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Entropy solutions for some noncoercive unilateral elliptic problems in anisotropic Sobolev spaces

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Abstract. In this paper, we will study the following class of non-coercive unilateral elliptic problems:

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \Gamma(x,\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^N$ $(N \geq 2)$, with $f(x,s)$ and $\Gamma(x,\xi)$ satisfying only some growth conditions. We demonstrate the existence of entropy solutions for the unilateral problem associated with the non-coercive elliptic equations, and we will conclude some regularity results.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. Boccardo et al., in [12], have proved the existence and some regularity results for the following quasilinear elliptic problem with degenerate coercivity condition

$$\begin{cases} -\text{div } (A(x,u)\nabla u) = f & \text{in } \quad \Omega, \\ u = \ 0 & \text{on } \quad \partial\Omega, \end{cases} \tag{1.1}$$

where the data $f \in L^m(\Omega)$ for $m \geq 1$. We refer the reader to [4, 14], and also [22] for the case of measure data.
In [3], Alvino et al. have studied the quasilinear degenerated elliptic problem

$$\begin{cases} -\text{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = f & \text{in } \quad \Omega, \\ u = \ 0 & \text{on } \quad \partial\Omega, \end{cases} \tag{1.2}$$

the existence and regularity of solutions were proved in the cases of $f \in L^m(\Omega)$ for $m \geq 1$ (see also [11]).

Zheng et al. have studied in [28] the obstacle problem associated with the degenerate elliptic equation

$$\begin{cases} -\text{ div }\left(\dfrac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\right) + b|u|^{r-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the data f is assumed to be in $L^1(\Omega)$, with $0 < \theta < 1$ and $1 \leq r \leq p$. They have proved the existence of entropy solution. For more details, we refer the reader [5, 6]. Also to [17] [18] and [21].

In [25], Porretta et al. have considered the following strongly nonlinear elliptic problem

$$\begin{cases} -\text{div}(a(x,u,\nabla u)) + g(x,u,\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $a(x,s,\xi)$ verifying some coercivity condition, and $g(x,s,\xi)$ is assumed to satisfying only some growth condition. They prove the existence of solutions using the rearrangement techniques, see also [13] and [26].

In this article, we have focused on the investigation of the nonlinear and non-coercive elliptic equations of the type

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \Gamma(x,\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where the Carathéodory functions $a_i(x,s,\xi)$ verifying the degenerate coercivity condition, and the lower-order terms $\Gamma(x,\xi)$ and $f(x,s)$ are Carathéodory functions and satisfies only some growth conditions.

The aims of this paper is to show the existence of entropy solutions for the unilateral problem associated to the strongly nonlinear elliptic equation with degenerate coercivity (1.5) in the setting of the anisotropic Sobolev spaces. For the prove of the existence of the main result, we use an approximation procedure and some a priori estimates. The study of this work is motivated by the approach utilized in [2] to obtain the existence and some regularity of solutions to unilateral problems for a nonlinear elliptic equation with degenerate coercivity in the classical Sobolev spaces.

In this article we use the following plan: In section 2, we announce some hypotheses on the Carathéodory functions $a_i(x,s,\xi)$, $\Gamma(x,\xi)$ and $f(x,s)$ for which our nonlinear elliptic equation associated with the unilateral problem (1.5) has at least one entropy solution. In section 3, we introduce the definition of the entropy solution for our elliptic equation associated to the unilateral problem, and we show the main result. Section 4 is completely devoted to proving the existence results, also some regularity results will be investigated. Finally, in Section 5, we will prove Lemma 5.1.

2 Preliminaries and essential assumptions

Let Ω be an open bounded domain in $I\!R^N$ $(N \geq 2)$ with boundary $\partial\Omega$.
Let $p_1, \ldots, p_N$ be N real constants numbers, with $1 < p_i < \infty$ for $i = 1, \ldots, N$.
We denote

$$\vec{p} = (1, p_1, \ldots, p_N), \qquad D^0 u = u \qquad \text{and} \qquad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for} \qquad i = 1, \ldots, N,$$

and we set

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad p_0 = \max\{p_1, p_2, \ldots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as follows :

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N \right\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|u\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (cf [24]).

