)(z) &= z\,{}_2F_1(1, 1+b, 2+b; z) * f(z) \\ (\mathcal{J}^{a,1,c}_{0,\lambda_1,0,\lambda_2} f)(z) &= z\,{}_2F_1(a, 1, c; z) * f(z) \\ (\mathcal{J}^{a,b,c}_{0,\lambda_1,0,\lambda_2} f)(z) &= z\,{}_2F_1(a, b, c; z) * f(z) \\ (\mathcal{J}^{a,1,c+1}_{0,\lambda_1,0,\lambda_2} f)(z) &= z\,{}_2F_1(a, 1, c+1; z) * f(z). \end{aligned}$$

Definition

Definition 1.5. Let $f \in \mathcal{A}(p)$, we say that f belongs to the class $\mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$ if the following differential subordination holds

$$(1-\gamma)z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z) + \frac{\gamma}{p} z^{-p+1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)'(z) \prec w(z) \tag{1.8}$$

where $\gamma \in \mathbb{C}$, w is holomorphic and convex univalent in $\mathbb{D}$ with $w(0) = 1$.

Throughout this paper, we discuss some essential properties related to the class $\mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$ in view of some differential subordination properties, convolution and Herglotz representation with applications owing to geometric function theory.

2 Certain Properties of $\mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$

Theorem 2.1. *Let* $0 \le \gamma_1 < \gamma_2$*, then* $\mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma_2; w) \subset \mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma_1; w)$.

Proof. Suppose that $f \in \mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$, and

$$L(z) = z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z), \tag{2.1}$$

one can obviously see that $L(z)$ is holomorphic in $\mathbb{D}$ with $L(0) = 1$. Now

$$\begin{aligned} &(1-\gamma_2)z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z) + \frac{\gamma_2}{p} z^{-p+1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)'(z) \\ &\quad = (1-\gamma_2)z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z) + \frac{\gamma_2}{p} z \left[L'(z) + p z^{-p-1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z) \right] \\ &\quad = L(z) + \frac{\gamma_2}{p} z L'(z). \end{aligned} \tag{2.2}$$

By Lemma 1, we imply that $L(z) \prec w(z)$. Here, under the given conditions on γ_1 and γ_2 we find that $\frac{\gamma_1}{\gamma_2} \leq 1$. Also, since $w(z)$ is convex univalent in $\mathbb{D}$, hence

$$\begin{aligned}
&(1-\gamma_1)z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z) + \frac{\gamma_1}{p}z^{-p+1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)'(z)\\
&= L(z) - \gamma_1 L(z) + \frac{\gamma_1}{p}z^{-p+1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)'(z)\\
&= \left(1-\frac{\gamma_1}{\gamma_2}\right)L(z) + \frac{\gamma_1}{\gamma_2}\left[(1-\gamma_2)z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z) + \frac{\gamma_2}{p}z^{-p+1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)'(z)\right]\\
&\prec w(z).
\end{aligned}$$

which gives $f \in \mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma_1; w)$. □

Theorem 2.2. *Let* $0 < a_1 < a_2$, *then* $\mathcal{Q}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w) \subset \mathcal{Q}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$.

Proof. Consider the function

$$L(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}}{(a_2)_{n-1}}\, z^n,\ \ z \in \mathbb{D}.$$

Immediately we note that $L(z) \in \mathcal{A}(1)$. In addition let $J_{a_2}(z) = \frac{z}{(1-z)^{a_2}}$ which belong to $S^*_{1-\frac{a_2}{2}}$, hence

$$\begin{aligned}
L(z) * J_{a_2}(z) &= z + \sum_{n=2}^{\infty}(a_1)_{n-1}z^n,\\
&= J_{a_1}(z) \in S^*_{1-\frac{a_2}{2}},
\end{aligned}$$

which gives, $S^*_{1-\frac{a_1}{2}} \subset S^*_{1-\frac{a_2}{2}}$, and since $S^*_{1-\frac{a_1}{2}} = \mathcal{R}_{1-\frac{a_1}{2}}$, then for $0 < a_1 < a_2$, we have

$$L(z) \in \mathcal{R}_{1-\frac{a_1}{2}}. \tag{2.3}$$

Now, let $f \in \mathcal{Q}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$, that is

$$(1-\gamma)z^{-p}(\mathcal{J}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z) + \frac{\gamma}{p}z^{-p+1}(\mathcal{J}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)'(z) \prec w(z).$$

