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EUCLIDEAN OPERATOR RADIUS INEQUALITIES FOR HILBERT C^*-MODULES

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Abstract In this paper, we present a new method for studying the Euclidean operator radius of two adjointable operators on Hilbert C^*-modules. Our method enables us to obtain some new results and generalize some known theorems for bounded operators on Hilbert spaces to two bounded adjointable operators on Hilbert C^*-module spaces.

1 Introduction

Let $(H; \langle .,. \rangle)$ be a complex Hilbert space and denote by $B(H)$ the set of all bounded linear operators on H. The numerical radius of $T \in B(H)$ is defined by [1]:

$$w(T) = \sup\{|\langle Tx, x\rangle| : \ x \in H, \ \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(.)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\|.\|$. In fact, for any $T \in B(H)$,

$$w(T) \leqslant \|T\| \leqslant 2w(T). \tag{1.2}$$

Kittaneh proved that for any $T \in B(H)$,

$$w(T) \leqslant \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}}), \tag{1.3}$$

and

$$\frac{1}{4}\|T^*T + TT^*\| \leqslant w^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|. \tag{1.4}$$

The above inequalities can be found in [2, 3], respectively. For other results on the numerical radius (see [4], [5], [2], [6], [7]).
Let (B, C) be a pair of bounded linear operators on H. The Euclidean operator radius is defined by [8]:

$$w_e(B, C) = \sup\left\{\left(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right)^{\frac{1}{2}} : \ x \in H, \ \|x\| = 1\right\}. \tag{1.5}$$

As pointed out in [9], the following inequality holds:

$$\frac{\sqrt{2}}{4}\|C^*C + D^*D\|^{\frac{1}{2}} \leqslant w_e(C, D) \leqslant \|C^*C + D^*D\|^{\frac{1}{2}}. \tag{1.6}$$

There are many inequalities involving the Euclidean operator radius (see [8], [9], [10]).
By a Hilbert C^*-module, we mean a linear space with an inner product that takes values in a C^*-algebra. This idea initially arose in a paper by Kaplansky (see [13]), who created the theory for commutative unital algebras. Paschke (see [14]) and Rieffel (see [15]) expanded the theory to

include general C^*-algebras. For further details (see [16], [11], [12]). The different structure of Hilbert C^*-modules makes it appear that different definitions of some concepts, which are natural extensions of some standard definitions, are required for studying some inequalities in Hilbert C^*-modules, even though it is possible to prove some inequalities in Hilbert C^*-module spaces using standard methods. Our new definitions of the Euclidean operator radius and numerical radius for bounded adjointable operators on Hilbert C^*-modules are the natural extensions of these concepts to operators on Hilbert spaces and they appear in this work. We establish some basic inequalities in the operational radius of adjointable bounded operators on Hilbert C^*-modules, using these definitions and specialized methods.
We recall some fundamental definitions in the theory of Hilbert modules that will be used in this paper.

Definition 1.1. ([17]). Let A be a C^*-algebra. An inner-product A-module is a linear space E which is a right A-module with compatible scalar multiplication:

$$\lambda(xa) = (\lambda x)a = x(\lambda a) \text{ for all } x \in E,\, a \in A \text{ and } \lambda \in \mathbb{C}$$

together with a map $\langle .,. \rangle : E \times E \longrightarrow A$, which has the following properties:

(i) $\langle x, x\rangle \geqslant 0$, if $\langle x, x\rangle = 0$ then $x = 0$,

(ii) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$,

(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,

(iv) $\langle x, y\rangle^* = \langle y, x\rangle$,

for all $x, y, z \in E$, $a \in A$, $\alpha, \beta \in \mathbb{C}$.

We can define a norm on E by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. An inner-product A-module that is complete concerning its norm is called a Hilbert A-module or a Hilbert C^*-module over the C^*-algebra A. We define $\mathcal{L}(E)$ which is a C^*-algebra to be the set of all maps $T : E \longrightarrow E$ for which there is a map $T^* : E \longrightarrow E$ which satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in E$. Let $\mathcal{L}^{-1}(E)$ denote the set of all invertible operators in $\mathcal{L}(E)$.

Definition 1.2. ([12, page 89]). A state on a C^*-algebra A is a positive linear functional on A of norm one. We denote the state space of A by $S(A)$.

Definition 1.3. ([18]). Let E is a Hilbert right A-module. We define the numerical radius of $T \in \mathcal{L}(E)$ by

$$w_A(T) = \sup\{|\varrho\langle x, Tx\rangle| : \ x \in E,\ \varrho \in S(A),\ \varrho\langle x, x\rangle = 1\}. \tag{1.7}$$

In fact, in this case, the C^*-algebra A is the set of complex numbers and $S(A)$ contains only the identity function on the set of complex numbers.
Moreover, we assume that A is a C^*-algebra and E is an inner product A-module.
In order to drive our main results, we need the following lemmas:

Lemma 1.4. *([18]). $w_A(T) = \|T\|$ for every self-adjoint element of $\mathcal{L}(E)$.*

Lemma 1.5. *([14]). For $T \in \mathcal{L}(E)$, we have $\langle Tx, Tx\rangle \leqslant \|T\|^2\langle x, x\rangle$ for every $x \in E$.*

Lemma 1.6. *([12, page 88, Theorem 3.3.2]). Let A be a C^*-algebra. If ϱ is a positive linear functional on A, then*

$$\varrho(a^*) = \overline{\varrho(a)}, \quad for \ \ all \ \ a \in A.$$

Lemma 1.7. *([18]). Let $T \in \mathcal{L}(E)$ and $\varrho \in S(A)$. The following statements are equivalent:*
a) $\varrho\langle x, Tx\rangle = 0$ for every $x \in E$ with $\varrho\langle x, x\rangle = 1$,
b) $\varrho\langle x, Tx\rangle = 0$ for every $x \in E$.

Lemma 1.8. *([18]). Let $T \in \mathcal{L}(E)$, then $T = 0$ if and only if $\varrho\langle x, Tx\rangle = 0$ for every $x \in E$ and $\varrho \in S(A)$.*

For $T \in \mathcal{L}(E)$, then T is self-adjoint if and only $\varrho\langle x, Tx\rangle$ is positive for every $x \in E$ and $\varrho \in S(A)$.

