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ON OCCASIONALLY WEAKLY BIASED MAPS OF TYPE $(\mathcal{A})$ IN d-METRIC SPACES

H. Bouhadjera

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Corresponding Author: H. Bouhadjera

Abstract The purposes of this paper are threefold; The first is to present a unique common fixed point theorem for two pairs of occasionally weakly biased maps of type $(\mathcal{A})$ in a dislocated metric space, the second is to improve this result by extending its constants, and the third and last purpose is to furnish two convinced examples and an application in order to highlight the credibility of our results.

1 Introduction

Fixed point theory is a very rich domain in mathematics. Many authors investigated the existence, uniqueness and approximation of fixed and common fixed points for single and both single and set-valued maps, under several conditions, and in different spaces (see for instance [1], [5], [7], [10], [11], [12], [13], [14], [16], [22], [23], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [38], [39], [40], [41], [42], [43]) and others.

Now, a generalization of the commuting maps concept is introduced in 1982 by Sessa [37] under the name of weakly commuting maps. Compatible maps were introduced in 1986 by Jungck [17] as a generalization of commuting and weakly commuting maps and have been useful as a tool for obtaining fixed point theorems. After nine years, Jungck and Pathak [20] presented a generalization of the concept of compatible maps called biased maps by softening the restrictions imposed by compatibility. Again, the same authors [20], suggested the concept of weakly biased maps which represents a convenient generalization of biased maps. In 2012, in [9], we gave the concept of occasionally weakly biased maps which is a legitimate generalization of weakly biased maps given by Jungck and Pathak in [20]. Let us return back to 1993, Jungck et al. [19] introduced the concept of compatible maps of type $(\mathcal{A})$ which is equivalent to compatible maps under the continuity condition. After two years, Pathak et al. [26] generalized the last notion by giving the concept of biased maps of type $(\mathcal{A})$ in order to prove fixed point theorems for certain contractions of four maps. According to them, the concept of biased maps of type $(\mathcal{A})$ appears to be a natural and effective generalization of compatible maps of type $(\mathcal{A})$. Again and in the same paper [26], the authors provided the definition of weakly g-biased of type $(\mathcal{A})$. In 1996, the notion of compatible maps was again generalized in [18] by Jungck himself by giving the weakly compatible maps concept. In 2008, Al-Thagafi and Shahzad [4] furnished the notion of occasionally weakly compatible maps (owc) as a generalization of weakly compatible maps. While the paper [4] was under review, Jungck and Rhoades [21] used the concept of owc and proved several results under different contractive conditions (see [3]). Since then, a lot of important common fixed point theorems of commuting, weakly commuting, compatible, biased, biased of type $(\mathcal{A})$, weakly compatible, weakly biased, occasionally weakly compatible, weakly biased of type $(\mathcal{A})$ and occasionally weakly biased maps under various contractive and expansive conditions have been obtained by several authors. Recently, in 2022, in [8] we delivered the concept of weakly f-biased of type $(\mathcal{A})$, and the concepts of occasionally weakly f-biased of type $(\mathcal{A})$ and occasionally weakly g-biased of type $(\mathcal{A})$, and we showed that the two last new definitions coincide with our concepts; occasionally weakly f-biased and occasionally weakly g-biased respectively given in [9]. We also asserted that our notion of occasionally weakly biased maps of type $(\mathcal{A})$ has an edge over weak and occasionally weak compatibility; i.e., weakly (respectively occasionally weakly) compatible maps are both occasionally weakly f-biased and g-biased of type $(\mathcal{A})$, however the converses are false in general.

On the other hand, in 1985, in his thesis [24], Matthews suggested the class of metric domains. According to him, metric domain has been introduced in order to promote the notion of completeness in domain theory and, he pointed out that there is a one to one correspondence between the class of metric domains and the class of metric spaces. In 1992, in his paper [25], the same author provided another generalisation of metric spaces under the name of partial metric spaces in which he keeps the symmetry axiom. In 2012, in his paper [2], Amini-Harandi introduced a new generalization of a partial metric space which is called a metric-like space. Then, he gave some fixed point theorems in such spaces which generalize and improve some well-known results in both metric-like and partial metric spaces. In fact, the notions of metric domains, metric-like spaces and dislocated metric spaces are exactly the same, and they also named d-metric spaces.

In this paper, we will prove unique common fixed point theorems for four occasionally weakly biased maps of type $(\mathcal{A})$ on a d-metric space. Our results improve the one's of Bennani et al. [6], and Jha and Panthi [15].

2 Preliminary notes

In this section, we only give the following definitions:

Definition 2.1. ([24]) A **metric domain** is a pair $<\mathcal{M}, m>$ where $\mathcal{M}$ is a non-empty set, and m is a function from $\mathcal{M}\times\mathcal{M}$ to $\mathbb{R}_+$ such that

(i) $\forall\, \alpha, \beta \in \mathcal{M}\ m(\alpha,\beta)=0 \Rightarrow \alpha=\beta$

(ii) $\forall\, \alpha, \beta \in \mathcal{M}\ m(\alpha,\beta)=m(\beta,\alpha)$

(iii) $\forall\, \alpha, \beta, \gamma \in \mathcal{M}\ m(\alpha,\beta)\le m(\alpha,\gamma)+m(\gamma,\beta)$.

Definition 2.2. ([25]) A **partial metric (pmetric)** is a function $\mathcal{P}:\mathcal{X}\times\mathcal{X}\to\mathbb{R}$, such that

$(\mathcal{P}_1)$ $\forall\, \delta,\theta\in\mathcal{X}$, $\delta=\theta \Leftrightarrow \mathcal{P}(\delta,\delta)=\mathcal{P}(\delta,\theta)=\mathcal{P}(\theta,\theta)$

$(\mathcal{P}_2)$ $\forall\, \delta,\theta\in\mathcal{X}$, $\mathcal{P}(\delta,\delta)\le\mathcal{P}(\delta,\theta)$

$(\mathcal{P}_3)$ $\forall\, \delta,\theta\in\mathcal{X}$, $\mathcal{P}(\delta,\theta)=\mathcal{P}(\theta,\delta)$

$(\mathcal{P}_4)$ $\forall\, \delta,\theta,\lambda\in\mathcal{X}$, $\mathcal{P}(\delta,\lambda)\le\mathcal{P}(\delta,\theta)+\mathcal{P}(\theta,\lambda)-\mathcal{P}(\theta,\theta)$.

