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FIXED POINT THEOREMS FOR MULTI-VALUED NON-SELF F-CONTRACTION MAPPINGS IN METRICALLY CONVEX PARTIAL METRIC SPACES

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Abstract In this study, we establish that a fixed point theorem is valid for multi-valued non-self F-contraction mappings within metrically convex partial metric spaces. Additionally, we provide an example to support and illustrate our findings.

1 Introduction

In 1965, Halpern [12] initiated research on fixed points for non-self mappings. Later in 1972, Assad and Krik [7] established fixed point theorems for multi-valued non-self mappings in metrically convex spaces. Gajic and Rakocevic [11] established a fixed point theorem for non-self mappings with a Takahashi convex structure in metric space. Khan [13] demonstrated hybrid pairings of non-self multi-valued mappings in a convex metric space.

In 1994, Matthews [19] introduced the concept of partial metric spaces as a generalization of metric spaces, relaxing the condition that the self-distance of a point must be zero. He also extended the Banach contraction principle to partial metric spaces, which have since found broad applications in areas such as computer networking, data organization, and programming. In recent years, various researchers extended fixed point theorems in metric spaces to partial metric spaces [see, [1, 2, 4, 16, 17, 25, 26]]. Also several fixed point theorems has been demonstrated in various spaces one can see [9, 20, 22, 24].

Wardowski [28] presented an intriguing generalisation of the Banach contraction principle in 2012, employing a distinct contraction known as the F-contraction. Since then, other academics have utilised F-contractions to demonstrate fixed point theorems in a variety of spaces. Some of which found in [3, 10, 14, 15, 21, 23, 27].

In this study we extend and generalize the concepts from Altun *et al.* [5], Assad and Kirk [7], Sgroi and Vetro [23] and Paesano and Vetro [21] to metrically convex partial metric spaces.

2 Preliminaries

In this section, we introduce definitions, lemmas, propositions, and some well-known results that will be used to create the new result.

In 1972, Assad and Kirk [7] defined the metrically convex space as follows:

Definition 2.1. [7] A metric space (X, d) is said to be metrically convex if for all $x, y \in X$ with $x \neq y$, there exists a point $z \in X$ $(x \neq z \neq y)$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Theorem 2.2. *[7] Let (X, d) be a complete metrically convex metric space, K a non empty closed subset of X and $T : K \to CB(X)$ be a mapping. Assume that the following conditions hold:*

(i) $Tx \in K$ for each $x \in \partial K$,

(ii) there exists $k \in (0,1)$ $\forall\, x, y \in K$,

$$d(Tx, Ty) \leq\ kd(x, y).$$

Then T has a unique fixed point in K.

They also proved the following lemma:

Lemma 2.3. *[7] If K is a nonempty closed subset of a complete and metrically convex metric spaces (X, d), then for any $x \in K$, $y \notin K$, there exists a point $z \in \partial K$ (the boundary of K) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Kumar and Rugumisa [18] gave the following definition in the context of non-self mapping in partial metric space.

Definition 2.4. [18] Let $T : K \to CB^p(X)$ be a multivalued mapping, where $K \subseteq X$. We say that T is a self-mapping if $K = X$, otherwise T is called a non-self mapping. If there an element $x \in K$ such that $x \in Tx$, we say that x is a fixed point of T in X.

Wardowski [28] defined the function F as follows:
Let F be a function defined as $F : \mathbb{R}^+ \to \mathbb{R}$, which satisfies the following conditions:

(F1) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}_+$ we have $\alpha < \beta$ implying $F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim\limits_{n\to\infty} \alpha_n = 0,$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (\alpha_n)^k F(\alpha_n) = 0.$$

Then the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the condition $(F1) - (F3)$ is denoted by $\mathfrak{F}$.
Some examples of $F \in \mathfrak{F}$ are:

(1) $F(\alpha) = \ln \alpha$;

(2) $F(\alpha) = \alpha + \ln \alpha$;

(3) $F(\alpha) = \ln(\alpha^2 + \alpha)$.

Definition 2.5. [28] Let (X, d) be a metric space. A self-mapping T on X is called an F-contraction mapping if there exists $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that for all $x, y \in X$,

$$d(Tx, Ty)\ > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

In 2012, Wardowski [28] proved the following fixed point theorem:

Theorem 2.6. *[28] Let (X, d) be a complete metric space and $T : X \to X$ be a F-contraction mapping. If there exists $\tau > 0$ such that for all $x, y \in X$, $d(Tx, Ty)\ > 0$, implies*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{2.1}$$

then T has a unique fixed point.

Cosentino *et al.* [10] gave the definition of multi-valued F-contractions in b-metric space as follows:

Definition 2.7. [10] Let (X, d, s) be a b-metric space. A multi-valued mapping $T : X \to CB(X)$ is called an F-contraction of Nadler type if there exists F $\in \mathfrak{F}$ and $\tau > \mathbb{R}^+$ such that

$$2\tau + F(sH(Tx, Ty)) \quad \leq \quad F(d(x, y)),$$

for all $x, y \in X$ with $Tx \neq Ty$.

Altun *et al.* [5] proved the following fixed point theorem for multi-valued non-self F-contractions on convex metric spaces.

Theorem 2.8. *[5] Let (X, d) be a complete and metrically convex metric space, K a non empty closed subset of X, $T : K \to CB(X)$ and $F \in \mathfrak{F}$. Assume that the following conditions hold:*

(i) $Tx \in K$ for each $x \in \partial K$,

(ii) there exists $\tau > 0$ such that for each $x, y \in K$ with $H(Tx, Ty) > 0$, it satisfies

$$\tau + F(H(Tx, Ty)) \leq F(d(x, y)).$$

Then T has a fixed point in K.