Alternatively,

$$\begin{aligned}
(1-\gamma)z^{-p}(\mathcal{J}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z) + \frac{\gamma}{p}z^{-p+1}(\mathcal{J}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)'(z) &= \frac{L(z)}{z} * G(z)\\
&= \frac{L(z) * [zG(z)]}{L(z) * z},
\end{aligned} \tag{2.4}$$

where,

$$\begin{aligned}
G(z) &= (1-\gamma)z^{-p}(\mathcal{J}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z) + \frac{\gamma}{p}z^{-p+1}(\mathcal{J}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)'(z)\\
&= 1 + \frac{1}{p}\sum_{n=2}^{\infty} \frac{[\gamma(n-1)+p]\,(a)_{n-1}(b)_{n-1}\Gamma(\lambda_1)\Gamma(\lambda_2)}{p\,(c)_{n-1}\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)(n-1)!}\ \alpha_{n+p-1}\, z^{n-1},
\end{aligned} \tag{2.5}$$

which is clearly holomorphic on $\mathbb{D}$. Now, by (2.3) to (2.5) we can apply Lemma 2, to get

$$\frac{L(z)}{z} * G(z) \subset \bar{co}[G(\mathbb{D})]$$

$$(1-\gamma)z^{-p}(\mathcal{J}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z) + \frac{\gamma}{p}z^{-p+1}(\mathcal{J}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)'(z) \subset \bar{co}[G(\mathbb{D})]. \tag{2.6}$$

That means, $f \in \mathcal{Q}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$. □

Theorem 2.3. *Let* $L(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}}{(a_2)_{n-1}} z^n$. *If* $Re\left(\frac{L(z)}{z}\right) > \frac{1}{2}$, *then* $\mathcal{Q}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w) \subset \mathcal{Q}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$.

Proof. Note that,

$$z^{-p}(\mathcal{J}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z) = \frac{L(z)}{z} * z^{-p}(\mathcal{J}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z),$$

and

$$z^{-p+1}(\mathcal{J}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)'(z) = \frac{L(z)}{z} * z^{-p+1}(\mathcal{J}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)'(z).$$

Let $f \in \mathcal{Q}^{a_2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$, hence

$$(1-\gamma)z^{-p}(\mathcal{J}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z) + \frac{\gamma}{p} z^{-p+1}(\mathcal{J}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)'(z) = \frac{L(z)}{z} * G(z) \tag{2.7}$$

where,$G(z)$ is given by (2.5).
Further, since $\frac{L(z)}{z}$ is holomorphic with $L(0) = 1$, and from the assumption $Re\left(\frac{L(z)}{z}\right) > \frac{1}{2}$, we observe that $\frac{L(z)}{z}$ can be expressed in terms of Herglotz representation as

$$\frac{L(z)}{z} = \int_{|T|=1} \frac{d\delta(T)}{1 - Tz},\ z \in \mathbb{D} \tag{2.8}$$

such that $\delta(T)$ represents the probability measure on the circle $|T| = 1$ with $\int_{|T|=1} d\delta(T) = 1$.
Here, since $w(z)$ is convex univalent in $\mathbb{D}$, and from (2.5), (2.7) and (2.8), yields

$$\frac{L(z)}{z} * G(z) = \int_{|T|=1} G(Tz)d\delta(T) \prec w(z)$$

Therefore, by returning to (2.7), we conclude $f \in \mathcal{Q}^{a_1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$. □

Theorem 2.4. *Let* $f \in \mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$ *and* $g(z) \in \mathcal{A}(p)$. *If* $Re[z^{-p}g(z)] > \frac{1}{2}$, *then* $f * g \in \mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$.

Proof. Consider

$$\begin{aligned}
&(1-\gamma)z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(f*g))(z) + \frac{\gamma}{p} z^{-p+1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(f*g))'(z) \\
&= (1-\gamma)\left[z^{-p}g(z)\right] * \left[z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)(z)\right] + \frac{\gamma}{p}\left[z^{-p}g(z)\right] * \left[z^{-p+1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f)'(z)\right] \\
&= \left[z^{-p}g(z)\right] * G(z),
\end{aligned} \tag{2.9}$$

where $G(z)$ is defined in (2.5). In addition, we have $z^{-p}g(z)$ is analytic with $Re(z^{-p}g(z)) > \frac{1}{2}$ that enable us to use Herglotz representation as

$$z^{-p}g(z) = \int_{|T|=1} \frac{d\delta(T)}{1 - Tz},\ z \in \mathbb{D} \tag{2.10}$$

such that $\delta(T)$ represents the probability measure on the circle $|T| = 1$ with $\int_{|T|=1} d\delta(T) = 1$.
Here, since $w(z)$ is convex univalent in $\mathbb{D}$, and from (2.5), (2.9) and (2.10), yields

$$z^{-p}g(z) * G(z) = \int_{|T|=1} G(Tz)d\delta(T) \prec w(z)$$

Therefore by returning to (2.9), we obtain $(f * g) \in \mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$. □