Lemma 1.9. *([20]). For $a, b \geq 0$ and $0 \leq \alpha \leq 1$,*

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}} \quad for \ \ r \geq 1.$$

Lemma 1.10. *([18]). Let $T \in \mathcal{L}(E)$, $T \geq 0$ and $x \in E$, then for every $\varrho \in S(A)$*

(i) $(\varrho\langle x, Tx\rangle)^r \leq \|x\|^{2(1-r)}\varrho\langle x, T^r x\rangle \quad for \ \ r \geq 1,$

(ii) $(\varrho\langle x, Tx\rangle)^r \geq \|x\|^{2(1-r)}\varrho\langle x, T^r x\rangle \quad for \ \ 0 < r \leq 1.$

Lemma 1.11. *([18, page 12]). Let $a, b, e \in E$ and $\varrho \in S(A)$ with $\varrho\langle e, e\rangle = 1$, then*

$$|\varrho\langle a, e\rangle\varrho\langle e, b\rangle| \leqslant \frac{1}{2}(\varrho\langle a, a\rangle^{\frac{1}{2}}\varrho\langle b, b\rangle^{\frac{1}{2}} + |\varrho\langle a, b\rangle|).$$

Lemma 1.12. *([19, Cauchy-Schwarz inequality]). Let $T \in B(H)$ and $0 \leq \alpha \leq 1$, then*

$$|\langle x, Ty\rangle|^2 \leq \langle x, |T|^{2\alpha}x\rangle\langle y, |T^*|^{2(1-\alpha)}y\rangle,$$

for all any $x, y \in H$.

The following result is a consequence of Lemma 1.12.

Corollary 1.13. *Let $x \in E$ and $\varrho \in S(A)$, $\varrho\langle ., .\rangle$ is a semi-inner product. Suppose that $T \in \mathcal{L}(E)$ and $0 \leq \alpha \leq 1$, then*

$$|\varrho\langle x, Ty\rangle|^2 \leq \varrho\langle x, |T|^{2\alpha}x\rangle\varrho\langle y, |T^*|^{2(1-\alpha)}y\rangle,$$

for all any $x, y \in E$.

In this section, we provide a new definition of the Euclidean operator radius for bounded adjointable operators on Hilbert C^*-modules, which are of course the natural generalizations of this concept to operators on Hilbert spaces. By using this definition and applying special techniques, we prove some fundamental inequalities in the Euclidean operator radius of two adjointable bounded operators on Hilbert C^*-modules.

2 Main results

We start with this definition.

Definition 2.1. Suppose that E is a Hilbert right A-module. We define the Euclidean operator radius of $B, C \in \mathcal{L}(E)$ by

$$w_e(B, C) = \sup\left\{\left(|\varrho\langle x, Bx\rangle|^2 + |\varrho\langle x, Cx\rangle|^2\right)^{\frac{1}{2}} : \ x \in E, \ \varrho \in S(A), \ \varrho\langle x, x\rangle = 1\right\}. \quad (2.1)$$

Note that our definition is a natural extension of the definition of Euclidean operator radius of two bounded operators on Hilbert spaces.

Lemma 2.2. *Let $B, C \in \mathcal{L}(E)$, then for every $x \in E$ and $\varrho \in S(A)$*

$$\left(|\varrho\langle x, Bx\rangle|^2 + |\varrho\langle x, Cx\rangle|^2\right)^{\frac{1}{2}} \leqslant w_e(B, C)\varrho\langle x, x\rangle.$$

Theorem 2.3. *The Euclidean operator radius $w_e(., .) : \mathcal{L}(E)\times\mathcal{L}(E) \to [0, \infty)$ for two operators satisfies the following properties:*

(i) $w_e(B, C) = 0$ *if and only if* $B = C = 0$,

(ii) $w_e(\lambda B, \lambda C) = |\lambda|w_e(B, C)$ *for any* $\lambda \in \mathbb{C}$,

(iii) $w_e(B + V, C + T) \leqslant w_e(B, C) + w_e(V, T)$,

(iv) $w_e(U^*BU, U^*CU) = w_e(B,C)$ *for any unitary operator* $U : E \longrightarrow E$,

(v) $w_e(T^*BT, T^*CT) \leqslant \|T\|^2 w_e(B,C)$ *for any operator* $T : E \longrightarrow E$,

(vi) $w_e^2(B,B) = 2w_A^2(B)$,

(vii) If $B \in \mathcal{L}(E)$ *and* $B = C + iD$ *is the Cartesian decomposition of* B*, then* $w_e^2(C,D) = w_A^2(B)$.

Proof. The first seven properties can be easily deduced using the definition of w_e.
Let $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle Tx, Tx\rangle \neq 0$. Then

$$\varrho\left\langle \frac{Tx}{(\varrho\langle Tx,Tx\rangle)^{\frac{1}{2}}}, \frac{Tx}{(\varrho\langle Tx,Tx\rangle)^{\frac{1}{2}}} \right\rangle = 1,$$

so that

$$\left(\left|\varrho\left\langle \frac{Tx}{(\varrho\langle Tx,Tx\rangle)^{\frac{1}{2}}}, B\left(\frac{Tx}{(\varrho\langle Tx,Tx\rangle)^{\frac{1}{2}}}\right)\right\rangle\right|^2 + \left|\varrho\left\langle \frac{Tx}{(\varrho\langle Tx,Tx\rangle)^{\frac{1}{2}}}, C\left(\frac{Tx}{(\varrho\langle Tx,Tx\rangle)^{\frac{1}{2}}}\right)\right\rangle\right|^2\right)^{\frac{1}{2}} \leqslant w_e(B,C).$$

Hence

$$\left(|\varrho\langle Tx, BTx\rangle|^2 + |\varrho\langle Tx, CTx\rangle|^2\right)^{\frac{1}{2}} \leqslant w_e(B,C)\varrho\langle Tx,Tx\rangle.$$

By Lemma 1.5,

$$\left(|\varrho\langle x, T^*BTx\rangle|^2 + |\varrho\langle x, T^*CTx\rangle|^2\right)^{\frac{1}{2}} \leqslant w_e(B,C)\|T\|^2\varrho\langle x,x\rangle.$$

By taking supremum over $\varrho\langle x,x\rangle = 1$,

$$w_e(T^*BT, T^*CT) \leqslant \|T\|^2 w_e(B,C).$$

Let $\varrho\langle Tx,Tx\rangle = 0$. By the Cauchy-Schwarz inequality, we have

$$|\varrho\langle x, T^*BTx\rangle|^2 \leqslant \varrho\langle Tx,Tx\rangle\varrho\langle BTx, BTx\rangle.$$

It follows that $|\varrho\langle x, T^*BTx\rangle| = |\varrho\langle x, T^*CTx\rangle| = 0$.
Therefore, we deduce (v). □

We can now generalize (1.6) for Hilbert C^*-modules.