The partial metric space and its properties was defined by Matthew [19] as follows:

Definition 2.9. [19] A partial metric on a non-empty set X is a mapping $p : X \times X \to \mathbb{R}_+$, such that for all $x, y, z \in X$,

(P0) $0 \leq p(x, x) \leq p(x, y)$,

(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,

(P2) $p(x, y) = p(y, x)$ and

(P3) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is said to be a partial metric space.

As an example, let $X = \mathbb{R}^+$ and let $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Lemma 2.10. *[19] If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}$ given by*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$, is a metric on X.

Furthermore, a sequence $\{x_n\}$ in (X, d_p) converges to a point $x \in X$ with respect to τ_p if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x, x_n) = p(x, x).$$

Definition 2.11. [19]

(i) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a p-Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(ii) A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only it is a p-Cauchy sequence in a metric (x, d_p).

(iii) A partial metric space (X, p) is said to be p-complete if every p-Cauchy sequence $\{x_n\}$ in X is p-convergent, with respect to τ_p, to a point $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x).$$

Aydi *et al.* [8] provide the following description and features of the partial Hausdorff metric. Let $CB^p(X)$ be the family of non-empty, closed, and bounded subsets of a partial metric space (X, p), induced by the partial metric p. A is a bounded subset in (X, p) if there exists $x_0 \in X$ and $N \in \mathbb{N}$ such that for any $a \in A$, we have $a \in B_p(x_0, N)$.

$$p(x_0, a) \leq p(a, a) + N.$$

For all $A, B \in CB^p(X)$ and $x \in X$, we define:

$$\begin{aligned} p(x, A) &= inf\{p(x, a) : a \in A\}; \\ \delta_p(A, B) &= \sup\{p(a, B) : a \in A\}; \\ \delta_p(B, A) &= \sup\{p(b, A) : b \in B\}. \end{aligned}$$

Lemma 2.12. *[6] Let (X, p) be partial metric space and A any non-empty set in (X, p), then*

$$a \in \bar{A} \Leftrightarrow p(a, A) = p(a, a), \tag{2.2}$$

where $\bar{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X, p) if and only if $A = \bar{A}$.

Define the partial Hausdorff metric $H_p : CB^p \times CB^p \to \mathbb{R}^+$ as

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

We provide the following properties of the partial Hausdorff metric H_p from Aydi *et al.* [8].

Proposition 2.13. *[8] Let (X, p) be a partial metric space, then for any $A, B, C \in CB^p(X)$, we have*

(i) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$;

(ii) $\delta_p(A, A) \leq \delta_p(A, B)$;

(iii) $\delta_p(A, B) = 0 \implies A \subseteq B$;

(iv) $\delta_p(A, B) = \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Note the following lemma from Aydi *et al.* [8].

Lemma 2.14. *[8] Let (X, p) be a partial metric space, for $A, B \in CB^p(X)$ and $h > 1$ for any $a \in A$, there exists $b = b(a) \in B$ such that*

$$p(a, b) \leq hH_p(A, B).$$

3 Main Results

This section commence with the following definition of a metrically convex partial metric space by Kumar and Rugumisa [18].

Definition 3.1. [18] A partial metric space (X, p) is said to be metrically convex if the corresponding metric space (X, d_p) is metrically convex in the sense of Lemma 2.3, where $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$.

Kumar and Rugumisa [18] formulated the following lemma:

Lemma 3.2. *[18] Let K be a non-empty subset of a metrically convex partial metric space (X, p) which is closed in (X, d_p). If $x \in K$ and $y \in X \backslash K$, then there exists a point $z \in \partial K$ (the boundary of K) such that*

$$p(x, z) + p(z, y) = p(x, y) + p(z, z). \tag{3.1}$$

We expand Cosentino *et al.* [10]'s definition to multi-valued F-contractions in partial metric space, as follows:

Definition 3.3. Let (X, p) be a complete partial metric space. A multi-valued mapping $T : X \to CB^p(X)$ is called an F-contraction of Assad and Kirk type if there exists $F \in \mathfrak{F}$ and $\tau > \mathbb{R}^+$ such that

$$2\tau + F(H_p(Tx, Ty)) \quad \leq \quad F(kd(x, y)),$$

for all $x, y \in X$ with $Tx \neq Ty$.

We prove the following theorem for non-self mappings using F-contraction mappings in metrically convex partial metric spaces.

Theorem 3.4. *Let (X, p) be a complete metrically convex partial metric space and K a non-empty closed subset of X. Suppose that $T : K \to CB^p(X)$ is a multi-valued F-contraction mapping. Assume that the following conditions hold:*

(i) $Tx \in K$ for each $x \in \partial K$,

(ii) for all $x, y \in K$ with $H_p(Tx, Ty) > 0$, there exists $k \in (0, 1)$ and $\tau > 0$ such that

$$2\tau + F(H_p(Tx, Ty)) \leq F(kp(x, y)). \tag{3.2}$$

If T satisfies Rothe's type condition that is $x \in \partial K \Rightarrow Tx \subset K$, then T has a fixed point z in K, such that $p(z, z) = 0$.