Definition 2.3. ([2]) A map $\varphi:\mathcal{Y}\times\mathcal{Y}\to\mathbb{R}_+$, where $\mathcal{Y}$ is a nonempty set, is said to be metric-like on $\mathcal{Y}$ if for any η, χ, $\mu\in\mathcal{Y}$, the following three conditions hold true:

(φ_1) $\varphi(\eta,\chi)=0\Rightarrow\eta=\chi$

(φ_2) $\varphi(\eta,\chi)=\varphi(\chi,\eta)$

(φ_3) $\varphi(\eta,\chi)\le\varphi(\eta,\mu)+\varphi(\mu,\chi)$.

The pair $(\mathcal{Y},\varphi)$ is then called a **metric-like space**. Then a metric-like on $\mathcal{Y}$ satisfies all of the conditions of a metric except that $\varphi(\eta,\eta)$ may be positive for $\eta\in\mathcal{Y}$.

Definition 2.4. ([8]) Let $\mathcal{S}$ and $\mathcal{T}$ be self-maps of a non-empty set $\mathcal{X}$. The pair $(\mathcal{S},\mathcal{T})$ is said to be **occasionally weakly $\mathcal{S}$-biased of type** $(\mathcal{A})$ and **occasionally weakly $\mathcal{T}$-biased of type** $(\mathcal{A})$, respectively, if and only if, there exists a point ι in $\mathcal{X}$ such that $\mathcal{S}\iota=\mathcal{T}\iota$ implies

$$d(\mathcal{S}\mathcal{S}\iota,\mathcal{T}\iota)\le d(\mathcal{T}\mathcal{S}\iota,\mathcal{S}\iota),$$
$$d(\mathcal{T}\mathcal{T}\iota,\mathcal{S}\iota)\le d(\mathcal{S}\mathcal{T}\iota,\mathcal{T}\iota),$$

respectively.

Before stating and proving our results, let us start by giving the main theorems of Jha and Panthi, and Bennani et al. with inevitable discussions. In 2012, Jha and Panthi [15] have established the following theorem:

Theorem 2.5. *([15]) Let $(\mathcal{X}, d)$ be a complete d-metric space. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be continuous maps satisfying,*

(i) $\mathcal{T}(\mathcal{X}) \subset \mathcal{A}(\mathcal{X})$, $\mathcal{S}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X})$.

(ii) The pairs $(\mathcal{S}, \mathcal{A})$ and $(\mathcal{T}, \mathcal{B})$ are weakly compatible and

(iii) $d(\mathcal{S}x, \mathcal{T}y) \leq \alpha d(\mathcal{A}x, \mathcal{T}y) + \beta d(\mathcal{B}y, \mathcal{S}x) + \gamma d(\mathcal{A}x, \mathcal{B}y)$

for all x, $y \in \mathcal{X}$ where $\alpha, \beta, \gamma \geq 0$, $0 \leq \alpha + \beta + \gamma < \frac{1}{2}$.

Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

In this theorem, we mention that the common fixed point does not exist if the four maps are not continuous, also, if we have not the inclusions between the range spaces. Again, the authors required the completeness of the space.

In 2015, Bennani et al. [6] improved the above theorem by removing the continuity. Furthermore, they gave some other results when $\alpha + \beta + \gamma \leq \frac{1}{2}$.

Theorem 2.6. *([6]) Let $(\mathcal{X}, d)$ be a d-metric space. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{S}$ be four self-maps of $\mathcal{X}$ such that*

(i) $\mathcal{T}(\mathcal{X}) \subset \mathcal{A}(\mathcal{X})$, $\mathcal{S}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X})$.

(ii) The pairs $(\mathcal{S}, \mathcal{A})$ and $(\mathcal{T}, \mathcal{B})$ are weakly compatible and

(iii) $d(\mathcal{S}x, \mathcal{T}y) \leq \alpha d(\mathcal{A}x, \mathcal{T}y) + \beta d(\mathcal{B}y, \mathcal{S}x) + \gamma d(\mathcal{A}x, \mathcal{B}y)$

for all x, $y \in \mathcal{X}$ where $\alpha, \beta, \gamma \geq 0$ satisfying $0 \leq \alpha + \beta + \gamma < \frac{1}{2}$.

(iv) The range of one of the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ or $\mathcal{T}$ is a complete subspace of $\mathcal{X}$.

Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point in $\mathcal{X}$.

In this theorem, we mention that the common fixed point does not exist if we have not the inclusions between the range spaces. Also, the authors required the completeness of the range of one of the maps.

In this investigation, we will use our new definition to prove the existence and uniqueness of common fixed points for quadruple maps in a d-metric space. These theorems improve and extend the above theorems and some similar results in (metric, partial metric and d-metric) spaces.

3 Our Main Results

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a d-metric space. Let $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$, $\mathfrak{M}_4 : \mathcal{X} \to \mathcal{X}$ be maps satisfying*

(i) the pairs $(\mathfrak{M}_1, \mathfrak{M}_3)$ and $(\mathfrak{M}_2, \mathfrak{M}_4)$ are occasionally weakly $\mathfrak{M}_1$-biased (respectively $\mathfrak{M}_2$-biased) of type $(\mathcal{A})$ and

(ii) $d(\mathfrak{M}_3x, \mathfrak{M}_4y) \leq \rho d(\mathfrak{M}_1x, \mathfrak{M}_4y) + \varrho d(\mathfrak{M}_2y, \mathfrak{M}_3x) + \sigma d(\mathfrak{M}_1x, \mathfrak{M}_2y)$

for all x, $y \in \mathcal{X}$, where $\rho, \varrho, \sigma \geq 0$, $0 \leq \rho + \varrho + \sigma < 1$. Then $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ and $\mathfrak{M}_4$ have a unique common fixed point.

Proof. By assumptions, there are two points θ and ϑ in $\mathcal{X}$ such that $\mathfrak{M}_1\theta = \mathfrak{M}_3\theta$ implies $d(\mathfrak{M}_1\mathfrak{M}_1\theta, \mathfrak{M}_3\theta) \leq d(\mathfrak{M}_3\mathfrak{M}_1\theta, \mathfrak{M}_1\theta)$ and $\mathfrak{M}_2\vartheta = \mathfrak{M}_4\vartheta$ implies $d(\mathfrak{M}_2\mathfrak{M}_2\vartheta, \mathfrak{M}_4\vartheta) \leq d(\mathfrak{M}_4\mathfrak{M}_2\vartheta, \mathfrak{M}_2\vartheta)$.