Theorem 2.4. *If* $C, D \in \mathcal{L}(E)$*, then*

$$\frac{1}{2}\|C^*C + D^*D\|^{\frac{1}{2}} \leqslant w_e(C,D) \leqslant \|C^*C + D^*D\|^{\frac{1}{2}}, \tag{2.2}$$

where the constants $\frac{1}{2}$ *and* 1 *are best possible in* (2.2).

Proof. Let $C, D \in \mathcal{L}(E)$. There are self-adjoint elements $a, b, c, d \in \mathcal{L}(E)$ such that $C = a + ib$ and $D = c + id$. For every vector $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x,x\rangle = 1$.
Since $\varrho\langle x, (a+b+c+d)x\rangle \in \mathbb{R}$, then $\left(\varrho\langle x, (a+b+c+d)x\rangle\right)^2 = |\varrho\langle x, (a+b+c+d)x\rangle|^2$.

We have

$$\begin{aligned}
|\varrho\langle x,(a+b+c+d)x\rangle|^2 &= \Big(\varrho\langle x,(a+b+c+d)x\rangle\Big)^2\\
&= \Big(\varrho\langle x,(a+b)x\rangle+\varrho\langle x,(c+d)x\rangle\Big)^2\\
&\leqslant 2\Big(\varrho\langle x,(a+b)x\rangle^2+\varrho\langle x,(c+d)x\rangle^2\Big)\\
&= 2\Big(\varrho\langle x,ax\rangle+\varrho\langle x,bx\rangle\Big)^2+2\Big(\varrho\langle x,cx\rangle+\varrho\langle x,dx\rangle\Big)^2\\
&\leqslant 4\Big(\varrho\langle x,ax\rangle^2+\varrho\langle x,bx\rangle^2\Big)+4\Big(\varrho\langle x,cx\rangle^2+\varrho\langle x,dx\rangle^2\Big)\\
&= 4\Big(|\varrho\langle x,(a+ib)x\rangle|^2+|\varrho\langle x,(c+id)x\rangle|^2\Big)\\
&= 4\Big(|\varrho\langle x,Cx\rangle|^2+|\varrho\langle x,Dx\rangle|^2\Big).
\end{aligned}$$

By taking supremum over all $x\in E$ and $\varrho\in S(A)$ with $\varrho\langle x,x\rangle=1$, we have

$$w_A^2(a+b+c+d)\leqslant 4w_e^2(C,D).$$

By lemma 1.4, we have $w_A(a+b+c+d)=\|a+b+c+d\|$. Thus

$$\|a+b+c+d\|^2\leqslant 4w_e^2(C,D).$$

Since

$$\begin{aligned}
\varrho\langle x,(a+b-c-d)x\rangle^2 &= \Big(\varrho\langle x,(a+b)x\rangle-\varrho\langle x,(c+d)x\rangle\Big)^2\\
&\leqslant \Big(|\varrho\langle x,(a+b)x\rangle|+|\varrho\langle x,(c+d)x\rangle|\Big)^2\\
&\leqslant 2\Big(\varrho\langle x,(a+b)x\rangle^2+\varrho\langle x,(c+d)x\rangle^2\Big)\\
&= 2\Big(\varrho\langle x,ax\rangle+\varrho\langle x,bx\rangle\Big)^2+2\Big(\varrho\langle x,cx\rangle+\varrho\langle x,dx\rangle\Big)^2\\
&\leqslant 4\Big(\varrho\langle x,ax\rangle^2+\varrho\langle x,bx\rangle^2\Big)+4\Big(\varrho\langle x,cx\rangle^2+\varrho\langle x,dx\rangle^2\Big)\\
&= 4\Big(|\varrho\langle x,(a+ib)x\rangle|^2+|\varrho\langle x,(c+id)x\rangle|^2\Big)\\
&= 4\Big(|\varrho\langle x,Cx\rangle|^2+|\varrho\langle x,Dx\rangle|^2\Big).
\end{aligned}$$

By taking supremum over all $x\in E$ and $\varrho\in S(A)$ with $\varrho\langle x,x\rangle=1$, we have

$$w_A^2(a+b-c-d)\leqslant 4w_e^2(C,D).$$

By lemma 1.4, we have $w_A(a+b-c-d)=\|a+b+c+d\|$. Thus

$$\|a+b-c-d\|^2\leqslant 4w_e^2(C,D).$$

Since

$$\begin{aligned}
\varrho\langle x,(a-b+c-d)x\rangle^2 &= \Big(\varrho\langle x,(a-b)x\rangle+\varrho\langle x,(c-d)x\rangle\Big)^2\\
&\leqslant 2\Big(\varrho\langle x,(a-b)x\rangle^2+\varrho\langle x,(c-d)x\rangle^2\Big)\\
&= 2\Big(\varrho\langle x,ax\rangle-\varrho\langle x,bx\rangle\Big)^2+2\Big(\varrho\langle x,cx\rangle-\varrho\langle x,dx\rangle\Big)^2\\
&\leqslant 2\Big(|\varrho\langle x,ax\rangle|+|\varrho\langle x,bx\rangle|\Big)^2+2\Big(|\varrho\langle x,cx\rangle|+|\varrho\langle x,dx\rangle|\Big)^2\\
&\leqslant 4\Big(\varrho\langle x,ax\rangle^2+\varrho\langle x,bx\rangle^2\Big)+4\Big(\varrho\langle x,cx\rangle^2+\varrho\langle x,dx\rangle^2\Big)\\
&= 4\Big(|\varrho\langle x,(a+ib)x\rangle|^2+|\varrho\langle x,(c+id)x\rangle|^2\Big)\\
&= 4\Big(|\varrho\langle x,Cx\rangle|^2+|\varrho\langle x,Dx\rangle|^2\Big).
\end{aligned}$$