Afterward, we recall Poincaré's type inequality (see [19]). Let $u \in W_0^{1,\vec{p}}(\Omega)$, there exists a constant $C_{p_i} > 0$, such that

$$\|u\|_{L^{p_i}(\Omega)} \leq C_{p_i} \, \|D^i u\|_{L^{p_i}(\Omega)} \qquad \text{for any} \quad i = 1, \ldots, N.$$

Moreover, for any $u \in W_0^{1,\vec{p}}(\Omega)$, there exists an other constant $C_s > 0$, (see [27]) such that

$$\|u\|_{L^q(\Omega)} \leq C_s \prod_{i=1}^{N} \Big\| \frac{\partial u}{\partial x_i} \Big\|^{\frac{1}{N}}_{L^{p_i}(\Omega)},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \text{if} \quad \overline{p} < N \\ q \in [1, +\infty[& \text{if} \quad \overline{p} \geq N \end{cases}$$

Definition 2.1. Let $k > 0$, we consider the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$, given by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$\mathcal{T}_0^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

We cite [9, 15] for more information on anisotropic Sobolev spaces.
We consider a Leray-Lions operator $A : W_0^{1,\vec{p}}(\Omega) \longmapsto W^{-1,\vec{p}'}(\Omega)$ given by

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u)$$

Here $a_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R^N$ is a function such that

- $x \longmapsto a_i(x, s, \xi)$ is a measurable function for every (s, ξ) in $I\!R \times I\!R^N$,
- $(s, \xi) \longmapsto a_i(x, s, \xi)$ is a continuous function for almost every x in Ω,

furthermore,

$$|a_i(x, s, \xi)| \leq \beta(K_i(x) + |s|^{p_i - 1 - \sigma_i} + |\xi_i|^{p_i - 1}), \tag{2.2}$$

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0 \quad \text{for any} \quad \xi_i \neq \xi_i', \tag{2.3}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in I\!R \times I\!R^N$, where $K_i(x)$ is a nonnegative function lying in $L^{p_i'}(\Omega)$, $\beta > 0$, $0 \leq \sigma_i < p_i - 1$.

$$a_i(x, s, \xi) \cdot \xi \geq b(|s|)|\xi|^{p_i}, \tag{2.4}$$

and

$$\frac{b_0}{(1 + |s|)^{\lambda}} \leq b(|s|) \qquad \text{for any} \quad s \in I\!R, \tag{2.5}$$

where $b(\cdot) : I\!R^+ \mapsto I\!R^+$ is a decreasing function that belongs to $L^1(R) \cap L^\infty(R)$, such that there exists a positive constant b_0 and $0 \leq \lambda < \min\Big\{1, p_0 - 1, \dfrac{1}{p_0 - 1}\Big\}$.
The lower order term $\Gamma(x, \xi)$ is a Carathéodory function that satisfies only the growth condition

$$|\Gamma(x, \xi)| \leq \sum_{i=1}^{N} c_i(x)|\xi_i|^{q_i}, \tag{2.6}$$

where $0 \leq q_i < \dfrac{p_i(p_0 - \lambda - 1)}{p_0}$ and $c_i(x) \in L^{r_i}(\Omega)$ with $r_i > \dfrac{p_i(p_0 - 1 - \lambda)}{p_i(p_0 - 1 - \lambda) - p_0 q_i}$ for $i = 1, \ldots, N$.

The Carathéodory function $f(x,s) : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ verifies only the growth condition:

$$|f(x,s)| \leq f_0(x) + d(x)|s|^{\gamma}, \tag{2.7}$$

where f_0 belong to $L^1(\Omega)$, with $0 \leq \gamma < p_0 - \lambda - 1$ and $d(x) \in L^m(\Omega)$ such that $m > \dfrac{p_0 - 1 - \lambda}{p_0 - 1 - \lambda - \gamma}$.