First of all, we are going to prove that $\mathfrak{M}_3\theta = \mathfrak{M}_4\vartheta$. Suppose that $\mathfrak{M}_3\theta \neq \mathfrak{M}_4\vartheta$, from inequality (ii) we have

$$\begin{aligned} d(\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta) &\leq \rho d(\mathfrak{M}_1\theta, \mathfrak{M}_4\vartheta) + \varrho d(\mathfrak{M}_2\vartheta, \mathfrak{M}_3\theta) + \sigma d(\mathfrak{M}_1\theta, \mathfrak{M}_2\vartheta) \\ &= (\rho + \varrho + \sigma) d(\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta) \\ &< d(\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta), \end{aligned}$$

which is a contradiction, thus, $\mathfrak{M}_3\theta = \mathfrak{M}_4\vartheta$.

Now, we assert that $\mathfrak{M}_3\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$. If not, then the use of condition (ii) gives

$$\begin{aligned} d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta) &\leq \rho d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta) + \varrho d(\mathfrak{M}_2\vartheta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\ &\quad + \sigma d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_2\vartheta); \end{aligned}$$

i.e.,

$$\begin{aligned} d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) &\leq \rho d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) + \varrho d(\mathfrak{M}_3\theta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\ &\quad + \sigma d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) \\ &= \rho d(\mathfrak{M}_1\mathfrak{M}_1\theta, \mathfrak{M}_3\theta) + \varrho d(\mathfrak{M}_3\theta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\ &\quad + \sigma d(\mathfrak{M}_1\mathfrak{M}_1\theta, \mathfrak{M}_3\theta) \\ &\leq \rho d(\mathfrak{M}_3\mathfrak{M}_1\theta, \mathfrak{M}_1\theta) + \varrho d(\mathfrak{M}_3\theta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\ &\quad + \sigma d(\mathfrak{M}_3\mathfrak{M}_1\theta, \mathfrak{M}_1\theta) \\ &= \rho d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) + \varrho d(\mathfrak{M}_3\theta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\ &\quad + \sigma d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) \\ &= (\rho + \varrho + \sigma) d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) \\ &< d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta), \end{aligned}$$

which is a contradiction, therefore, $\mathfrak{M}_3\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$, consequently, $\mathfrak{M}_1\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$.

Now, suppose that $\mathfrak{M}_4\mathfrak{M}_4\vartheta \neq \mathfrak{M}_4\vartheta$. Using inequality (ii) we obtain

$$\begin{aligned} d(\mathfrak{M}_3\theta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) &\leq \rho d(\mathfrak{M}_1\theta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) + \varrho d(\mathfrak{M}_2\mathfrak{M}_4\vartheta, \mathfrak{M}_3\theta) \\ &\quad + \sigma d(\mathfrak{M}_1\theta, \mathfrak{M}_2\mathfrak{M}_4\vartheta); \end{aligned}$$

i.e.,

$$\begin{aligned} d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) &\leq \rho d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) + \varrho d(\mathfrak{M}_2\mathfrak{M}_4\vartheta, \mathfrak{M}_4\vartheta) \\ &\quad + \sigma d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_4\vartheta) \\ &= \rho d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) + \varrho d(\mathfrak{M}_2\mathfrak{M}_2\vartheta, \mathfrak{M}_4\vartheta) \\ &\quad + \sigma d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_2\vartheta) \\ &\leq \rho d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) + \varrho d(\mathfrak{M}_4\mathfrak{M}_2\vartheta, \mathfrak{M}_2\vartheta) \\ &\quad + \sigma d(\mathfrak{M}_2\vartheta, \mathfrak{M}_4\mathfrak{M}_2\vartheta) \\ &= \rho d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) + \varrho d(\mathfrak{M}_4\mathfrak{M}_4\vartheta, \mathfrak{M}_4\vartheta) \\ &\quad + \sigma d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) \\ &= (\rho + \varrho + \sigma) d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) \\ &< d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta), \end{aligned}$$

a contradiction, which implies that $\mathfrak{M}_4\mathfrak{M}_4\vartheta = \mathfrak{M}_4\vartheta$ and so $\mathfrak{M}_2\mathfrak{M}_4\vartheta = \mathfrak{M}_4\vartheta$; i.e., $\mathfrak{M}_4\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$ and $\mathfrak{M}_2\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$. Putting $\mathfrak{M}_1\theta = \mathfrak{M}_3\theta = \mathfrak{M}_2\vartheta = \mathfrak{M}_4\vartheta = \mu$, therefore μ is a common fixed point of maps $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ and $\mathfrak{M}_4$.

Finally, let μ and ν be two distinct common fixed points of maps $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ and $\mathfrak{M}_4$. Then, $\mu = \mathfrak{M}_1\mu = \mathfrak{M}_2\mu = \mathfrak{M}_3\mu = \mathfrak{M}_4\mu$ and $\nu = \mathfrak{M}_1\nu = \mathfrak{M}_2\nu = \mathfrak{M}_3\nu = \mathfrak{M}_4\nu$. From (ii) we have

$$d(\mathfrak{M}_3\nu, \mathfrak{M}_4\mu) \leq \rho d(\mathfrak{M}_1\nu, \mathfrak{M}_4\mu) + \varrho d(\mathfrak{M}_2\mu, \mathfrak{M}_3\nu) + \sigma d(\mathfrak{M}_1\nu, \mathfrak{M}_2\mu);$$

i.e.,

$$\begin{aligned} d(\nu, \mu) &\leq \rho d(\nu, \mu) + \varrho d(\mu, \nu) + \sigma d(\nu, \mu) \\ &= (\rho + \varrho + \sigma) d(\nu, \mu) \\ &< d(\nu, \mu), \end{aligned}$$

which is a contradiction, thus, $\nu = \mu$. □

Now, we give an illustrative example which supports our result.