By taking supremum over all $x\in E$ and $\varrho\in S(A)$ with $\varrho\langle x,x\rangle=1$, we have

$$w_A^2(a-b+c-d)\leqslant 4w_e^2(C,D).$$

By lemma 1.4, we have $w_A(a-b+c-d)=\|a-b+c-d\|$.
Thus

$$\|a-b+c-d\|^2\leqslant 4w_e^2(C,D).$$

Replacing c by d in the above inequality, we obtain that

$$\|a-b+d-c\|^2\leqslant 4w_e^2(C,D).$$

Moreover,

$$\begin{aligned}
16w_e^2(C,D) &\geqslant \|(a+b+c+d)^2+(a+b-c-d)^2+(a-b+c-d)^2+(a-b+d-c)^2\|\\
&\geqslant 4\|a^2+b^2+c^2+d^2\|.
\end{aligned}$$

Therefore,

$$\frac{1}{4}\|C^*C+D^*D\|\leqslant w_e^2(C,D).$$

For every vector $x\in E$ and $\varrho\in S(A)$ with $\varrho\langle x,x\rangle=1$, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|\varrho\langle x,Cx\rangle|^2+|\varrho\langle x,Dx\rangle|^2 &\leqslant \varrho\langle x,x\rangle\varrho\langle Cx,Cx\rangle+\varrho\langle x,x\rangle\varrho\langle Dx,Dx\rangle\\
&\leqslant \varrho\langle x,C^*Cx\rangle+\varrho\langle x,D^*Dx\rangle\\
&\leqslant \varrho\langle x,(C^*C+D^*D)x\rangle\\
&\leqslant \|C^*C+D^*D\|.
\end{aligned}$$

By taking supremum over all $\varrho\langle x,x\rangle=1$, we have

$$w_e^2(C,D)\leqslant\|C^*C+D^*D\|.$$

□

Remark 2.5. (i) The lower bound of $w_e(B,C)$ obtained in Theorem 2.4 is stronger than the lower bound in (1.6).

(ii) If we take $C = D$ in (2.2) and Theorem 2.3 (vi), then we get the following upper bound (see [18]) for the numerical radius of a bounded linear operator C on $\mathcal{L}(E)$:

$$\frac{1}{2}\|C\| \leqslant w_A(C) \leqslant \|C\|.$$

(iii) We observe that, if C and D are self-adjoint operators, then (2.2) becomes

$$\frac{1}{2}\|C^2 + D^2\|^{\frac{1}{2}} \leqslant w_e(C, D) \leqslant \|C^2 + D^2\|^{\frac{1}{2}}. \tag{2.3}$$

(iv) If $C = B + iD$ is the cartesian decomposition of C, then

$$C^*C + CC^* = 2(B^2 + D^2),$$

by the inequality (2.3) and Theorem 2.3 (vii), then we have

$$\frac{1}{8}\|C^*C + CC^*\| \leqslant w_A^2(C) \leqslant \frac{1}{2}\|C^*C + CC^*\|.$$

Theorem 2.6. *For any $B, C \in \mathcal{L}(E)$ and $r \geqslant 1$, we have the inequality:*

$$w_e^2(B, C) \leqslant w_A^2(B - C) + 2^{-\frac{1}{r}}\|(CC^*)^r + (B^*B)^r\|^{\frac{1}{r}} + w_A(CB). \tag{2.4}$$

Proof. Let $B, C \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle = 1$,

$$\begin{aligned}
|\varrho\langle x, Bx\rangle|^2 - 2Re(\varrho\langle x, Bx\rangle\overline{\varrho\langle x, Cx\rangle}) + |\varrho\langle x, Cx\rangle|^2 &= |\varrho\langle x, Bx\rangle - \varrho\langle x, Cx\rangle|^2 \\
&= |\varrho\langle x, (B - C)x\rangle|^2 \\
&\leqslant w_A^2(B - C).
\end{aligned}$$

Thus,

$$\begin{aligned}
|\varrho\langle x, Bx\rangle|^2 + |\varrho\langle x, Cx\rangle|^2 &\leqslant w_A^2(B - C) + 2Re\Big(\varrho\langle x, Bx\rangle\overline{\varrho\langle x, Cx\rangle}\Big) \\
&\leqslant w_A^2(B - C) + 2|\varrho\langle x, Bx\rangle\varrho\langle x, Cx\rangle| \\
&= w_A^2(B - C) + 2|\varrho\langle C^*x, x\rangle\varrho\langle x, Bx\rangle| \\
&\leqslant w_A^2(B - C) + (\varrho\langle C^*x, C^*x\rangle)^{\frac{1}{2}}(\varrho\langle Bx, Bx\rangle)^{\frac{1}{2}} + |\varrho\langle C^*x, Bx\rangle| \\
&\quad (by\ Lemma\ 1.11) \\
&\leqslant w_A^2(B - C) + \left(\frac{1}{2}\varrho\langle x, CC^*x\rangle^r + \frac{1}{2}\varrho\langle x, B^*Bx\rangle^r\right)^{\frac{1}{r}} + |\varrho\langle x, CBx\rangle| \\
&\quad (by\ Lemma\ 1.9) \\
&\leqslant w_A^2(B - C) + 2^{-\frac{1}{r}}\left(\varrho\langle x, (CC^*)^r x\rangle + \varrho\langle x, (B^*B)^r x\rangle\right)^{\frac{1}{r}} + |\varrho\langle x, CBx\rangle| \\
&\quad (by\ Lemma\ 1.10) \\
&\leqslant w_A^2(B - C) + 2^{-\frac{1}{r}}\left(\varrho\langle x, (CC^*)^r + (B^*B)^r x\rangle\right)^{\frac{1}{r}} + |\varrho\langle x, CBx\rangle| \\
&\leqslant w_A^2(B - C) + 2^{-\frac{1}{r}}\|(CC^*)^r + (B^*B)^r\|^{\frac{1}{r}} + w_A(CB).
\end{aligned}$$

By taking supremum over all $\varrho\langle x, x\rangle = 1$, we have

$$w_e^2(B, C) \leqslant w_A^2(B - C) + 2^{-\frac{1}{r}}\|(CC^*)^r + (B^*B)^r\|^{\frac{1}{r}} + w_A(CB).$$