Finally, let $\psi : \Omega \longmapsto \overline{\mathbb{R}}$ be a measurable function such that

$$\psi^+ \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega), \tag{2.8}$$

and we define the closed convex set

$$K_\psi = \{u \in W_0^{1,\vec{p}}(\Omega), u \geq \psi \quad \text{a.e. in } \Omega\}. \tag{2.9}$$

Note that the set K_ψ has a non empty intersection with $L^\infty(\Omega)$, (since $\psi^+ \in K_\psi \cap L^\infty(\Omega)$). We consider the unilateral problem associated to the anisotropic quasilinear elliptic equation

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \Gamma(x,\nabla u) = f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Now, we recall some important Lemma useful to prove our main result.

Lemma 2.2. *(see [7]) Let $u \in L^q(\Omega)$ and $u_n \in L^q(\Omega)$ with $\|u_n\|_{L^q(\Omega)} \leq C$ for $1 < q < \infty$. If $u_n(x) \to u(x)$ almost everywhere in Ω, then $u_n \rightharpoonup u$ weakly in $L^q(\Omega)$.*

Lemma 2.3. *(see. [8]) Assuming that (2.2) – (2.4) hold, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,\vec{p}}(\Omega)$ and*

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \Big(a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u)\Big)(D^i u_n - D^i u)\,dx \\ &+ \int_\Omega \Big(|u_n|^{\underline{p}-2}u_n - |u|^{\underline{p}-2}u\Big)(u_n - u)\,dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty, \end{aligned} \tag{2.11}$$

then $u_n \to u$ strongly in $W_0^{1,\vec{p}}(\Omega)$ for a subsequence.

3 Main results : existence of entropy solutions

Definition 3.1. A measurable function u is an entropy solution for the unilateral problem associated to the strongly nonlinear elliptic equation (2.10) if

$$T_k(u) \in K_\psi \quad \text{for any} \quad k > \|\psi^+\|_\infty, \quad \Gamma(x,\nabla u) \in L^1(\Omega), \quad f(x,u) \in L^1(\Omega)$$

and satisfy

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)\cdot D^i T_k(u-v)dx + \int_\Omega \Gamma(x,\nabla u) T_k(u-v)dx \leq \int_\Omega f(x,u) T_k(u-v)dx, \tag{3.1}$$

for any $v \in K_\psi \cap L^\infty(\Omega)$.

Our aim in this paper is to prove the following existence theorem.

Theorem 3.2. *Assuming that the conditions (2.2) – (2.7) hold true, then the unilateral problem associated to the strongly nonlinear elliptic equation (2.10) has at least one entropy solution u. Moreover, this solution u verifies*

$$|u|^r \in L^1(\Omega) \qquad \textit{for any} \quad 0 < r < p_0 - \lambda - 1,$$

and

$$\lim_{h\to\infty} \frac{1}{h} \int_{\{|u|\leq h\}} a_i(x,u,\nabla u)\cdot D^i u\,dx = 0.$$

4 Proof of Theorem 3.2

Step 1 : Approximate problems

Let $n \in \mathbb{N}^*$, we consider the sequence of approximate Dirichlet problems

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_n), \nabla u_n) + \Gamma_n(x, \nabla u_n) = f_n(x, u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Gamma_n(x,\xi) = T_n(\Gamma(x,\xi))$ and $f_n(x,s) = T_n(f(x,s))$.
The operators A_n and G_n acted from $W_0^{1,\vec{p}}(\Omega)$ into its dual $W^{-1,\vec{p}'}(\Omega)$ are defined by

$$\langle A_n u, v\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) \cdot D^i v \, dx,$$

and

$$\langle G_n u, v\rangle = \int_\Omega \Gamma_n(x, \nabla u) v \, dx - \int_\Omega f_n(x,u) v \, dx,$$

for any $u, v \in W_0^{1,\vec{p}}(\Omega)$. It's clear that

$$\begin{aligned} |\langle G_n u, v\rangle| &\le \int_\Omega |\Gamma_n(x,\nabla u) v| \, dx + \int_\Omega |f_n(x,u) v| \, dx \\ &\le 2n \int_\Omega |v| \, dx \\ &\le 2n \|v\|_{1,\vec{p}}. \end{aligned} \tag{4.2}$$

We conclude from lemma 5.1 (see Appendix) that: for all fixed $n > 0$, there exists at least one weak solution $u_n \in K_\psi$ for the unilateral problem (4.1) (cf. [23], Theorem 2.7, p. 180). i.e.