Proof. Suppose that T has no fixed points, then $p(x_0, Tx_0) > 0$ for all $x \in K$. We proceed by constructing two sequences $\{x_n\} \in K$ and $\{y_n\} \in K$ in the following way: Let $x_0 \in \partial K$ and $y_1 \in Tx_0$, if $y_1 \in K$, let $x_1 = y_1$, then our proof would be completed. Since T is a multivalued F-contraction mapping $x_1 \in Tx_0$. If $y_1 \notin K$, then, by Lemma 3.2 there exists $x_1 \in \partial K$ such that

$$p(x_0, x_1) + p(x_1, y_1) \quad = \quad p(x_0, y_1) + p(x_1, x_1). \tag{3.3}$$

Thus for $x_1 \in K$ and using Lemma 2.14, we can choose $y_2 \in Tx_1$ such that

$$p(x_1, Tx_1) < hH_p(Tx_0, Tx_1).$$

We deduce that

$$p(x_1, y_2) \leq hH_p(Tx_0, Tx_1).$$

As a results, we get

$$p(x_1, y_2) \leq p(y_1, y_2) \leq hH_p(Tx_0, Tx_1).$$

By the continuity of F, there exists a real number $h > 1$ such that

$$F\big(p(y_1, y_2)\big) \leq F\big(hH_p(Tx_0, Tx_1)\big) < F\big(H_p(Tx_0, Tx_1)\big) + \tau. \tag{3.4}$$

Using Lemma 2.14, Equation 3.4 and (3.2), we get

$$2\tau + F\big(p(y_1, y_2)\big) \leq 2\tau + F\big(H_p(Tx_0, Tx_1)\big) + \tau.$$

However, if $y_2 \in K$, let $x_2 = y_2$. If $y_2 \notin K$. Then, using Lemma 3.2 there exists $x_2 \in \partial K$ such that

$$p(x_1, x_2) + p(x_2, y_2) \quad = \quad p(x_1, y_2) + p(x_2, x_2).$$

Thus $x_2 \in K$ and using Lemma 2.14, we can choose $y_3 \in Tx_2$ such that

$$p(x_2, Tx_2) < hH_p(Tx_1, Tx_2).$$

We deduce that

$$p(x_2, y_3) \leq hH_p(Tx_1, Tx_2).$$

As a results, we get

$$p(x_2, y_3) \le p(y_2, y_3) \le hH_p(Tx_1, Tx_2).$$

Since $\tau \in \mathbb{R}^+$ and F is continuous, there exists a real number $h > 1$ such that

$$F\big(p(y_2, y_3)\big) \le F\big(hH_p(Tx_1, Tx_2)\big) < F\big(H_p(Tx_1, Tx_2)\big) + \tau. \tag{3.5}$$

By Lemma 2.14, Equation 3.7 and (3.2), we obtain

$$2\tau + F\big(p(y_2, y_3)\big) \le 2\tau + F\big(H_p(Tx_1, Tx_2)\big) + \tau.$$

Continuing this way, we constructing the sequences $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 1}$ such that:

(i) If $x_n \in Tx_n$, then x_n is a fixed point of T and we have completed the proof.

(ii) If $y_n \in K$, we set $x_n = y_n$;

(iii) If $y_n \notin K$, then $x_n \neq y_n$ by Lemma 3.2, there exists $x_n \in \partial K$ such that

$$p(x_n, x_{n+1}) + p(x_{n+1}, y_{n+1}) = p(x_n, y_{n+1}) + p(x_{n+1}, x_{n+1}).$$

(iv) If $x_n \notin Tx_n$, then by Lemma 2.14, we can choose $y_{n+1} \in Tx_n$ such that $p(x_n, Tx_n) < hH_p(Tx_{n-1}, Tx_n)$. Consequently, we get

$$p(x_n, y_{n+1}) \le p(y_n, y_{n+1}) \le hH_p(Tx_{n-1}, Tx_n).$$

By the property of F, we have

$$F\big(p(y_n, y_{n+1})\big) \le F\big(hH_p(Tx_{n-1}, Tx_n)\big) < F\big(H_p(Tx_{n-1}, Tx_n)\big) + \tau.$$

Therefore,

$$\begin{aligned} p(x_n, y_{n+1}) &\le hH_p(Tx_{n-1}, Tx_n), \\ \Longrightarrow p(x_n, y_{n+1}) &\le H_p(Tx_{n-1}, Tx_n) + \tau. \end{aligned}$$

Let us consider the situation where $x_n \notin Tx_n$ for all $n \in \mathbb{N}$. We will show that there is $z \in K$ such that $x_n \to z$ as $n \to \infty$.

As Tx_n is a closed set, we have

$$p(x_n, Tx_n) > p(y_n, y_{n+1}) \ge 0. \tag{3.6}$$

We partition the sequence $\{x_n\}$ into sets P and Q. Let $P = \{x_i \in x_n : x_i = y_i, i = 1, 2, \dots\}$ and $Q = \{x_i \in x_n : x_i \neq y_i, i = 1, 2, \dots\}$.

Note that by the construction of sequence, that $x_n \in Q \Rightarrow x_n \in \partial K$. From the construction of proof, we note that if $x_n \in Q$ for some n, then $x_{n-1}, x_{n+1} \in P$.