□

Remark 2.7. If we take $B = C$ and $r = 1$ in (2.4) and Theorem 2.3 (vi), we get the following upper bound (see [18]) for the numerical radius of a bounded linear operator B on $\mathcal{L}(E)$:

$$w_A^2(B) \leqslant \frac{1}{2}(w_A(B^2) + \|B\|^2).$$

Theorem 2.8. *For any $B, C \in \mathcal{L}(E)$ and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have the inequality:*

$$w_e^2(B, C) \leqslant \left\| |B|^p + |C|^p \right\|^{\frac{1}{p}} . \left\| |B^*|^q + |C^*|^q \right\|^{\frac{1}{q}} . \tag{2.5}$$

Proof. Let $B, C \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned} |\varrho\langle x, Bx\rangle|^2 + |\varrho\langle x, Cx\rangle|^2 &\leqslant \varrho\langle x, |B|x\rangle \varrho\langle x, |B^*|x\rangle + \varrho\langle x, |C|x\rangle \varrho\langle x, |C^* \mid x\rangle \;\; (Corollary\ 1.13) \\ &\leqslant \left(\varrho\langle x, |B|x\rangle^p + \varrho\langle x, |C|x\rangle^p\right)^{\frac{1}{p}} . \left(\varrho\langle x, |B^*|x\rangle^q + \varrho\langle x, |C^*|x\rangle^q\right)^{\frac{1}{q}} \\ &\leqslant \left(\varrho\langle x, |B|^p x\rangle + \varrho\langle x, |C|^p x\rangle\right)^{\frac{1}{p}} . \left(\varrho\langle x, |B^*|^q x\rangle + \varrho\langle x, |C^*|^q x\rangle\right)^{\frac{1}{q}} \\ &\leqslant \left(\varrho\langle x, (|B|^p + |C|^p)x\rangle\right)^{\frac{1}{p}} . \left(\varrho\langle x, (|B^*|^q + |C^*|^q)x\rangle\right)^{\frac{1}{q}} \\ &\leqslant \left\| |B|^p + |C|^p \right\|^{\frac{1}{p}} . \left\| |B^*|^q + |C^*|^q \right\|^{\frac{1}{q}} \end{aligned}$$

By taking supremum over all $\varrho\langle x, x\rangle = 1$, we have

$$w_e^2(B, C) \leqslant \left\| |B|^p + |C|^p \right\|^{\frac{1}{p}} . \left\| |B^*|^q + \||C^*|^q \right\|^{\frac{1}{q}} .$$

□

Lemma 2.9. *([19]). Let $T \in B(H)$, and f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\varrho\langle x, Ty\rangle| \leqslant \|f(|T|)x\| \|g(|T^*|)y\|,$$

for all any $x, y \in H$.

The following result is a consequence of Lemma 2.9.

Corollary 2.10. *For $\varrho \in S(A)$, $\varrho\langle ., .\rangle$ is a semi-inner product. Suppose that $T \in \mathcal{L}(E)$, and f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\varrho\langle x, Ty\rangle| \leqslant \varrho\langle f(|T|)x, f(|T|)x\rangle^{\frac{1}{2}} \varrho\langle g(|T^*|)y, g(|T^*|)y\rangle^{\frac{1}{2}},$$

for all any $x, y \in E$.

The above results enable us to state the following.

Theorem 2.11. *Let $B, T, C, D, S, M \in \mathcal{L}(E)$ and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$\begin{aligned} w_e^{2r}(B^*TC, D^*SM) \leqslant 2^{r-1} \Big\| &\alpha\left(B^* f^2(|T|)B\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(C^* g^2(|T^*|)C\right)^{\frac{r}{1-\alpha}} \\ &+ \alpha\left(D^* f^2(|S|)D\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(M^* g^2(|S^*|)M\right)^{\frac{r}{1-\alpha}} \Big\|, \end{aligned} \tag{2.6}$$

for $r \geqslant 1$ and $0 < \alpha < 1$.

Proof. For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned}
|\varrho\langle x, B^*TCx\rangle|^2 &= |\varrho\langle Bx, TCx\rangle|^2 \\
&\leqslant \varrho\langle f(|T|)Bx, f(|T|)Bx\rangle \varrho\langle g(|T^*|)Cx, g(|T^*|)Cx\rangle \\
&\quad (by\ Corollary\ 2.10) \\
&= \varrho\langle x, B^* f^2(|T|)Bx\rangle \varrho\langle x, C^* g^2(|T^*|)Cx\rangle \\
&= \varrho\langle x, \left((B^* f^2(|T|)B)^{\frac{1}{\alpha}}\right)^{\alpha} x\rangle \varrho\langle x, \left((C^* g^2(|T^*|)C)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} x\rangle \\
&\leqslant \varrho\langle x, \left(B^* f^2(|T|)B\right)^{\frac{1}{\alpha}} x\rangle^{\alpha} \varrho\langle x, \left(C^* g^2(|T^*|)C\right)^{\frac{1}{1-\alpha}} x\rangle^{1-\alpha} \\
&\quad (by\ Lemma\ 1.10) \\
&\leqslant \left(\alpha\varrho\langle x, \left(B^* f^2(|T|)B\right)^{\frac{1}{\alpha}} x\rangle^{r} + (1-\alpha)\varrho\langle x, \left(C^* g^2(|T^*|)C\right)^{\frac{1}{1-\alpha}} x\rangle^{r}\right)^{\frac{1}{r}} \\
&\quad (by\ Lemma\ 1.9) \\
&\leqslant \left(\alpha\varrho\langle x, \left(B^* f^2(|T|)B\right)^{\frac{r}{\alpha}} x\rangle + (1-\alpha)\varrho\langle x, \left(C^* g^2(|T^*|)C\right)^{\frac{r}{1-\alpha}} x\rangle\right)^{\frac{1}{r}} \\
&\quad (by\ Lemma\ 1.10).
\end{aligned}$$