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \cdot D^i(u_n - v) dx + \int_\Omega \Gamma_n(x, \nabla u_n)(u_n - v) dx \le \int_\Omega f_n(x, u_n)(u_n - v) dx, \tag{4.3}$$

for any $v \in K_\psi$.

Step 2 : Weak convergence of truncations

Taking $1 < \theta < p_0 - \lambda$ is small enough such that

$$\frac{p_0 - \theta - \lambda}{p_0 - \theta - \lambda - \gamma} < m \qquad \text{and} \qquad \frac{p_i(p_0 - \theta - \lambda)}{p_i(p_0 - \theta - \lambda) - q_i p_0} < r_i.$$

We set $B(s) = \int_0^s \frac{1}{(1+|\tau|)^\theta} d\tau$, note that $0 \le B(s) \le B(\infty) := \int_0^\infty \frac{1}{(1+|\tau|)^\theta} \, d\tau < \infty$ is finite real number.
Let $k \ge \max(1, \|\psi^+\|_\infty)$, $M = k + \|\psi^+\|_\infty$, and $\eta > 0$ small enough such that $v = u_n - \eta T_k(u_n - \psi^+) e^{B(|u_n|)} \in K_\psi$. Thus, by taking v as a test function in the approximate problem (4.3), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{|u_n - \psi^+| \le k\}} a_i(x, T_n(u_n), \nabla u_n) \cdot (D^i u_n - D^i \psi^+) e^{B(|u_n|)} \, dx \\ &+ \sum_{i=1}^{N} \int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n}{(1+|u_n|)^\theta} \big|T_k(u_n - \psi^+)\big| e^{B(|u_n|)} dx \\ &+ \int_\Omega \Gamma_n(x, \nabla u_n) T_k(u_n - \psi^+) e^{B(|u_n|)} \, dx \le \int_\Omega f_n(x, u_n) T_k(u_n - \psi^+) e^{B(|u_n|)} \, dx. \end{aligned} \tag{4.4}$$

In view of (2.4), (2.6) and (2.7) we obtain

$$
\begin{aligned}
& b_0 \sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\le k\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda}} dx + b_0 \sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}} \big|T_k(u_n-\psi^+)\big| dx \\
& \le \int_\Omega |f_n(x,u_n)|\, |T_k(u_n-\psi^+)| e^{B(|u_n|)}\, dx + \int_\Omega |\Gamma_n(x,\nabla u_n)|\, |T_k(u_n-\psi^+)| e^{B(|u_n|)}\, dx \\
& \quad + \sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\le k\}} a_i(x,T_n(u_n),\nabla u_n)\cdot D^i\psi^+ e^{B(|u_n|)}\, dx \\
& \le e^{B(\infty)} \int_\Omega |f_0||T_k(u_n-\psi^+)| dx + e^{B(\infty)} \int_\Omega d(x)|u_n|^{\gamma}|T_k(u_n-\psi^+)| dx \\
& \quad + e^{B(\infty)} \sum_{i=1}^{N} \int_\Omega c_i(x)|D^i u_n|^{q_i}|T_k(u_n-\psi^+)| dx \\
& \quad + e^{B(\infty)} \sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\le k\}} |a_i(x,T_n(u_n),\nabla u_n)|\, |D^i\psi^+|\, dx.
\end{aligned} \tag{4.5}
$$

We have $f_0 \in L^1(\Omega)$, then

$$
e^{B(\infty)} \int_\Omega |f_0||T_k(u_n-\psi^+)| dx \le k e^{B(\infty)} \|f_0\|_{L^1(\Omega)}. \tag{4.6}
$$