Applications for the Class $\mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$

Proposition 1. Let $a, b, c, \tau_1, \tau_2, \lambda_1, \lambda_2, \gamma \in \mathbb{R}^+$ and $\pi(z) = 1 + r_1 z + r_2 z^2 + ...$ with $r_1 \neq 0$. If $f \in \mathcal{A}(p)$ given by (1) belongs to the class $\mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$, then for $\mu \in \mathbb{C}$ we have

$$|\alpha_{p+2} - \mu\alpha_{p+1}^2| \leq \frac{2p\, c(c+1)\,\Gamma(2\tau_1+\lambda_1)\Gamma(2\tau_2+\lambda_2)}{a(a+1)b(b+1)(p+2\gamma)\Gamma(\lambda_1)\Gamma(\lambda_2)} \max\left\{|r_1|, r_2 - \frac{\mu p\, r_1^2 c\,(a+1)(b+1)(p+2\gamma)(\Gamma(\tau_1+\lambda_1))^2(\Gamma(\tau_2+\lambda_2))^2}{2ab(c+1)(p+\gamma)\Gamma(\lambda_1)\Gamma(\lambda_2)(\Gamma(2\tau_1+\lambda_1))^2(\Gamma(2\tau_2+\lambda_2))^2}\right\}.$$

Proof. Let $f \in \mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; w)$, then for $\mu \in \mathbb{C}$, then f satisfies (1.8). Hence, there exist Schwartz function $\phi(z)$, that is holomorphic in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, such that

$$(1-\gamma)z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z) + \frac{\gamma}{p}z^{-p+1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)'(z) = \phi(w(z)). \tag{2.11}$$

Define

$$L(z) = \frac{1+\phi(z)}{1-\phi(z)} = 1 + \beta_1 z + \beta_2 z^2 + ...,$$

which is obviously holomorphic with $L(0) = 1$ and $Re(L(z)) > 0$. So,

$$w(\phi(z)) = w\left(\frac{L(z)-1}{L(z)+1}\right) = 1 + \frac{r_1\beta_1}{2}z + \frac{1}{2}\left(r_1\beta_2 - \frac{r_1\beta_1^2}{2} + \frac{r_2\beta_1^2}{2}\right)z^2 + ... \tag{2.12}$$

put (2.12) in (2.11), implies

$$(1-\gamma)z^{-p}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z) + \frac{\gamma}{p}z^{-p+1}(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)'(z) = 1 + \frac{r_1\beta_1}{2}z + \frac{1}{2}\left(r_1\beta_2 - \frac{r_1\beta_1^2}{2} + \frac{r_2\beta_1^2}{2}\right)z^2 + ...$$

That is,

$$1 + \frac{(\gamma+p)\,ab\Gamma(\lambda_1)\Gamma(\lambda_2)}{pc\,\Gamma(\tau_1+\lambda_1)\Gamma(\tau_2+\lambda_2)}\alpha_{p+1}z + \frac{(2\gamma+p)\,a(a+1)b(b+1)\Gamma(\lambda_1)\Gamma(\lambda_2)}{pc(c+1)\,\Gamma(2\tau_1+\lambda_1)\Gamma(2\tau_2+\lambda_2)}\alpha_{p+2}z^2 + ... = \frac{p\,r_1\beta_1 c\Gamma(\tau_1+\lambda_1)\Gamma(\tau_2+\lambda_2)}{2ab(p+\gamma)\Gamma(\lambda_1)\Gamma(\lambda_2)}.$$