Thus,

$$|\varrho\langle x, B^*TCx\rangle|^{2r} \leqslant \varrho\langle x, (\alpha\left(B^* f^2(|T|)B\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(C^* g^2(|T^*|)C\right)^{\frac{r}{1-\alpha}})x\rangle,$$

by convexity of the function $f(t) = t^r$ on $[0, \infty)$, we have

$$\begin{aligned}
\left(|\varrho\langle x, B^*TCx\rangle|^2 + |\varrho\langle x, D^*SMx\rangle|^2\right)^{r} &\leqslant 2^{r-1}\left(|\varrho\langle x, B^*TCx\rangle|^{2r} + |\varrho\langle x, D^*SMx\rangle|^{2r}\right) \\
&\leqslant 2^{r-1}\Big[\varrho\langle x, [\alpha\left(B^* f^2(|T|)B\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(C^* g^2(|T^*|)C\right)^{\frac{r}{1-\alpha}} \\
&\quad + \alpha\left(D^* f^2(|S|)D\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(M^* g^2(|S^*|)M\right)^{\frac{r}{1-\alpha}}]x\rangle\Big]. \quad (2.7)
\end{aligned}$$

Now, taking the supremum over all $x \in E$ with $\varrho\langle x, x\rangle = 1$, we obtain

$$\begin{aligned}
w_e^{2r}(B^*TC, D^*SM) \leqslant 2^{r-1}\Big\|&\alpha\left(B^* f^2(|T|)B\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(C^* g^2(|T^*|)C\right)^{\frac{r}{1-\alpha}} \\
&+ \alpha\left(D^* f^2(|S|)D\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(M^* g^2(|S^*|)M\right)^{\frac{r}{1-\alpha}}\Big\|.
\end{aligned}$$

□

Choosing $B = C = D = M = I$ and $\alpha = \frac{1}{2}$, wet get:

Corollary 2.12. *Let $T, C, S, M \in \mathcal{L}(E)$ and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$w_e^{2r}(T, S) \leqslant 2^{r-2}\left\|f^{4r}(|T|) + g^{4r}(|T^*|) + f^{4r}(|S|) + g^{4r}(|S^*|)\right\|, \quad (2.8)$$

for $r \geqslant 1$.

Letting $f(t) = g(t) = t^{\frac{1}{2}}$, wet get:

Corollary 2.13. *Let* $B,T,C,D,S,M \in \mathcal{L}(E)$. *Then*

$$w_e^{2r}(B^*TC, D^*SM) \leqslant 2^{r-1}\Big\|\alpha(B^*|T|B)^{\frac{r}{\alpha}} + (1-\alpha)(C^*|T^*|C)^{\frac{r}{1-\alpha}} + \alpha(D^*|S|D)^{\frac{r}{\alpha}} + (1-\alpha)(M^*|S^*|M)^{\frac{r}{1-\alpha}}\Big\|, \quad (2.9)$$

for $r \geqslant 1$ *and* $0 < \alpha < 1$.

Corollary 2.14. *Let* $B,C,D,M \in \mathcal{L}(E)$. *Then*

$$w_e^{2r}(B^*C, D^*M) \leqslant 2^{r-1}\Big\|\alpha(B^*B)^{\frac{r}{\alpha}} + (1-\alpha)(C^*C)^{\frac{r}{1-\alpha}} + \alpha(D^*D)^{\frac{r}{\alpha}} + (1-\alpha)(M^*M)^{\frac{r}{1-\alpha}}\Big\|, \quad (2.10)$$

for $r \geqslant 1$ *and* $0 < \alpha < 1$.

Choosing $B = D$ and $C = M$, we get:

Corollary 2.15. *Let* $B,C \in \mathcal{L}(E)$. *Then*

$$w_e^{2r}(B^*C, B^*C) \leqslant 2^{r}\Big\|\alpha(B^*B)^{\frac{r}{\alpha}} + (1-\alpha)(C^*C)^{\frac{r}{1-\alpha}}\Big\|, \quad (2.11)$$

for $r \geqslant 1$ *and* $0 < \alpha < 1$.

In particular, if we choose $w_e^{2r}(B^*C, B^*C) = 2^r w_A^{2r}(B^*C)$, we have

$$w_A^{2r}(B^*C) \leqslant \Big\|\alpha(B^*B)^{\frac{r}{\alpha}} + (1-\alpha)(C^*C)^{\frac{r}{1-\alpha}}\Big\|, \quad (2.12)$$

for $r \geqslant 1$ and $0 < \alpha < 1$.
We remark that, in [5] Dragomir, has proved the inequality (2.12) in a Hilbert space.
In this theorem, we show that the previous findings enable us to generalize some results about the Euclidean operator radius of the operators on Hilbert spaces to the Euclidean operator radius on Hilbert C^*-modules.

Theorem 2.16. *Let* $B,C \in \mathcal{L}(E)$, $r \geqslant 1$ *and* $0 \leqslant \alpha \leqslant 1$, *then*

$$w_e^{2r}(B,C) \leqslant 2^{r-1}\Big\|\alpha|B|^{2r} + (1-\alpha)|B^*|^{2r} + \alpha|C|^{2r} + (1-\alpha)|C^*|^{2r}\Big\|. \quad (2.13)$$

Proof. For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x,x\rangle = 1$, we have

$$\begin{aligned} |\varrho\langle x,Bx\rangle|^2 &\leqslant \varrho\langle x,|B|^{2\alpha}x\rangle\varrho\langle x,|B^*|^{2(1-\alpha)}x\rangle \ \ (by\ Corollary\ 1.13) \\ &\leqslant \varrho\langle x,|B|^2x\rangle^{\alpha}\varrho\langle x,|B^*|^2x\rangle^{1-\alpha} \ \ (by\ Lemma\ 1.10) \\ &\leqslant \left(\alpha\varrho\langle x,|B|^2x\rangle^r + (1-\alpha)\varrho\langle x,|B^*|^2x\rangle^r\right)^{\frac{1}{r}} \ \ (by\ Lemma\ 1.9) \\ &\leqslant \left(\alpha\varrho\langle x,|B|^{2r}x\rangle + (1-\alpha)\varrho\langle x,|B^*|^{2r}x\rangle\right)^{\frac{1}{r}} \ \ (by\ Lemma\ 1.10) \\ &= \left(\varrho\langle x,(\alpha|B|^{2r} + (1-\alpha)|B^*|^{2r})x\rangle\right)^{\frac{1}{r}}. \end{aligned}$$