Example 3.2. Let $(\mathcal{X} = (-20, 20), d)$ be a d-metric space such that $d(x, y) = \max\{|x|, |y|\}$. Consider the following maps:

$$\mathfrak{M}_3 x = \begin{cases} 0 \text{ if } x \in (-20, 0] \\ -\dfrac{1}{100} \text{ if } x \in (0, 20), \end{cases} \quad \mathfrak{M}_4 x = \begin{cases} -\dfrac{x}{20} \text{ if } x \in (-20, 0] \\ -\dfrac{1}{50} \text{ if } x \in (0, 20), \end{cases}$$

$$\mathfrak{M}_1 x = \begin{cases} -x \text{ if } x \in (-20, 0] \\ 18 \text{ if } x \in (0, 20), \end{cases} \quad \mathfrak{M}_2 x = \begin{cases} -x \text{ if } x \in (-20, 0] \\ 19 \text{ if } x \in (0, 20). \end{cases}$$

First of all, we mention that the condition of occasionally weakly biased maps of type $(\mathcal{A})$ is satisfied. Taking $\rho = \frac{1}{9}$, $\varrho = \frac{1}{10}$ and $\sigma = \frac{3}{4}$, we get

First case: for x, $y \in (-20, 0]$, we have $\mathfrak{M}_3 x = 0$, $\mathfrak{M}_4 y = -\frac{y}{20}$, $\mathfrak{M}_1 x = -x$, $\mathfrak{M}_2 y = -y$ and

$$\begin{aligned} d(\mathfrak{M}_3 x, \mathfrak{M}_4 y) &= -\frac{y}{20} \\ &\leq \frac{1}{9} \max\left\{-x, -\frac{y}{20}\right\} + \frac{1}{10}(-y) + \frac{3}{4} \max\{-x, -y\} \\ &= \rho d(\mathfrak{M}_1 x, \mathfrak{M}_4 y) + \varrho d(\mathfrak{M}_2 y, \mathfrak{M}_3 x) + \sigma d(\mathfrak{M}_1 x, \mathfrak{M}_2 y). \end{aligned}$$

Second case: for x, $y \in (0, 20)$, we have $\mathfrak{M}_3 x = -\frac{1}{100}$, $\mathfrak{M}_4 y = -\frac{1}{50}$, $\mathfrak{M}_1 x = 18$, $\mathfrak{M}_2 y = 19$ and

$$\begin{aligned} d(\mathfrak{M}_3 x, \mathfrak{M}_4 y) &= \frac{1}{50} \\ &\leq \frac{1}{9} \times (18) + \frac{1}{10} \times (19) + \frac{3}{4} \times (19) \\ &= \frac{363}{20} \\ &= \rho d(\mathfrak{M}_1 x, \mathfrak{M}_4 y) + \varrho d(\mathfrak{M}_2 y, \mathfrak{M}_3 x) + \sigma d(\mathfrak{M}_1 x, \mathfrak{M}_2 y). \end{aligned}$$

Third case: for $x \in (-20, 0]$ and $y \in (0, 20)$, we have $\mathfrak{M}_3 x = 0$, $\mathfrak{M}_4 y = -\frac{1}{50}$, $\mathfrak{M}_1 x = -x$, $\mathfrak{M}_2 y = 19$ and

$$\begin{aligned} d(\mathfrak{M}_3 x, \mathfrak{M}_4 y) &= \frac{1}{50} \\ &\leq \frac{1}{9} \max\left\{\frac{1}{50}, -x\right\} + \frac{1}{10} \times (19) + \frac{3}{4} \max\{-x, 19\} \\ &= \rho d(\mathfrak{M}_1 x, \mathfrak{M}_4 y) + \varrho d(\mathfrak{M}_2 y, \mathfrak{M}_3 x) + \sigma d(\mathfrak{M}_1 x, \mathfrak{M}_2 y). \end{aligned}$$

Fourth case: for $x \in (0, 20)$ and $y \in (-20, 0]$, we have $\mathfrak{M}_3 x = -\frac{1}{100}$, $\mathfrak{M}_4 y = -\frac{y}{20}$, $\mathfrak{M}_1 x = 18$, $\mathfrak{M}_2 y = -y$ and

$$\begin{aligned} d(\mathfrak{M}_3 x, \mathfrak{M}_4 y) &= \max\left\{\frac{1}{100}, -\frac{y}{20}\right\} \\ &\leq \frac{1}{9} \times (18) + \frac{1}{10} \max\left\{\frac{1}{100}, -y\right\} + \frac{3}{4} \max\{18, -y\} \\ &= \rho d(\mathfrak{M}_1 x, \mathfrak{M}_4 y) + \varrho d(\mathfrak{M}_2 y, \mathfrak{M}_3 x) + \sigma d(\mathfrak{M}_1 x, \mathfrak{M}_2 y), \end{aligned}$$

so, all the requirements of Theorem 3.1 are satisfied and 0 is the unique common fixed point of maps $\mathfrak{M}_3$, $\mathfrak{M}_4$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

Remark 3.3. Note that Theorem 2.5 of [15] and Theorem 2.6 of [6] are not applicable because the four maps are discontinuous, the space is incomplete and we have $\mathfrak{M}_3 \mathcal{X} = \left\{-\frac{1}{100}, 0\right\} \not\subseteq \mathfrak{M}_2 \mathcal{X} = [0, 20)$ and $\mathfrak{M}_4 \mathcal{X} = [0, 1) \cup \left\{-\frac{1}{50}\right\} \not\subseteq \mathfrak{M}_1 \mathcal{X} = [0, 20)$.

In the next, we will extend the constants of the above theorem.

Theorem 3.4. *Let $(\mathcal{X}, d)$ be a d-metric space. Let $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$, $\mathfrak{M}_4 : \mathcal{X} \to \mathcal{X}$ be maps such that the pairs $(\mathfrak{M}_1, \mathfrak{M}_3)$ and $(\mathfrak{M}_2, \mathfrak{M}_4)$ are occasionally weakly $\mathfrak{M}_1$-biased (respectively $\mathfrak{M}_2$-biased) of type $(\mathcal{A})$ and*

$$\begin{aligned} d(\mathfrak{M}_3 x, \mathfrak{M}_4 y) \leq\ & \rho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y)) d(\mathfrak{M}_1 x, \mathfrak{M}_4 y) \\ & + \varrho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y)) d(\mathfrak{M}_2 y, \mathfrak{M}_3 x) \\ & + \sigma(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y)) d(\mathfrak{M}_1 x, \mathfrak{M}_2 y) \end{aligned} \tag{3.1}$$

for all x, $y \in \mathcal{X}$, where ρ, ϱ, $\sigma : [0, \infty) \to [0, 1)$ are non-decreasing functions which satisfying the following condition

$$\rho(z) + \varrho(z) + \sigma(z) < 1 \ \forall z > 0.$$

Then $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ and $\mathfrak{M}_4$ have a unique common fixed point.