Directly, we find that

$$\alpha_{p+1} = \frac{p\,r_1\beta_1 c\,\Gamma(\tau_1+\lambda_1)\Gamma(\tau_2+\lambda_2)}{2ab(p+\gamma)\Gamma(\lambda_1)\Gamma(\lambda_2)},$$

and

$$\alpha_{p+2} = \left(r_1\beta_2 - \frac{r_1\beta_1^2}{2} + \frac{r_2\beta_1^2}{2}\right)\frac{p\,c(c+1)\,\Gamma(2\tau_1+\lambda_1)\Gamma(2\tau_2+\lambda_2)}{a(a+1)b(b+1)(p+2\gamma)\Gamma(\lambda_1)\Gamma(\lambda_2)}.$$

Therefore,

$$\alpha_{p+2} - \mu\alpha_{p+1}^2 = \frac{p\,c(c+1)\,\Gamma(2\tau_1+\lambda_1)\Gamma(2\tau_2+\lambda_2)}{a(a+1)b(b+1)(p+2\gamma)\Gamma(\lambda_1)\Gamma(\lambda_2)}(\beta_2 - q\beta_1^2),$$

where

$$q = \frac{\mu p\, B_1 c\,(a+1)(b+1)(p+2\gamma)(\Gamma(\tau_1+\lambda_1))^2(\Gamma(\tau_2+\lambda_2))^2}{2ab(c+1)(p+\gamma)\Gamma(\lambda_1)\Gamma(\lambda_2)(\Gamma(2\tau_1+\lambda_1))^2(\Gamma(2\tau_2+\lambda_2))^2} - \frac{r_2}{r_1}.$$

Apply Lemma 3, yields the acquired result. □

Proposition 2. Let $-1 \leq A < B \leq 1, \gamma > 0$. If $f \in \mathcal{Q}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(\gamma; \frac{1+Az}{1+Bz})$, then

$$Re\left\{\frac{(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z)}{z^p}\right\} > \frac{p}{\gamma}\int_0^1 y^{\frac{p}{\gamma}-1}\left(\frac{1-Ay}{1-By}\right)dy,$$

the result is sharp.

Proof. Assume $L(z)$ that defined in (2.1), which is analytic in $\mathbb{D}$. In addition, from (2.2) we have

$$L(z) + \frac{\gamma}{p}z\acute{L}(z) \prec \frac{1+Az}{1+Bz}.$$

From Lemma 1, we conclude that $L(z) \prec \frac{1+Az}{1+Bz}$. in addition, an application of Lemma 4 we conclude

$$L(z) \prec \frac{p}{\gamma}z^{-\frac{p}{\gamma}}\int_0^z x^{\frac{p}{\gamma}-1}\left(\frac{1+Ax}{1+Bx}\right)dx.$$

Now, one can say that there is a Schwarz function $\psi(z)$ with $\psi(0) = 0$ and $|\psi(z) < 1$ in $\mathbb{D}$, such that

$$L(z) = \frac{p}{\gamma}z^{-\frac{p}{\gamma}}\int_0^1 y^{\frac{p}{\gamma}-1}\left(\frac{1+Ay\psi(z)}{1+By\psi(z)}\right)dy. \tag{2.13}$$

Here, under the conditions $-1 \leq A < B < 1$ and $\gamma > 0$, then (2.13) implies

$$Re\{L(z)\} > \frac{p}{\gamma}\int_0^1 y^{\frac{p}{\gamma}-1}\left(\frac{1-Ay}{1-By}\right)dy, \quad z \in \mathbb{D}.$$

□

For the sharpness, assume

$$\frac{(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z)}{z^p} = \frac{p}{\gamma}z^{-\frac{p}{\gamma}}\int_0^1 y^{\frac{p}{\gamma}-1}\left(\frac{1+Ayz}{1+Byz}\right)dy,$$

Therefore, when $z \to -1$, we find that

$$\frac{(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z)}{z^p} \to \frac{p}{\gamma}z^{-\frac{p}{\gamma}}\int_0^1 y^{\frac{p}{\gamma}-1}\left(\frac{1-Ay}{1-By}\right)dy.$$

3 Discussion and Conclusion

In this paper we assumed the convolution technique to define a new operator by terms of seven-parameter Mittag-Leffler function (1.4). The central idea was to suggest new class associated with the operator (1.7) in view of differential subordination concept, then look into the basic properties in certain directions with an application. On the other hand, this work performs an important motivation to discuss a similar properties by virtue of the other parameters of the function (1.7), especially the parameters used for gamma function. Moreover, we achieved a necessary recurrence relation of that operator as

$$z(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)\acute{}(z) = a(\mathcal{J}^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z) - (a-p)(\mathcal{J}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f)(z),$$

which can be useful tool to study second-order differential subordination as a future work.

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