Thus,

$$|\varrho\langle x, Bx\rangle|^{2r} \leqslant \varrho\langle x, (\alpha|B|^{2r} + (1-\alpha)|B^*|^{2r})x\rangle,$$

by the convexity of the function $f(t) = t^r$ on $[0, \infty)$, we have

$$\begin{aligned}(|\varrho\langle x, Bx\rangle|^2 + |\varrho\langle x, Cx\rangle|^2)^r &\leqslant 2^{r-1}(|\varrho\langle x, Bx\rangle|^{2r} + |\varrho\langle x, Cx\rangle|^{2r}) \\ &\leqslant 2^{r-1}\varrho\langle x, (\alpha|B|^{2r} + (1-\alpha)|B^*|^{2r} + \alpha|C|^{2r} + (1-\alpha)|C^*|^{2r})x\rangle.\end{aligned}$$

By taking supremum over all $\varrho\langle x, x\rangle = 1$, we have

$$w_e^{2r}(B, C) \leqslant 2^{r-1}\left\|\alpha|B|^{2r} + (1-\alpha)|B^*|^{2r} + \alpha|C|^{2r} + (1-\alpha)|C^*|^{2r}\right\|. \tag{2.14}$$

□

Using this theorem we give the following corollary.

Corollary 2.17. *If $B \in \mathcal{L}(E)$, $r \geqslant 1$ and $0 \leqslant \alpha \leqslant 1$, then*

$$w_A^{2r}(B) \leqslant \left\|\alpha|B|^{2r} + (1-\alpha)|B^*|^{2r}\right\|. \tag{2.15}$$

Proof. If in Theorem 2.16, we choose $B = C$, then by Theorem 2.3 (vi) we get

$$w_e^{2r}(B, B) = 2^r w_A^{2r}(B),$$

which implies the desired result. □

In particular, if we choose $r = 1$, $\alpha = \frac{1}{2}$, we have

$$w_A^2(B) \leqslant \frac{1}{2}\left\|BB^* + B^*B\right\|. \tag{2.16}$$

Corollary 2.18. *Let $B = C + iD$ be the Cartesian decomposition of B and $r \geqslant 1$. Then*

$$w_A^{2r}(B) \leqslant 2^{r-1}\left\||C|^{2r} + |D|^{2r}\right\|. \tag{2.17}$$

The proof is obvious by the inequality (2.13) on choosing $\alpha = \frac{1}{2}$ and by Theorem 2.3 (vii).

Theorem 2.19. *For any $B, C \in \mathcal{L}(E)$ and $0 \leqslant \alpha \leqslant 1$, we have*

$$w_e(B, C) \leqslant \frac{1}{2}\left\|(|B|^{2\alpha} + |B^*|^{2(1-\alpha)})^2 + (|C|^{2\alpha} + |C^*|^{2(1-\alpha)})^2\right\|^{\frac{1}{2}}. \tag{2.18}$$

Proof. For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle = 1$. We have

$$\begin{aligned}
|\varrho\langle x, Bx\rangle|^2 + |\varrho\langle x, Cx\rangle|^2 &\leqslant \left(\varrho\langle x, |B|^{2\alpha}x\rangle^{\frac{1}{2}}\varrho\langle x, |B^*|^{2(1-\alpha)}x\rangle^{\frac{1}{2}}\right)^2 \\
&+ \left(\varrho\langle x, |C|^{2\alpha}x\rangle^{\frac{1}{2}}\varrho\langle x, |C^*|^{2(1-\alpha)}x\rangle^{\frac{1}{2}}\right)^2 \ (by\ Corollary\ 1.13) \\
&\leq \frac{1}{4}\left(\varrho\langle x, |B|^{2\alpha}x\rangle + \varrho\langle x, |B^*|^{2(1-\alpha)}x\rangle\right)^2 \\
&+ \frac{1}{4}\left(\varrho\langle x, |C|^{2\alpha}x\rangle + \varrho\langle x, |C^*|^{2(1-\alpha)}x\rangle\right)^2 \ (by\ Lemma\ 1.9) \\
&= \frac{1}{4}\left(\varrho\langle x, (|B|^{2\alpha} + |B^*|^{2(1-\alpha)})x\rangle^2 + \varrho\langle x, (|C|^{2\alpha} + |C^*|^{2(1-\alpha)})x\rangle^2\right) \\
&\leq \frac{1}{4}\left(\varrho\langle x, (|B|^{2\alpha} + |B^*|^{2(1-\alpha)})^2x\rangle + \varrho\langle x, (|C|^{2\alpha} + |C^*|^{2(1-\alpha)})^2x\rangle\right) \\
&\quad (by\ Lemma\ 1.10) \\
&\leqslant \frac{1}{4}\varrho\langle x, \left((|B|^{2\alpha} + |B^*|^{2(1-\alpha)})^2 + (|C|^{2\alpha} + |C^*|^{2(1-\alpha)})^2\right)x\rangle \\
&\leqslant \frac{1}{4}\left\|(|B|^{2\alpha} + |B^*|^{2(1-\alpha)})^2 + (|C|^{2\alpha} + |C^*|^{2(1-\alpha)})^2\right\|.
\end{aligned}$$

Taking supremum over all $x \in E$ with $\varrho\langle x, x\rangle = 1$, we get

$$w_e^2(B, C) \leqslant \frac{1}{4}\left\|(|B|^{2\alpha} + |B^*|^{2(1-\alpha)})^2 + (|C|^{2\alpha} + |C^*|^{2(1-\alpha)})^2\right\|.$$

□

Theorem 2.20. *Let* $B, C \in \mathcal{L}(E)$ *and* $0 \leqslant \alpha \leqslant 1$. *Then*

$$w_e^2(B, C) \leqslant w_A(|B|^{2\alpha} + i|C|^{2\alpha})w_A(|B^*|^{2(1-\alpha)} + i|C^*|^{2(1-\alpha)}). \tag{2.19}$$