Proof. Again, by conditions, there are two elements θ and ϑ in $\mathcal{X}$ such that $\mathfrak{M}_1\theta = \mathfrak{M}_3\theta$ implies that $d(\mathfrak{M}_1\mathfrak{M}_1\theta, \mathfrak{M}_3\theta) \leq d(\mathfrak{M}_3\mathfrak{M}_1\theta, \mathfrak{M}_1\theta)$ and $\mathfrak{M}_2\vartheta = \mathfrak{M}_4\vartheta$ implies that $d(\mathfrak{M}_2\mathfrak{M}_2\vartheta, \mathfrak{M}_4\vartheta) \leq d(\mathfrak{M}_4\mathfrak{M}_2\vartheta, \mathfrak{M}_2\vartheta)$.

Firstly, we are going to prove that $\mathfrak{M}_3\theta = \mathfrak{M}_4\vartheta$. Suppose that $\mathfrak{M}_3\theta \neq \mathfrak{M}_4\vartheta$, from inequality (3.1) we have

$$\begin{aligned} d(\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta) &\leq \rho(d(\mathfrak{M}_1\theta, \mathfrak{M}_2\vartheta)) d(\mathfrak{M}_1\theta, \mathfrak{M}_4\vartheta) \\ &\quad + \varrho(d(\mathfrak{M}_1\theta, \mathfrak{M}_2\vartheta)) d(\mathfrak{M}_2\vartheta, \mathfrak{M}_3\theta) \\ &\quad + \sigma(d(\mathfrak{M}_1\theta, \mathfrak{M}_2\vartheta)) d(\mathfrak{M}_1\theta, \mathfrak{M}_2\vartheta) \\ &= (\rho(d(\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta)) + \varrho(d(\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta)) \\ &\quad + \sigma(d(\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta))) d(\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta) \\ &< d(\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta), \end{aligned}$$

which is a contradiction, thus $\mathfrak{M}_3\theta = T\vartheta$.

Secondly, we assert that $\mathfrak{M}_3\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$. If not, then, the use of condition (3.1) gives

$$\begin{aligned} d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta) &\leq \rho(d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_2\vartheta)) d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_4\vartheta) \\ &\quad + \varrho(d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_2\vartheta)) d(\mathfrak{M}_2\vartheta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\ &\quad + \sigma(d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_2\vartheta)) d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_2\vartheta); \end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) &\leq \rho(d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_3\theta))d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) \\
&\quad +\varrho(d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_3\theta))d(\mathfrak{M}_3\theta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\
&\quad +\sigma(d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_3\theta))d(\mathfrak{M}_1\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) \\
&= \rho(d(\mathfrak{M}_1\mathfrak{M}_1\theta, \mathfrak{M}_3\theta))d(\mathfrak{M}_1\mathfrak{M}_1\theta, \mathfrak{M}_3\theta) \\
&\quad +\varrho(d(\mathfrak{M}_1\mathfrak{M}_1\theta, \mathfrak{M}_3\theta))d(\mathfrak{M}_3\theta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\
&\quad +\sigma(d(\mathfrak{M}_1\mathfrak{M}_1\theta, \mathfrak{M}_3\theta))d(\mathfrak{M}_1\mathfrak{M}_1\theta, \mathfrak{M}_3\theta) \\
&\leq \rho(d(\mathfrak{M}_3\mathfrak{M}_1\theta, \mathfrak{M}_1\theta))d(\mathfrak{M}_3\mathfrak{M}_1\theta, \mathfrak{M}_1\theta) \\
&\quad +\varrho(d(\mathfrak{M}_3\mathfrak{M}_1\theta, \mathfrak{M}_1\theta))d(\mathfrak{M}_3\theta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\
&\quad +\sigma(d(\mathfrak{M}_3\mathfrak{M}_1\theta, \mathfrak{M}_1\theta))d(\mathfrak{M}_3\mathfrak{M}_1\theta, \mathfrak{M}_1\theta) \\
&= \rho(d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta))d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) \\
&\quad +\varrho(d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta))d(\mathfrak{M}_3\theta, \mathfrak{M}_3\mathfrak{M}_3\theta) \\
&\quad +\sigma(d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta))d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) \\
&= (\rho(d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta)) + \varrho(d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta)) \\
&\quad +\sigma(d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta)))d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta) \\
&< d(\mathfrak{M}_3\mathfrak{M}_3\theta, \mathfrak{M}_3\theta),
\end{aligned}$$

which is a contradiction, therefore $\mathfrak{M}_3\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$, consequently, $\mathfrak{M}_1\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$.

Thirdly, suppose that $\mathfrak{M}_4\mathfrak{M}_4\vartheta \neq \mathfrak{M}_4\vartheta$. Using inequality (3.1) we obtain