According to Young's inequality we have

$$
e^{B(\infty)} \int_\Omega d(x)|u_n|^{\gamma}|T_k(u_n-\psi^+)| dx \quad \le C_0 k \int_\Omega |d(x)|^{\frac{p_0-\lambda-\theta}{p_0-\lambda-\theta-\gamma}} dx + \varepsilon k \int_\Omega |u_n|^{p_0-\lambda-\theta}\, dx, \tag{4.7}
$$

since $p_0 = \max\{p_1, \cdots, p_N\}$, then there exists some $i_0 = \{1, 2, \cdots, N\}$ such that $p_0 = p_{i_0}$ and thanks to Poincaré's inequality, it's clear that

$$
\begin{aligned}
\int_\Omega |u_n|^{p_0-\lambda-\theta}\, dx & = \int_{\{|u_n|\le M\}} |u_n|^{p_0-\lambda-\theta}\, dx + \int_{\{|u_n|>M\}} |u_n|^{p_0-\lambda-\theta}\, dx \\
& \le C_2 k^{p_0-\lambda-\theta} + 2^{\lambda+\theta} \int_{\{|u_n|>M\}} \frac{|u_n|^{p_0}}{(1+|u_n|)^{\lambda+\theta}}\, dx \\
& \le C_2 k^{p_0-\lambda-\theta} + 2^{p_0+\lambda+\theta-1} \left(\int_\Omega \left| \frac{|u_n|-|T_M(u_n)|}{(1+|u_n|)^{\frac{\lambda+\theta}{p_0}}} \right|^{p_0} dx + \int_\Omega \frac{|T_M(u_n)|^{p_0}}{(1+|u_n|)^{\lambda+\theta}}\, dx \right) \\
& \le C_3 k^{p_0-\lambda-\theta} + 2^{p_0+\lambda+\theta-1} \int_\Omega \left| \int_{|T_M(u_n)|}^{|u_n|} \frac{ds}{(1+|s|)^{\frac{\lambda+\theta}{p_0}}} \right|^{p_0} dx \\
& \le C_3 k^{p_0-\lambda-\theta} + 2^{p_0+\lambda+\theta-1} C_{p_{i_0}}^{p_{i_0}} \int_\Omega \left| D^{i_0} \int_{|T_M(u_n)|}^{|u_n|} \frac{ds}{(1+|s|)^{\frac{\lambda+\theta}{p_{i_0}}}} \right|^{p_{i_0}} dx \\
& \le C_3 k^{p_0-\lambda-\theta} + 2^{p_0+\lambda+\theta-1} C_{p_{i_0}}^{p_{i_0}} \int_{\{|u_n|>M\}} \frac{|D^{i_0} u_n|^{p_{i_0}}}{(1+|u_n|)^{\lambda+\theta}}\, dx \\
& \le C_3 k^{p_0-\lambda-\theta} + 2^{p_0+\lambda+\theta-1} C_{p_{i_0}}^{p_{i_0}} \sum_{i=1}^{N} \int_{\{|u_n|>M\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+\theta}}\, dx.
\end{aligned} \tag{4.8}
$$

We deduce by taking $\varepsilon > 0$ small enough that

$$
\begin{aligned}
e^{B(\infty)} \int_\Omega d(x)|u_n|^{\gamma}|T_k(u_n-\psi^+)| dx & \le C_4 k^{p_0-\lambda-\theta+1} + \frac{b_0 k}{4} \sum_{i=1}^{N} \int_{\{|u_n|>M\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+\theta}}\, dx \\
& \le C_4 k^{p_0-\lambda-\theta+1} + \frac{b_0}{4} \sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+\theta}} |T_k(u_n-\psi^+)|\, dx.
\end{aligned} \tag{4.9}
$$