Proof. Let $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle = 1$. Then

$$\begin{aligned}
|\varrho\langle x, Bx\rangle|^2 + |\varrho\langle x, Cx\rangle|^2 &\leqslant \varrho\langle x, |B|^{2\alpha}x\rangle\varrho\langle x, |B^*|^{2(1-\alpha)}x\rangle + \varrho\langle x, |C|^{2\alpha}x\rangle\varrho\langle x, |C^*|^{2(1-\alpha)}x\rangle \\
&\quad (by\ Corollary\ 1.13) \\
&\leqslant \left[\left(\varrho\langle x, |B|^{2\alpha}x\rangle^2 + \varrho\langle x, |C|^{2\alpha}x\rangle^2\right)\right. \\
&\times \left.\left(\varrho\langle x, |B^*|^{2(1-\alpha)}x\rangle^2 + \varrho\langle x, |C^*|^{2(1-\alpha)}x\rangle^2\right)\right]^{\frac{1}{2}} \\
&\quad (by\ the\ inequality\ (ab + cd)^2 \leqslant (a^2 + c^2)(b^2 + d^2)\ for\ a,\ b,\ c,\ d \in \mathbb{R}) \\
&= \left|\varrho\langle x, |B|^{2\alpha}x\rangle + i\varrho\langle x, |C|^{2\alpha}x\rangle\right|\left|\varrho\langle x, |B^*|^{2(1-\alpha)}x\rangle + i\varrho\langle x, |C^*|^{2(1-\alpha)}x\rangle\right| \\
&= \left|\varrho\langle x, (|B|^{2\alpha} + i|C|^{2\alpha})x\rangle\right|\left|\varrho\langle x, (|B^*|^{2(1-\alpha)} + i|C^*|^{2(1-\alpha)})x\rangle\right| \\
&\leqslant w_A(|B|^{2\alpha} + i|C|^{2\alpha})w_A(|B^*|^{2(1-\alpha)} + i|C^*|^{2(1-\alpha)})
\end{aligned}$$

Taking supremum over all $x \in E$ with $\varrho\langle x, x\rangle = 1$, we get

$$w_e^2(B, C) \leqslant w_A(|B|^{2\alpha} + i|C|^{2\alpha})w_A(|B^*|^{2(1-\alpha)} + i|C^*|^{2(1-\alpha)}).$$

□

Remark 2.21. It is not difficult to verify that $w_A^2(|B|^{2\alpha}+i|C|^{2\alpha}) \leqslant \left\|(B^*B)^{2\alpha}+(C^*C)^{2\alpha}\right\|$ and $w_A^2(|B^*|^{2(1-\alpha)}+i|C^*|^{2(1-\alpha)}) \leqslant \|(BB^*)^{2(1-\alpha)}+(CC^*)^{2(1-\alpha)}\|$.
Therefore,

$$w_A(|B|^{2\alpha}+i|C|^{2\alpha})w_A(|B^*|^{2(1-\alpha)}+i|C^*|^{2(1-\alpha)}) \leqslant \left\|(B^*B)^{2\alpha}+(C^*C)^{2\alpha}\right\|^{\frac{1}{2}} \cdot\left\|(BB^*)^{2(1-\alpha)}+(CC^*)^{2(1-\alpha)}\right\|^{\frac{1}{2}}. \quad (2.20)$$

Theorem 2.22. *Let $B,C \in \mathcal{L}(E)$ and let f and g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ for all $t \in [0,\infty)$. Then*

$$\frac{1}{2}\max\{w_A^2(B+C), w_A^2(B-C)\} \leqslant w_e(f^2(|B|), f^2(|C|))w_e(g^2(|B^*|), g^2(|C^*|)). \quad (2.21)$$

Proof. For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x,x\rangle = 1$, we have

$$\begin{aligned}
|\varrho\langle x,(B+C)x\rangle|^2 &= |\varrho\langle x,Bx\rangle + \varrho\langle x,Cx\rangle|^2\\
&\leqslant 2(|\varrho\langle x,Bx\rangle|^2 + |\varrho\langle x,Cx\rangle|^2)\\
&\leqslant 2(\varrho\langle f(|B|)x, f(|B|)x\rangle\varrho\langle g(|B^*|)x, g(|B^*|)x\rangle\\
&+ \varrho\langle f(|C|)x, f(|C|)x\rangle\varrho\langle g(|C^*|)x, g(|C^*|)x\rangle) \;\; (by\ Corollary\ 2.10)\\
&\leqslant 2(\varrho\langle x, f^2(|B|)x\rangle\varrho\langle x, g^2(|B^*|)x\rangle + \varrho\langle x, f^2(|C|)x\rangle\varrho\langle x, g^2(|C^*|)x\rangle)\\
&\leqslant 2(\varrho\langle x, f^2(|B|)x\rangle^2 + \varrho\langle x, f^2(|C|)x\rangle^2)^{\frac{1}{2}}(\varrho\langle x, g^2(|B^*|)x\rangle^2 + \varrho\langle x, g^2(|C^*|)x\rangle^2)^{\frac{1}{2}}\\
&\leqslant 2w_e(f^2(|B|), f^2(|C|))w_e(g^2(|B^*|), g^2(|C^*|)).
\end{aligned}$$

Taking supremum over $\varrho\langle x,x\rangle = 1$, we get

$$\frac{1}{2}w_A^2(B+C) \leqslant w_e(f^2(|B|), f^2(|C|))w_e(g^2(|B^*|), g^2(|C^*|)). \quad (2.22)$$

Similarly, we can prove that:

$$\frac{1}{2}w_A^2(B-C) \leqslant w_e(f^2(|B|), f^2(|C|))w_e(g^2(|B^*|), g^2(|C^*|)). \quad (2.23)$$

Combining the inequalities (2.22) and (2.23), we get

$$\frac{1}{2}\max\{w_A^2(B+C), w_A^2(B-C)\} \leqslant w_e(f^2(|B|), f^2(|C|))w_e(g^2(|B^*|), g^2(|C^*|)).$$

□

In particular, if we take $f(t)=g(t)=t^{\frac{1}{2}}$, then

$$\frac{1}{2}\max\{w_A^2(B+C), w_A^2(B-C)\} \leqslant w_e(|B|,|C|)w_e(|B^*|,|C^*|). \quad (2.24)$$

Corollary 2.23. *For any self-adjoint bounded linear operators $B,C \in \mathcal{L}(E)$, we have*

$$\frac{1}{2}\max\{\|B+C\|^2, \|B-C\|^2\} \leqslant w_e^2(B,C). \quad (2.25)$$

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