Now for $n \ge 2$, we have the following three cases:

Case 1. If $x_n, x_{n+1} \in P$, then $y_n = x_n, y_{n+1} = x_{n+1}$. Assume that $x_n = y_n = Tx_{n-1}$, $x_{n+1} = y_{n+1} = Tx_n$. Then, we have

$$p(x_n, x_{n+1}) = p(x_n, y_{n+1}) = p(y_n, y_{n+1}).$$

Since $\{x_n\} \in K$ for all $n \in \mathbb{N}$, we shows that $x_n, x_{n+1} \in \partial K$. Now,

$$p(x_n, x_{n+1}) \quad = \quad p(Tx_{n-1}, Tx_n).$$

Using Lemma 2.14, we have

$$p(x_n, x_{n+1}) \quad \le \quad hH_p(Tx_{n-1}, Tx_n).$$

Consequently, we get

$$\begin{aligned} p(x_n, x_{n+1}) &\le p(x_n, y_{n+1}) \le p(y_n, y_{n+1}), \\ &\le hH_p(Tx_{n-1}, Tx_n) < H_p(Tx_{n-1}, Tx_n) + \tau. \end{aligned}$$

Letting $x = x_{n-1}$ and $y = x_n$ in (3.2), we have

$$\begin{aligned} 2\tau + F\big(p(x_n, x_{n+1})\big) &= 2\tau + F\big(p(x_n, y_{n+1})\big), \\ &\le 2\tau + F\big(H_p(Tx_{n-1}, Tx_n)\big) + \tau, \\ &\le F\big(kp(x_{n-1}, x_n)\big) + \tau. \end{aligned} \tag{3.7}$$

The expression (3.7) implies

$$\begin{aligned} 2\tau + F\big(p(x_n, x_{n+1})\big) &\le F\big(kp(x_{n-1}, x_n)\big) + \tau, \\ \tau + F\big(p(x_n, x_{n+1})\big) &\le F\big(kp(x_{n-1}, x_n)\big). \end{aligned}$$

As F is increasing, by $(F1)$ we have

$$p(x_n, x_{n+1}) < kp(x_{n-1}, x_n).$$

From $k < 1$, we deduce

$$\begin{aligned} p(x_n, x_{n+1}) &< p(x_{n-1}, x_n), \\ \tau + F(p(x_n, x_{n+1})) &\le F(p(x_{n-1}, x_n)) - \tau. \end{aligned}$$

Which is equivalent to

$$F(p(x_n, x_{n+1})) \le F(p(x_{n-1}, x_n)) - \tau.$$

Case II

If $x_n \in P$, $x_{n+1} \in Q$. Assume that $x_n = y_n = Tx_{n-1}$, $x_{n+1} \neq y_{n+1} = Tx_n$. Then, we have

$$p(x_n, x_{n+1}) = p(x_n, y_{n+1}) = p(y_n, y_{n+1}).$$

Since $\{x_n\} \in K$ for all $n \in \mathbb{N}$, we shows that $x_n, x_{n-1} \in \partial K$.

Using Lemma 2.14, we have

$$p(x_n, x_{n+1}) \quad \le \quad hH_p(Tx_{n-1}, Tx_n).$$

Consequently, we have

$$p(x_n, x_{n+1}) \quad \le \quad hH_p(Tx_{n-1}, Tx_n) < H_p(Tx_{n-1}, Tx_n) + \tau.$$

Apply $x = x_{n-1}, y = x_n$ in (3.2), we get

$$\begin{aligned} 2\tau + F\big(p(x_n, x_{n+1})\big) &\le 2\tau + F\big(H_p(Tx_{n-1}, Tx_n)\big) + \tau, \\ &\le F\big(kp(x_{n-1}, x_n)\big) + \tau. \end{aligned} \tag{3.8}$$

The expression (3.8) implies

$$\begin{aligned} 2\tau + F\big(p(x_n, x_{n+1})\big) &\le F\big(kp(x_{n-1}, x_n)\big) + \tau, \\ \tau + F\big(p(x_n, x_{n+1})\big) &\le F\big(kp(x_{n-1}, x_n)\big). \end{aligned}$$

As $\tau \in \mathbb{R}^+$ and F is strictly increasing, by $(F1)$ we have

$$p(x_n, x_{n+1}) < kp(x_{n-1}, x_n).$$

As $k < 1$, we deduce that

$$p(x_n, x_{n+1}) \le p(x_{n-1}, x_n).$$

Implies that

$$F(p(x_n, x_{n+1})) \le F(p(x_{n-1}, x_n)) - \tau. \tag{3.9}$$

we obtain the similar result as in case I.

Case III

If $x_n \in Q$, $x_{n+1} \in P$. In this case, we have $y_n \neq x_n, x_{n-1} \in P$, $x_{n+1} \in P$, $x_{n-1} = y_{n-1}$, $x_{n+1} = y_{n+1}$, $y_n \in Tx_{n-1}$.

Assume that

$$x_{n+1} = Tx_n, \; x_n \neq y_n = Tx_{n-1}.$$

Since $x_n \in Q$, by Lemma 2.14, we have

$$\begin{aligned} p(y_n, y_{n+1})) &\leq hH_p(Tx_{n-1}, Tx_n), \\ &\leq H_p(Tx_{n-1}, Tx_n) + \tau. \end{aligned} \tag{3.10}$$