$$\begin{aligned}
d(\mathfrak{M}_3\theta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) &\leq \rho(d(\mathfrak{M}_1\theta, \mathfrak{M}_2\mathfrak{M}_4\vartheta))d(\mathfrak{M}_1\theta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) \\
&\quad +\varrho(d(\mathfrak{M}_1\theta, \mathfrak{M}_2\mathfrak{M}_4\vartheta))d(\mathfrak{M}_2\mathfrak{M}_4\vartheta, \mathfrak{M}_3\theta) \\
&\quad +\sigma(d(\mathfrak{M}_1\theta, \mathfrak{M}_2\mathfrak{M}_4\vartheta))d(\mathfrak{M}_1\theta, \mathfrak{M}_2\mathfrak{M}_4\vartheta);
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) &\leq \rho(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_4\vartheta))d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) \\
&\quad +\varrho(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_4\vartheta))d(\mathfrak{M}_2\mathfrak{M}_4\vartheta, \mathfrak{M}_4\vartheta) \\
&\quad +\sigma(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_4\vartheta))d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_4\vartheta) \\
&= \rho(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_2\vartheta))d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) \\
&\quad +\varrho(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_2\vartheta))d(\mathfrak{M}_2\mathfrak{M}_2\vartheta, \mathfrak{M}_4\vartheta) \\
&\quad +\sigma(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_2\vartheta))d(\mathfrak{M}_4\vartheta, \mathfrak{M}_2\mathfrak{M}_2\vartheta) \\
&\leq \rho(d(\mathfrak{M}_2\vartheta, \mathfrak{M}_4\mathfrak{M}_2\vartheta))d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) \\
&\quad +\varrho(d(\mathfrak{M}_2\vartheta, \mathfrak{M}_4\mathfrak{M}_2\vartheta))d(\mathfrak{M}_4\mathfrak{M}_2\vartheta, \mathfrak{M}_2\vartheta) \\
&\quad +\sigma(d(\mathfrak{M}_2\vartheta, \mathfrak{M}_4\mathfrak{M}_2\vartheta))d(\mathfrak{M}_2\vartheta, \mathfrak{M}_4\mathfrak{M}_2\vartheta) \\
&= \rho(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta))d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) \\
&\quad +\varrho(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta))d(\mathfrak{M}_4\mathfrak{M}_4\vartheta, \mathfrak{M}_4\vartheta) \\
&\quad +\sigma(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta))d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) \\
&= (\rho(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta)) + \varrho(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta)) \\
&\quad +\sigma(d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta)))d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta) \\
&< d(\mathfrak{M}_4\vartheta, \mathfrak{M}_4\mathfrak{M}_4\vartheta),
\end{aligned}$$

this contradiction implies that $\mathfrak{M}_4\mathfrak{M}_4\vartheta = \mathfrak{M}_4\vartheta$ and so $\mathfrak{M}_2\mathfrak{M}_4\vartheta = \mathfrak{M}_4\vartheta$; i.e., $\mathfrak{M}_4\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$ and $\mathfrak{M}_2\mathfrak{M}_3\theta = \mathfrak{M}_3\theta$. Putting $\mathfrak{M}_1\theta = \mathfrak{M}_3\theta = \mathfrak{M}_2\vartheta = \mathfrak{M}_4\vartheta = \mu$, therefore, μ is a common fixed point of maps $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ and $\mathfrak{M}_4$.

Fourthly, let μ and ν be two different common fixed points of maps $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ and $\mathfrak{M}_4$. Then, $\mu = \mathfrak{M}_1\mu = \mathfrak{M}_2\mu = \mathfrak{M}_3\mu = \mathfrak{M}_4\mu$ and $\nu = \mathfrak{M}_1\nu = \mathfrak{M}_2\nu = \mathfrak{M}_3\nu = \mathfrak{M}_4\nu$. From (3.1) we have

$$\begin{aligned} d(\mathfrak{M}_3\nu, \mathfrak{M}_4\mu) &\leq \rho(d(\mathfrak{M}_1\nu, \mathfrak{M}_2\mu))d(\mathfrak{M}_1\nu, \mathfrak{M}_4\mu) \\ &\quad +\varrho(d(\mathfrak{M}_1\nu, \mathfrak{M}_2\mu))d(\mathfrak{M}_2\mu, \mathfrak{M}_3\nu) \\ &\quad +\sigma(d(\mathfrak{M}_1 A\nu, \mathfrak{M}_2\mu))d(A\nu, \mathfrak{M}_2\mu); \end{aligned}$$

i.e.,

$$\begin{aligned} d(\nu,\mu) &\leq \rho(d(\nu,\mu))d(\nu,\mu) + \varrho(d(\nu,\mu))d(\mu,\nu) + \sigma(d(\nu,\mu))d(\nu,\mu) \\ &= (\rho(d(\nu,\mu)) + \varrho(d(\nu,\mu)) + \sigma(d(\nu,\mu)))d(\nu,\mu) \\ &< d(\nu,\mu), \end{aligned}$$

which is a contradiction, thus, $\nu = \mu$. □

Again, we give an example which illustrates our above theorem.

Example 3.5. Let $\left(\mathcal{X} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), d\right)$ be a d-metric space such that $d(x,y) = \max\{|x|,|y|\}$. Consider the four maps

$$\mathfrak{M}_3 x = \begin{cases} 0 \text{ if } x \in \left(-\frac{\pi}{2}, 0\right] \\ -\frac{\pi}{10} \text{ if } x \in \left(0, \frac{\pi}{2}\right), \end{cases} \quad \mathfrak{M}_4 x = \begin{cases} -\frac{y}{10} \text{ if } x \in \left(-\frac{\pi}{2}, 0\right] \\ -\frac{\pi}{20} \text{ if } x \in \left(0, \frac{\pi}{2}\right), \end{cases}$$

$$\mathfrak{M}_1 x = \begin{cases} -x \text{ if } x \in \left(-\frac{\pi}{2}, 0\right] \\ \frac{\pi}{3} \text{ if } x \in \left(0, \frac{\pi}{2}\right), \end{cases} \quad \mathfrak{M}_2 x = \begin{cases} -x \text{ if } x \in \left(-\frac{\pi}{2}, 0\right] \\ \frac{\pi}{6} \text{ if } x \in \left(0, \frac{\pi}{2}\right). \end{cases}$$

Of course maps $\mathfrak{M}_3$ and $\mathfrak{M}_1$ are occasionally weakly $\mathfrak{M}_1$-biased of type $(\mathcal{A})$ and $\mathfrak{M}_4$ and $\mathfrak{M}_2$ are occasionally weakly $\mathfrak{M}_2$-biased of type $(\mathcal{A})$. Taking $\rho = \frac{|\sin(z)|}{9}$, $\varrho = \frac{|\sin(z)|}{10}$ and $\sigma = \frac{3}{4}$, we get

Case one: for $x, y \in \left(-\frac{\pi}{2}, 0\right]$, we have $\mathfrak{M}_3 x = 0$, $\mathfrak{M}_4 y = -\frac{y}{10}$, $\mathfrak{M}_1 x = -x$, $\mathfrak{M}_2 y = -y$ and

$$\begin{aligned} d(\mathfrak{M}_3 x, \mathfrak{M}_4 y) &= -\frac{y}{10} \\ &\leq \frac{1}{9}\left|\sin\left(\max\{-x,-y\}\right)\right| \times \max\left\{-x, -\frac{y}{10}\right\} \\ &\quad + \frac{1}{10}\left|\sin\left(\max\{-x,-y\}\right)\right| \times (-y) + \frac{3}{4}\max\{-x,-y\} \\ &= \rho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y))d(\mathfrak{M}_1 x, \mathfrak{M}_4 y) \\ &\quad +\varrho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y))d(\mathfrak{M}_2 y, \mathfrak{M}_3 x) \\ &\quad +\sigma(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y))d(\mathfrak{M}_1 x, \mathfrak{M}_2 y). \end{aligned}$$