We can observe that $\dfrac{q_i}{p_i}+\dfrac{p_i-q_i}{p_i}=1$ and $\dfrac{(\theta+\lambda)q_i}{(p_i-q_i)(p_0-\theta-\lambda)}+\dfrac{p_i(p_0-\theta-\lambda)-q_ip_0}{(p_i-q_i)(p_0-\theta-\lambda)}=1$, then thanks to Young's inequality and (4.8) we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega c_i(x)|D^iu_n|^{q_i}|T_k(u_n-\psi^+)|e^{B(|u_n|)}dx\\&\le C_5\sum_{i=1}^N\int_\Omega|c_i(x)|^{\frac{p_i}{p_i-q_i}}(1+|u_n|)^{\frac{(\lambda+\theta)q_i}{p_i-q_i}}|T_k(u_n-\psi^+)|dx+\frac{b_0}{4}\sum_{i=1}^N\int_\Omega\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\big|T_k(u_n-\psi^+)\big|dx\\&\le C_6k\sum_{i=1}^N\int_\Omega|c_i(x)|^{\frac{p_i(p_0-\theta-\lambda)}{p_i(p_0-\theta-\lambda)-q_ip_0}}dx+\varepsilon k\int_\Omega|u_n|^{p_0-\theta-\lambda}dx+\frac{b_0}{4}\sum_{i=1}^N\int_\Omega\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\big|T_k(u_n-\psi^+)\big|dx\\&\le C_7k^{p_0-\lambda-\theta+1}+C_6k\sum_{i=1}^N\int_\Omega|c_i(x)|^{\frac{p_i(p_0-\theta-\lambda)}{p_i(p_0-\theta-\lambda)-q_ip_0}}dx+\frac{b_0}{2}\sum_{i=1}^N\int_\Omega\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\big|T_k(u_n-\psi^+)\big|dx.\end{aligned}\tag{4.10}$$

By (2.2) and Young's inequality, the last behavior on the right-hand side of (4.5) is estimated as follows

$$\begin{aligned}&e^{B(\infty)}\int_{\{|u_n-\psi^+|\le k\}}|a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx\\&\le\frac{b_0}{2\beta}\int_{\{|u_n-\psi^+|\le k\}}\frac{|a_i(x,T_n(u_n),\nabla u_n)|^{p_i'}}{(1+|u_n|)^\lambda}\,dx+C_8\int_{\{|u_n-\psi^+|\le k\}}|D^i\psi^+|^{p_i}(1+|u_n|)^{\frac{p_i\lambda}{p_i'}}\,dx\\&\le\frac{b_0}{2}\int_{\{|u_n-\psi^+|\le k\}}\frac{|a_0|^{p_i'}}{(1+|u_n|)^\lambda}\,dx+\frac{b_0}{2}\int_{\{|u_n-\psi^+|\le k\}}\frac{|u_n|^{p_i-\sigma_ip_i'}}{(1+|u_n|)^\lambda}\,dx\\&\quad+\frac{b_0}{2}\int_{\{|u_n-\psi^+|\le k\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\lambda}\,dx+C_9(1+k)^{\frac{\lambda p_i}{p_i'}}\int_{\{|u_n-\psi^+|\le k\}}|D^i\psi^+|^{p_i}dx\\&\le\frac{b_0}{2}\int_{\{|u_n-\psi^+|\le k\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\lambda}\,dx+C_9k^{p_i-\sigma_ip_i'-\lambda}+C_{10}k^{\frac{\lambda p_i}{p_i'}}.\end{aligned}\tag{4.11}$$

Thus by combining (4.5), (4.6) and (4.9) – (4.11) we conclude that

$$\begin{aligned}&\frac{b_0}{2}\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\lambda}dx+\frac{b_0}{4}\sum_{i=1}^N\int_\Omega\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\big|T_k(u_n-\psi^+)\big|dx\\&\le C_{11}k^{p_0-\lambda-\theta+1}+C_9k^{p_0-\underline{\sigma}p_0'-\lambda}+C_{10}k^{\lambda p_0-\lambda},\end{aligned}\tag{4.12}$$

with $\underline{\sigma}=\min\{\sigma_1,\sigma_2,\dots,\sigma_N\}$.
Thanks to (4.8) and (4.12), there exists a constant C_{12} not depending on k and n, such that

$$\frac{1}{(1+k)^\lambda}\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}}|D^iu_n|^{p_i}dx+k\int_\Omega|u_n|^{p_0-\theta-\lambda}dx\le C_{12}(k^{p_0-\lambda-\theta+1}+k^{p_0-\underline{\sigma}p_0'-\lambda}+k^{\lambda p_0-\lambda}).\tag{4.13}$$

It follows that

$$\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}}|D^iu_n|^{p_i}\,dx\le C_{13}(k^{p_0-\theta+1}+k^{p_0-\underline{\sigma}p_0'}+k^{\lambda p_0}).\tag{4.14}$$