Using (3.10) in (3.2), we get

$$\begin{aligned} 2\tau + F\big(p(y_n, y_{n+1})\big) &\leq 2\tau + F\big(H_p(Tx_{n-1}, Tx_n)\big) + \tau, \\ &\leq F\big(kp(x_{n-1}, x_n)\big) + \tau, \\ F\big(p(y_n, y_{n+1})\big) &\leq F\big(kp(x_{n-1}, x_n)\big) - \tau, \end{aligned}$$

then, taking $k < 1$ we have

$$p(y_n, y_{n+1}) < p(x_{n-1}, x_n). \tag{3.11}$$

Similarly for

$$\begin{aligned} p(y_{n-1}, y_n) &\leq hH_p(Tx_{n-2}, Tx_{n-1}), \\ &\leq H_p(Tx_{n-2}, Tx_{n-1}) + \tau. \end{aligned} \tag{3.12}$$

Using (3.12) in (3.2), taking $k < 1$, we get

$$\begin{aligned} 2\tau + F\big(p(y_{n-1}, y_n)\big) &\leq 2\tau + F\big(H_p(Tx_{n-2}, Tx_{n-1})\big) + \tau, \\ &\leq F\big(p(x_{n-2}, x_{n-1})\big) + \tau, \\ F\big(p(y_{n-1}, y_n)\big) &\leq F\big(p(x_{n-2}, x_{n-1})\big) - \tau. \end{aligned} \tag{3.13}$$

$$\begin{aligned} F\big(p(x_n, x_{n+1})\big) &\leq F\big(p(x_n, y_n) + p(y_n, x_{n+1})\big), \\ &\leq F\big(p(x_n, y_n) + p(y_n, y_{n+1})\big), \\ &\leq F\big(p(x_n, y_n) + p(x_{n-1}, x_n)\big), \\ &\leq F\big(p(x_{n-1}, y_n)\big), \\ &\leq F\big(p(y_{n-1}, y_n)\big). \end{aligned} \tag{3.14}$$

Again, using (3.13) in (3.14) we obtain

$$F\big(p(x_n, x_{n+1})\big) \leq F\big(p(x_{n-2}, x_{n-1})\big) - \tau.$$

As F is increasing, by $(F1)$ we have

$$p(x_n, x_{n+1}) < p(x_{n-2}, x_{n-1}).$$

Equivalent to

$$F(p(x_n, x_{n+1})) \leq F(p(x_{n-2}, x_{n-1})) - \tau.$$

The only other possibility, $x_n \in Q, x_{n+1} \in Q$ not occur. Thus, combining cases I, II and III, for $n \geq 2$, we have the following possible cases;

$$F(p(x_n, x_{n+1})) \quad \leq \quad F(p(x_{n-1}, x_n)) - \tau,$$

and

$$F(p(x_n, x_{n+1})) \quad \leq \quad F(p(x_{n-2}, x_{n-1})) - \tau.$$

Now, we claim that

$$F(p(x_n, x_{n+1})) \leq F(\max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n)\}) - \left(\frac{n}{2}\right)\tau, \tag{3.15}$$

for all $n \geq 2$.
Denotes

$$\begin{aligned} \alpha_n &= p(x_n, x_{n+1}), \\ \beta_n &= \max\{\alpha_{n-2}, \alpha_{n-1}\}. \end{aligned}$$

Equation (3.15) is equivalent to

$$F(\alpha_n) \leq F(\beta_n) - (\frac{n}{2})\tau. \tag{3.16}$$

Following the method by Assad and Kirk [7], Inductively, for all these cases, there exists a sequence $x_n \in K$ such that $x_{n+1} \in Tx_n$, for $n \geq 2$. Define

$$\beta_2 = \max\{\alpha_0, \alpha_1\}.$$

We have

$$\begin{aligned} F(\alpha_2) &\leq F(\max\{\alpha_0, \alpha_1\}) - \tau, \\ &\leq F(\beta_2) - \tau. \end{aligned}$$

For $n = 3$, we have

$$\begin{aligned} F(\alpha_3)) &\leq F(\max\{\alpha_1, \alpha_2\}) - \frac{3}{2}\tau, \\ &\leq F(\beta_3) - \frac{3}{2}\tau. \end{aligned}$$

For $n = 4$, we have

$$\begin{aligned} F(\alpha_4) &\leq F(\max\{\alpha_2, \alpha_3\}) - 2\tau, \\ &\leq F(\beta_4) - 2\tau. \end{aligned}$$

Using condition (F_1), we obtain

$$F(\alpha_n) \leq F(\beta_2) - \tau \leq \cdots \leq F(\beta_n) - \left(\frac{n}{2}\right)\tau. \tag{3.17}$$

Hence,

$$\lim_{n\to\infty} F(\alpha_n) = -\infty,$$

by property $(F2)$, we obtain

$$\lim_{n\to\infty} (\alpha_n) = 0.$$

From $(F3)$, there exist $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0.$$

Multiplying α_n^k in (3.17) for all $n \in \mathbb{N}$, we get

$$\alpha_n^k F(\alpha_n) \leq \alpha_n^k F(\beta_2) - \alpha_n^k \tau \leq \cdots \leq \alpha_n^k F(\beta_n) - (\alpha_n)^k \left(\frac{n}{2}\right)\tau.$$

$$\alpha_n^k (F(\alpha_n) - \alpha_n^k F(\beta_n) \leq -\alpha_n^k \left(\frac{n}{2}\right)\tau \leq 0.$$

$$\alpha_n^k \Big[F(\alpha_n) - F(\beta_n)\Big] \leq -(\alpha_n)^k \left(\frac{n}{2}\right)\tau \leq 0. \tag{3.18}$$