Case two: for $x, y \in \left(0, \frac{\pi}{2}\right)$, we have $\mathfrak{M}_3 x = -\frac{\pi}{10}$, $\mathfrak{M}_4 y = -\frac{\pi}{20}$, $\mathfrak{M}_1 x = \frac{\pi}{3}$, $\mathfrak{M}_2 y = \frac{\pi}{6}$ and

$$\begin{aligned} d(\mathfrak{M}_3 x, \mathfrak{M}_4 y) &= \frac{\pi}{10} \\ &\leq \frac{1}{9}\left|\sin\left(\frac{\pi}{3}\right)\right| \times \frac{\pi}{3} + \frac{1}{10}\left|\sin\left(\frac{\pi}{3}\right)\right| \times \frac{\pi}{6} + \frac{3}{4} \times \frac{\pi}{3} \\ &= \frac{\sqrt{3}\pi}{54} + \frac{\sqrt{3}\pi}{60} + \frac{\pi}{4} \\ &= \rho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y))d(\mathfrak{M}_1 x, \mathfrak{M}_4 y) \\ &\quad +\varrho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y))d(\mathfrak{M}_2 y, \mathfrak{M}_3 x) \\ &\quad +\sigma(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y))d(\mathfrak{M}_1 x, \mathfrak{M}_2 y). \end{aligned}$$

Case three: for $x \in \left(-\frac{\pi}{2}, 0\right]$ and $y \in \left(0, \frac{\pi}{2}\right)$, we have $\mathfrak{M}_3 x = 0$, $\mathfrak{M}_4 y = -\frac{\pi}{20}$, $\mathfrak{M}_1 x = -x$, $\mathfrak{M}_2 y = \frac{\pi}{6}$ and

$$\begin{aligned} d(\mathfrak{M}_3 x, \mathfrak{M}_4 y) &= \frac{\pi}{20} \\ &\leq \frac{1}{9}\left|\sin\left(\max\left\{-x, \frac{\pi}{6}\right\}\right)\right| \times \max\left\{-x, \frac{\pi}{20}\right\} \\ &\quad + \frac{1}{10}\left|\sin\left(\max\left\{-x, \frac{\pi}{6}\right\}\right)\right| \times \frac{\pi}{6} + \frac{3}{4}\max\left\{-x, \frac{\pi}{6}\right\} \\ &= \rho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y)) d(\mathfrak{M}_1 x, \mathfrak{M}_4 y) \\ &\quad + \varrho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y)) d(\mathfrak{M}_2 y, \mathfrak{M}_3 x) \\ &\quad + \sigma(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y)) d(\mathfrak{M}_1 x, \mathfrak{M}_2 y). \end{aligned}$$

Case four: for $x \in \left(0, \frac{\pi}{2}\right)$ and $y \in \left(-\frac{\pi}{2}, 0\right]$, we have $\mathfrak{M}_3 x = -\frac{\pi}{10}$, $\mathfrak{M}_4 y = -\frac{y}{10}$, $\mathfrak{M}_1 x = \frac{\pi}{3}$, $\mathfrak{M}_2 y = -y$ and

$$\begin{aligned} d(\mathfrak{M}_3 x, \mathfrak{M}_4 y) &= \max\left\{\frac{\pi}{10}, -\frac{y}{10}\right\} = \frac{\pi}{10} \\ &\leq \frac{1}{9}\left|\sin\left(\max\left\{\frac{\pi}{3}, -y\right\}\right)\right| \times \frac{\pi}{3} \\ &\quad + \frac{1}{10}\left|\sin\left(\max\left\{\frac{\pi}{3}, -y\right\}\right)\right| \times \max\left\{\frac{\pi}{10}, -y\right\} \\ &\quad + \frac{3}{4}\max\left\{\frac{\pi}{3}, -y\right\} \\ &= \rho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y)) d(\mathfrak{M}_1 x, \mathfrak{M}_4 y) \\ &\quad + \varrho(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y)) d(\mathfrak{M}_2 y, \mathfrak{M}_3 x) \\ &\quad + \sigma(d(\mathfrak{M}_1 x, \mathfrak{M}_2 y)) d(\mathfrak{M}_1 x, \mathfrak{M}_2 y), \end{aligned}$$

so, all the requirements of Theorem 3.4 are satisfied and 0 is the unique common fixed point of maps $\mathfrak{M}_3$, $\mathfrak{M}_4$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

Remark 3.6. Note that $\mathfrak{M}_3 \mathcal{X} = \left\{-\frac{\pi}{10}, 0\right\} \nsubseteq \mathfrak{M}_2 \mathcal{X} = \left[0, \frac{\pi}{2}\right)$ and $\mathfrak{M}_4 \mathcal{X} = \left[0, \frac{\pi}{10}\right) \cup \left\{-\frac{\pi}{20}\right\} \nsubseteq \mathfrak{M}_1 \mathcal{X} = \left[0, \frac{\pi}{2}\right)$.

4 Application to an Integral Equation

Consider the integral equation

$$u(x) = \int_a^x l(x,t) n_i(t, u(t)) dt + \int_a^b p(x,t) q_i(t, u(t)) dt, \text{ for all } x \in [a,b], \tag{4.1}$$

where $n_i, q_i : [a,b] \times \mathbb{R} \to [0, +\infty), i = 1, 2$ are non-negative continuous functions.

Let $\mathcal{X} = C([a,b], [0, +\infty))$ be the set of non-negative real continuous functions defined on $[a,b]$. Take the d-metric $d : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ defined by

$$\begin{aligned} d(u,v) &= \|u\|_\infty + \|v\|_\infty \\ &= \max_{x \in [a,b]} u(x) + \max_{x \in [a,b]} v(x) \end{aligned}$$

for $u, v \in \mathcal{X}$, hence, $(\mathcal{X}, d)$ is a d-metric space.