We have

$$\{x\in\Omega\,,|u_n|\le k\}\subset\{x\in\Omega\,,|u_n-\psi^+|\le k+\|\psi^+\|_\infty\},$$

and since $\|\psi^+\|_\infty\le k$, therefore

$$\begin{aligned}\sum_{i=1}^N\int_\Omega|D^iT_k(u_n)|^{p_i}\,dx&=\sum_{i=1}^N\int_{\{|u_n|\le k\}}|D^iu_n|^{p_i}\,dx\\&\le\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k+\|\psi^+\|_\infty\}}|D^iu_n|^{p_i}\,dx\\&\le C_{14}(k^{p_0-\theta+1}+k^{p_0-\underline{\sigma}p_0'}+k^{\lambda p_0}).\end{aligned}\tag{4.15}$$

We deduce from Poincaré's inequality that

$$\|T_k(u_n)\|_{1,\vec{p}} \leq C_{15}(\lambda, k),$$

where $C_{15}(\lambda, k)$ is a positive constant not depending on n. Thus, the sequence $(T_k(u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $W_0^{1,\vec{p}}(\Omega)$, and there exists a subsequence, still denoted by $(T_k(u_n))_{n\in\mathbb{N}}$ and a measurable function $v_k \in W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k & \text{weakly in } \quad W_0^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to v_k & \text{strongly in } \quad L^{\underline{p}}(\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega. \end{cases} \tag{4.16}$$

Moreover, using Poincaré's inequality we have

$$\begin{aligned} k^{p_0} \operatorname{meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|^{p_{i_0}}\, dx \leq \int_\Omega |T_k(u_n)|^{p_{i_0}}\, dx \\ &\leq C_{p_{i_0}}^{p_{i_0}} \int_\Omega |D^{i_0} T_k(u_n)|^{p_{i_0}}\, dx \\ &\leq C_{p_{i_0}}^{p_{i_0}} C_{13}(k^{p_0-\theta+1} + k^{p_0-\underline{\sigma} p_0'} + k^{\lambda p_0}). \end{aligned} \tag{4.17}$$

We conclude from $0 \leq \lambda < \min(1, p_0 - 1, \dfrac{1}{p_0-1})$, and $1 < \theta < p_0 - \lambda$ that

$$\operatorname{meas}\{|u_n| > k\} \leq \frac{C_{p_0}^{p_0} C_{13}(k^{p_0-\theta+1} + k^{p_0-\underline{\sigma} p_0'} + k^{\lambda p_0})}{k^{p_0}} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{4.18}$$

Therefore, similarly as in [1] and [2], we show that $(u_n)_n$ is a Cauchy sequence in measure, therefore, the sequence converges almost everywhere, for a subsequence, to some measurable function u. Thanks to (4.16), we deduce that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in} \quad W_0^{1,\vec{p}}(\Omega) \\ T_k(u_n) \to T_k(u) & \text{in} \quad L^{\underline{p}}(\Omega) \quad \text{and a.e. in} \quad \Omega. \end{cases} \tag{4.19}$$

In addition, we deduce from (4.13) that for any $0 < r < p_0 - 1 - \lambda$, then there exists a positive constant not depending on n such that

$$\int_\Omega |u_n|^r dx \leq C_{16}. \tag{4.20}$$

Step 3 : The equi-integrability of the sequences $(f_n(x, u_n))_n$ in $L^1(\Omega)$

In this step, we will use Vitali's theorem to show that $f_n(x, u_n)$ converges to $f(x, u)$ strongly in $L^1(\Omega)$.
Evidently, $f_n(x, u_n) \to f(x, u)$ a.e. in Ω. Then, it is sufficient to show that the sequence $(f_n(x, u_n))_n$ is uniformly equi-integrable.
Let $h > 0$, for any measurable subset $E \subset \Omega$, we have

$$\int_E |f_n(x, u_n)|\, dx \leq \int_E |f_n(x, T_h(u_n))|\, dx + \int_{\{|u_n|>h\}} |f_n(x, u_n)|\, dx. \tag{4.21}$$