Taking $n \to \infty$ in (3.18), we get

$$\lim_{n\to\infty} \left(\frac{n}{2}\right)\tau(\alpha_n)^k = 0. \tag{3.19}$$

Since (3.19) is true, therefore there exist $n_1 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$, for all $n \geq n_1$

$$\alpha_n \leq (n)^{-\frac{1}{k}}, \forall n \geq n_1. \tag{3.20}$$

Now, we can show that $\{x_n\}$ is a Cauchy sequence. Consider $n, m \in N$ such that m, n $\geq N_1$, using $(P3)$ of Definition 2.9 and from (3.20), we have

$$\begin{aligned}
p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+3}) + \ldots \\
&\quad + p(x_{m-1}, x_m) - \sum_{j=n+1}^{m-1} p(x_j, x_j), \\
&\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+3}) + \ldots + p(x_{m-1}, x_m), \\
&\leq \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \ldots + \alpha_{m-1}, \\
&= \sum_{i=n}^{m-1} \alpha_i, \\
&\leq \sum_{i=n}^{\infty} \alpha_i, \\
&\leq \sum_{i=n}^{m-1} i^{-\frac{1}{k}}.
\end{aligned} \tag{3.21}$$

The results (3.21) ensures that the series $\sum_{i=n}^{m-1} i^{-\frac{1}{k}}$ is convergent. Hence a Cauchy sequence. Which implies that

$$\lim_{n\to\infty} p(x_n, x_m) = 0.$$

For that case if $n, m \in N$, we obtain

$$p^s(x_n, x_m) \leq 2p(x_n, x_m) \to 0 \; as \; n \to \infty,$$

which shows that $\{x_n\}, \forall n \in \mathbb{N}$ is a Cauchy sequence. Since (X, p) is complete partial metric space, also (X, p^s) is complete. Let a point $z \in X$ such that

$$\begin{aligned}
p(z, z) &= \lim_{n\to\infty} p(x_n, z) = 0, \\
&= \lim_{n,m\to\infty} p(x_n, x_m) = 0, \\
p(z, z) &= 0.
\end{aligned} \tag{3.22}$$

Hence, z is a fixed point of T in K. □

We give an example for illustrating our results. The example is a simple application of Theorem 3.4.

Example 3.5. Consider $X = [0, 1]$ endowed with the partial metric defined by $p(x, y) = |x - y|$. Distinctly, (X, p) is completely metrically convex partial metric space and K is closed subset of X. Also define a multivalued mapping $T : K \to CB^p(X)$ by

$$Tx = \begin{cases} \frac{x}{2}, & 0 \leq x \leq \frac{2}{3}, \\ -\frac{x}{2} + 1, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

Define $T : K \to 2^{[0,1]}$ and $K = \left[0, \frac{2}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.

Proof. Since ∂K (boundary of K), $\partial K = \left\{0, \frac{2}{3}, 1\right\}$. Which shows that the fixed point of T are 0 and $\frac{2}{3}$.
By (i) we have

$$0 \in \partial K \Rightarrow T0 = 0,$$
$$\frac{2}{3} \in \partial K \Rightarrow T\frac{2}{3} = \frac{1}{3}.$$

Also,

$$1 \in \partial C \Rightarrow T1 = \frac{1}{2},$$
$$\frac{2}{3} \in \partial C \Rightarrow T\frac{2}{3} = \frac{2}{3}.$$

We note that $\{0\}$ and $\{0, \frac{2}{3}\}$ are bounded sets in (X, p). By Lemma 2.12, if $z \in \{0, 1\}$, then

$$\begin{aligned} z \in \overline{\{0\}} &\Leftrightarrow p(z, \{0\}) = p_s(z, z), \\ &\Leftrightarrow p(z, 0) = |z - 0|, \\ &\Leftrightarrow z = 0 \Leftrightarrow z \in \{0\}. \end{aligned}$$

Next,

$$\begin{aligned} z \in \overline{\left\{0, \frac{2}{3}\right\}} &\Leftrightarrow p\Big(z, \Big\{0, \frac{2}{3}\Big\}\Big) = p(z, z), \\ &\Leftrightarrow \min\left\{p(z, 0), p\left(z, \frac{2}{3}\right)\right\}, \\ &\Leftrightarrow \min\left\{|z - 0|, |z - \frac{2}{3}|\right\}, \\ &\Leftrightarrow \left|z - \frac{2}{3}\right|, \\ &\Leftrightarrow z \in \left\{0, \frac{2}{3}\right\}. \end{aligned}$$

Hence, $\left\{0, \frac{2}{3}\right\}$ is also closed with respect to metrically convex partial metric p.

Since ∂K (boundary of K), $\partial K = \left\{0, \frac{2}{3}, 1\right\}$. For each $x \in \partial K, Tx \subset K$ and $\partial K \subset TK$. Now we show that the contractive condition (3.2) of Theorem 3.4 is satisfied for F-contractive condition through taking $F(\alpha) = \ln \alpha + \alpha$ and $\tau = \dfrac{1}{2}$ for $k < 1$ in (3.2) turns to,

$$\frac{H_p(Tx, Ty)}{p(x, y)} e^{H_p(Tx,Ty) - p(x,y)} \leq e^{-2\tau}, \tag{3.23}$$

for all $x, y \in K$.