Theorem 4.1. *Let $\mathcal{R}$, $\mathcal{W}$, $\mathcal{Y}$ and $\mathcal{Z}$ be functions defined by*

$$\begin{aligned} \mathcal{Y}u(x) &= \int_a^x l(x,t)n_1(t,u(t))dt \\ \mathcal{Z}u(x) &= \int_a^x l(x,t)n_2(t,u(t))dt \\ \mathcal{C}u(x) &= \int_a^b p(x,t)q_1(t,u(t))dt \\ \mathcal{D}u(x) &= \int_a^b p(x,t)q_2(t,u(t))dt \\ \mathcal{R}u(x) &= (\mathcal{I}-\mathcal{C})u(x) \\ \mathcal{W}u(x) &= (\mathcal{I}-\mathcal{D})u(x), \end{aligned}$$

where $\mathcal{I}$ is the identity function on $\mathcal{X}$. Suppose that the following conditions hold:

(i) there exists $\tau \in (a,b)$ such that for all $t \in [a,b]$ and $u \in \mathcal{X}$, we have $|n_i(t,u(t))| \leq \tau|u(t)|$ for $i = 1, 2$,

(ii) $\int_a^b \max_{x\in[a,b]} |l(x,t)|dt = \eta_1 < +\infty$,

(iii) there exists $\rho \in (a,b)$ such that for all $t \in [a,b]$ and $u \in \mathcal{X}$, we have $|q_i(t,u(t))| \leq \rho|u(t)|$ for $i = 1, 2$,

(iv) $\int_a^b \max_{x\in[a,b]} |p(x,t)|dt = \eta_2 < +\infty$,

(v) the functions commute at their each coincidence points,

then, equation (4.1) has a unique solution in $\mathcal{X}$ if and only if functions $\mathcal{R}$, $\mathcal{W}$, $\mathcal{Y}$ and $\mathcal{Z}$ have a unique common fixed point for τ, $\rho \in (a,b)$ with $\rho\eta_2 < 1$ and $\frac{\tau\eta_1}{1-\rho\eta_2} = \alpha < \frac{1}{2}$.

Proof. First of all, we mention that by the fifth condition, we can see that maps $\mathcal{R}$ and $\mathcal{Y}$ as well as $\mathcal{W}$ and $\mathcal{Z}$ are occasionally weakly $\mathcal{R}$-biased (respectively $\mathcal{W}$-biased) of type $(\mathcal{A})$.

Next, we have

$$\begin{aligned} \mathcal{Y}u(x)| &= \left|\int_a^x l(x,t)n_1(t,u(t))dt\right| \\ &\leq \int_a^x |l(x,t)|\,|n_1(t,u(t))|\,dt \\ &\leq \tau\int_a^x |l(x,t)|\,|u(t)|\,dt \\ &\leq \tau\int_a^b |l(x,t)| \max_{t\in[a,b]} |u(t)|\,dt \\ &\leq \tau\|u\|_\infty \int_a^b \max_{x\in[a,b]} |l(x,t)|\,dt, \end{aligned}$$

implies that

$$\|\mathcal{Y}u\|_\infty \leq \tau\eta_1\|u\|_\infty.$$

It follows that, for all u, $v \in \mathcal{X}$

$$d(\mathcal{Y}u, \mathcal{Z}v) \leq \tau\eta_1 d(u,v). \tag{4.2}$$

Similarly, we obtain

$$d(\mathcal{C}u, \mathcal{D}v) \leq \rho\eta_2 d(u,v). \tag{4.3}$$

consequently, we have

$$\begin{aligned} d(\mathcal{R}u, \mathcal{W}v) &= \|\mathcal{R}u\|_\infty + \|\mathcal{W}v\|_\infty \\ &= \max_{x\in[a,b]} \mathcal{R}u(x) + \max_{x\in[a,b]} \mathcal{W}v(x) \\ &= \max_{x\in[a,b]} \left[(\mathcal{I}-\mathcal{C})\, u(x) + (\mathcal{I}-\mathcal{D})\, v(x)\right] \\ &= \left[\max_{x\in[a,b]} u(x) + \max_{x\in[a,b]} v(x)\right] - \left[\max_{x\in[a,b]} \mathcal{C}u(x) + \max_{x\in[a,b]} \mathcal{D}v(x)\right] \\ &= d(u,v) - d(\mathcal{C}u, \mathcal{D}v) \\ &\geq d(u,v) - \rho\eta_2 d(u,v) \\ &= (1-\rho\eta_2) d(u,v), \end{aligned}$$

which implies that

$$d(u,v) \leq \frac{1}{1-\rho\eta_2} d(\mathcal{R}u, \mathcal{W}v). \tag{4.4}$$

From (4.2) and (4.4), we obtain

$$\begin{aligned} d(\mathcal{Y}u, \mathcal{Z}v) &\leq \tau\eta_1 \left(\frac{1}{1-\rho\eta_2}\right) d(\mathcal{R}u, \mathcal{W}v) \\ &= \frac{\tau\eta_1}{1-\rho\eta_2} d(\mathcal{R}u, \mathcal{W}v) \\ &= \alpha d(\mathcal{R}u, \mathcal{W}v), \end{aligned}$$

which amounts to say that

$$d(\mathcal{Y}u, \mathcal{Z}v) \leq \alpha d(\mathcal{R}u, \mathcal{W}v) + \beta d(\mathcal{W}v, \mathcal{Y}u) + \gamma d(\mathcal{R}u, \mathcal{Z}v).$$

Thus, all the conditions of the theorem are satisfied. Therefore, there exists a unique $u^* \in \mathcal{X}$ such that $\mathcal{R}u^* = \mathcal{W}u^* = \mathcal{Y}u^* = \mathcal{Z}u^* = u^*$, and consequently, u^* is a unique solution of (4.1). □

5 Conclusion

In this work, we could improve the main results of Bennani et al. [6], and Jha and Panthi [15] by removing some conditions. In other words, we could find unique common fixed points with neither continuity nor completeness and inclusions, under the new concept of occasionally weakly biased maps of type $(\mathcal{A})$.

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Author information

H. Bouhadjera, Laboratory of Applied Mathematics, Badji Mokhtar-Annaba University, P.O. Box 12, 23000 Annaba, Algeria.
E-mail: b_hakima2000@yahoo.fr

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