By (3.23) we have three cases to investigate:

Case 1

Let $x, y \in \left[0, \frac{2}{3}\right]$, for $x < y$. We define

$$\begin{aligned} H_p(Tx, Ty) &= \max\left\{\delta_p(Tx, Ty), \delta_p(Ty, Tx)\right\}, \\ \delta_p(Tx, Ty) &= \max\left\{\sup_{x \in Tx}(x, Ty), \sup_{y \in Ty}(Ty, Tx)\right\}, \end{aligned}$$

and

$$\sup_{x \in Tx}(x, Ty) = \inf\left\{p(x, y), p(x, y)\right\}.$$

Now, we have

$$H_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{y}{2},\frac{y}{2}\Big]\Big) = \max\Big\{\delta_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{y}{2},\frac{y}{2}\Big]\Big),\delta_p\Big(\Big[\frac{y}{2},\frac{y}{2}\Big],\Big[\frac{x}{2},\frac{x}{2}\Big]\Big)\Big\},$$

$$\delta_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{y}{2},\frac{y}{2}\Big]\Big) = \sup_{x\in Tx}\Big\{p\Big(\frac{x}{2},\Big[\frac{y}{2},\frac{y}{2}\Big]\Big),p\Big(\frac{x}{2},\Big[\frac{y}{2},\frac{y}{2}\Big]\Big)\Big\}. \tag{3.24}$$

It follows that

$$\begin{aligned} p\Big(\frac{x}{2},\Big[\frac{y}{2},\frac{y}{2}\Big]\Big) &= \inf\Big\{p\Big(\frac{x}{2},\frac{y}{2}\Big),p\Big(\frac{x}{2},\frac{y}{2}\Big)\Big\},\\ &= \inf\Big\{\Big|\frac{x-y}{2}\Big|,\Big|\frac{x-y}{2}\Big|\Big\} = \Big|\frac{x-y}{2}\Big|, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} p\Big(\frac{x}{2},\Big[\frac{y}{2},\frac{y}{2}\Big]\Big) &= \inf\Big\{p\Big(\frac{x}{2},\frac{y}{2}\Big),p\Big(\frac{x}{2},\frac{y}{2}\Big)\Big\},\\ &= \inf\left\{\Big|\frac{x-y}{2}\Big|,\Big|\frac{x-y}{2}\Big|\right\} = \Big|\frac{x-y}{2}\Big|. \end{aligned} \tag{3.26}$$

Using (3.25) and (3.26) in (3.24) we obtain

$$\delta_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{y}{2},\frac{y}{2}\Big]\Big) = \sup_{x\in Tx}\left\{\Big|\frac{x-y}{2}\Big|,\Big|\frac{x-y}{2}\Big|\right\} = \Big|\frac{x-y}{2}\Big|. \tag{3.27}$$

Similarly, we calculate

$$\delta_p\Big(\Big[\frac{y}{2},\frac{y}{2}\Big],\Big[\frac{x}{2},\frac{x}{2}\Big]\Big) = \sup_{y\in Ty}\left\{\Big|\frac{y-x}{2}\Big|,\Big|\frac{y-x}{2}\Big|\right\} = \Big|\frac{y-x}{2}\Big|. \tag{3.28}$$

Applying (3.27) and (3.28) in (3.24) we get

$$H_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{y}{2},\frac{y}{2}\Big]\Big) = \max\left\{\Big|\frac{x-y}{2}\Big|,\Big|\frac{y-x}{2}\Big|\right\} = \Big|\frac{x-y}{2}\Big|. \tag{3.29}$$

Also, we calculate

$$p(x,y) = |x-y|. \tag{3.30}$$

As a results we have

$$\begin{aligned} \frac{|\frac{x-y}{2}|}{|x-y|}e^{|\frac{x-y}{2}|-|x-y|} &\le e^{-2\tau},\\ \frac{|x-y|}{2|x-y|}e^{\frac{|x-y|-2|x-y|}{2}} &\le e^{-2\tau},\\ \frac{1}{2}e^{-|\frac{x-y}{2}|} &\le e^{-2\tau}. \end{aligned}$$

Case 2

Let $x\in\Big[0,\frac{2}{3}\Big]$ and $y\in\Big[\frac{2}{3},1\Big]$, for $x<y$. We now calculate $H_p(Tx.Ty)>0$.

$$\begin{aligned} H_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{2-y}{2},\frac{2-y}{2}\Big]\Big) = \max\Bigg\{&\delta_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{2-y}{2},\frac{2-y}{2}\Big]\Big),\\ &\delta_p\Big(\Big[\frac{2-y}{2},\frac{2-y}{2}\Big],\Big[\frac{x}{2},\frac{x}{2}\Big]\Big)\Bigg\}. \end{aligned} \tag{3.31}$$

In a similar way we calculate $\delta_p(Tx,Ty) > 0$.

$$\begin{aligned}\delta_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{2-y}{2},\frac{2-y}{2}\Big]\Big) &= \sup_{x\in Tx}\Big\{p\Big(\frac{x}{2},\Big[\frac{2-y}{2},\frac{2-y}{2}\Big]\Big),\\ &\quad p\Big(\frac{x}{2},\Big[\frac{2-y}{2},\frac{2-y}{2}\Big]\Big)\Big\}. \qquad (3.32)\end{aligned}$$

It follows that

$$\begin{aligned}p\Big(\frac{x}{2},\Big[\frac{2-y}{2},\frac{2-y}{2}\Big]\Big) &= \inf\Big\{p\Big(\frac{x}{2},\frac{2-y}{2}\Big),p\Big(\frac{x}{2},\frac{2-y}{2}\Big)\Big\},\\ &= \inf\Big\{\Big|\frac{x+y-2}{2}\Big|,\Big|\frac{x+y-2}{2}\Big|\Big\},\\ &= \Big|\frac{x+y-2}{2}\Big|. \qquad (3.33)\end{aligned}$$

and

$$\begin{aligned}p\Big(\frac{x}{2},\Big[\frac{2-y}{2},\frac{2-y}{2}\Big]\Big) &= \inf\Big\{p\Big(\frac{x}{2},\frac{2-y}{2}\Big),p\Big(\frac{x}{2},\frac{2-y}{2}\Big)\Big\},\\ &= \Big|\frac{x+y-2}{2}\Big|. \qquad (3.34)\end{aligned}$$

Using (3.33) and (3.34) in (3.32) we obtain

$$\begin{aligned}\delta_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{2-y}{2},\frac{2-y}{2}\Big]\Big) &= \sup_{x\in Tx}\Big\{\Big|\frac{x+y-2}{2}\Big|,\Big|\frac{x+y-2}{2}\Big|\Big\},\\ &= \Big|\frac{x+y-2}{2}\Big|. \qquad (3.35)\end{aligned}$$

Similarly, we calculate $\delta_p(Ty,Tx) > 0$.

$$\delta_p\Big(\Big[\frac{2-y}{2},\frac{2-y}{2}\Big],\Big[\frac{x}{2},\frac{x}{2}\Big]\Big) = \Big|\frac{-x-y+2}{2}\Big|. \qquad (3.36)$$

Applying (3.35) and (3.36) in (3.31) we get

$$\begin{aligned}H_p\Big(\Big[\frac{x}{2},\frac{x}{2}\Big],\Big[\frac{2-y}{2},\frac{2-y}{2}\Big]\Big) &= \max\Big\{\Big|\frac{x+y-2}{2}\Big|,\Big|\frac{-x-y-2}{2}\Big|\Big\},\\ &= \Big|\frac{x+y-2}{2}\Big|. \qquad (3.37)\end{aligned}$$

Likewise, we calculate

$$p(x,y) = |x-y|. \qquad (3.38)$$

From (3.23), we obtain

$$\begin{aligned}\frac{|\frac{x+y-2}{2}|}{|x-y|}e^{|\frac{x+y-2}{2}|-|x-y|} &\leq e^{-2\tau},\\ \frac{|x+y-2|}{2|x-y|}e^{|\frac{x+y-2}{2}|-|x-y|} &\leq e^{-2\tau}.\end{aligned}$$

Case 3
For $x, y \in \left[\frac{2}{3}, 1\right]$, we now calculate $H_p(Tx.Ty) > 0$.

$$
\begin{aligned}
H_p\Big(\Big[\frac{2-x}{2}, \frac{2-x}{2}\Big], \Big[\frac{2-y}{2}, \frac{2-y}{2}\Big]\Big) &= \max\Big\{\delta_p\Big(\Big[\frac{2-x}{2}, \frac{2-x}{2}\Big], \\
&\Big[\frac{2-y}{2}, \frac{2-y}{2}\Big]\Big) \\
&, \delta_p\Big(\Big[\frac{2-y}{2}, \frac{2-y}{2}\Big], \\
&\Big[\frac{2-x}{2}, \frac{2-x}{2}\Big]\Big)\Big\}.
\end{aligned} \tag{3.39}
$$

It follows that

$$
\delta_p\Big(\Big[\frac{2-x}{2}, \frac{2-x}{2}\Big], \Big[\frac{2-y}{2}, \frac{2-y}{2}\Big]\Big) = \Big|\frac{y-x}{2}\Big|, \tag{3.40}
$$

and

$$
\delta_p\Big(\Big[\frac{2-y}{2}, \frac{2-y}{2}\Big], \Big[\frac{2-x}{2}, \frac{2-x}{2}\Big]\Big) = \Big|\frac{x-y}{2}\Big|. \tag{3.41}
$$

Applying (3.40) and (3.41) in (3.39) we get

$$
\begin{aligned}
H_p\Big(\Big[\frac{2-x}{2}, \frac{2-x}{2}\Big], \Big[\frac{2-y}{2}, \frac{2-y}{2}\Big]\Big) &= \max\left\{\Big|\frac{x-y}{2}\Big|, \Big|\frac{y-x}{2}\Big|\right\}, \\
&= \Big|\frac{x-y}{2}\Big|.
\end{aligned} \tag{3.42}
$$

Also, we calculate

$$
p(x, y) = |x-y|. \tag{3.43}
$$

As a results we have

$$
\begin{aligned}
\frac{|\frac{x-y}{2}|}{|x-y|} e^{|\frac{x-y}{2}|-|x-y|} &\leq e^{-2\tau}, \\
\frac{|x-y|}{2|x-y|} e^{\frac{|x-y|-2|x-y|}{2}} &\leq e^{-2\tau}, \\
\frac{1}{2} e^{-|\frac{x-y}{2}|} &\leq e^{-2\tau}.
\end{aligned}
$$

From case1, case 2 and case 3. T is a multivalued no-self F-contraction mapping with $F(\alpha) = \ln \alpha + \alpha$ and $\tau = \frac{1}{2}$. We conclude that Theorem 3.4 holds true. The mapping T has the fixed points at $x = 0$ and $x = \frac{2}{3}$. □

4 Conclusion remarks

This paper aims is to obtain new generalized concepts from Altun *et al.* [5], Assad and Kirk [7], Sgroi and Vetro [23] and Paesano and Vetro [21] to metrically convex partial metric spaces. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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