Let $\delta, s > 0$ such that $\gamma < s < s + \delta < p_0 - 1 - \lambda$, thanks to (4.20) we have

$$\int_\Omega |u_n|^{s+\delta} dx \leq C_{16}. \tag{4.22}$$

We have $f_0 \in L^1(\Omega)$ and $d(x) \in L^{\frac{s}{s-\gamma}}(\Omega)$. In view of (2.6), we conclude from meas$\{|u_n| > h\} \to 0$ as h tends to infinity that

$$\begin{aligned}\int_{\{|u_n|>h\}} |f_n(x,u_n)|\,dx &\le \int_{\{|u_n|>h\}} |f_0|\,dx + \int_{\{|u_n|>h\}} d(x)|u_n|^{\gamma}\,dx \\ &\le \int_{\{|u_n|>h\}} |f_0|\,dx + \int_{\{|u_n|>h\}} |d(x)|^{\frac{s}{s-\gamma}}\,dx + \int_{\{|u_n|>h\}} |u_n|^s\,dx \\ &\le \int_{\{|u_n|>h\}} |f_0|\,dx + \int_{\{|u_n|>h\}} |d(x)|^{\frac{s}{s-\gamma}}\,dx + \frac{1}{h^\delta}\int_{\{|u_n|>h\}} |u_n|^{s+\delta}\,dx \\ &\le \int_{\{|u_n|>h\}} |f_0|\,dx + \int_{\{|u_n|>h\}} |d(x)|^{\frac{s}{s-\gamma}}\,dx + \frac{1}{h^\delta}C_{16} \to 0 \text{ as } h\to\infty.\end{aligned} \tag{4.23}$$

Thus, for all $\varepsilon > 0,$ there exists $h_0(\varepsilon) > 0$ such that

$$\int_{\{|u_n|>h\}} |f_n(x,u_n)|\,dx \le \frac{\varepsilon}{2} \quad \text{for any} \quad h \ge h_0(\varepsilon). \tag{4.24}$$

On the other hand, there exists $\beta(\varepsilon,h) > 0$ small enough such that : for any $E \subset \Omega$ with meas$(E) \le \beta(\varepsilon,h)$ we have

$$\int_E |f_n(x,T_h(u_n))|\,dx \le \frac{\varepsilon}{2}. \tag{4.25}$$

We deduce by combining (4.21), (4.24) and (4.25) that

$$\int_E |f_n(x,u_n)|\,dx \le \varepsilon, \qquad \text{with } E \subseteq \Omega \quad \text{such that} \quad \text{meas}(E) \le \beta(\varepsilon). \tag{4.26}$$

Thus, the sequence $(f_n(x,u_n))_n$ is uniformly equi-integrable. Therefore, we conclude that

$$f_n(x,u_n) \longrightarrow f(x,u) \qquad \text{strongly in} \quad L^1(\Omega). \tag{4.27}$$

Moreover, we have $|u_n|^s \to |u|^s$ almost everywhere in Ω, and since

$$\begin{aligned}\int_E |u_n|^s\,dx &\le \int_E |T_h(u_n)|^s\,dx + \int_{\{|u_n|>h\}} |u_n|^s\,dx \\ &\le h\cdot \text{meas}(E) + \frac{1}{h^\delta}\int_{\{|u_n|>h\}} |u_n|^{s+\delta}\,dx.\end{aligned} \tag{4.28}$$

It follows that : for any $\varepsilon > 0$, there exists a positive constant $\beta(\varepsilon) > 0$ such that

$$\int_E |u_n|^s\,dx \le \varepsilon, \qquad \text{with } E \subseteq \Omega \quad \text{such that} \quad \text{meas}(E) \le \beta(\varepsilon). \tag{4.29}$$

Then, the sequence $(|u_n|^s)_n$ is uniformly equi-integrable. We deduce that

$$|u_n|^s \longrightarrow |u|^s \quad \text{strongly in} \quad L^1(\Omega) \qquad \text{for any} \quad \gamma < s < p_0 - 1 - \lambda. \tag{4.30}$$

Step 4 : Some regularity results

In the remaining part, we have $\lim\limits_{n\to\infty} \varepsilon_j(n) = 0$ for $j = 1,\, 2, \ldots,$ where $\varepsilon_j(n)$ are some different functions of real numbers. Similarly, $\lim\limits_{h\to\infty} \varepsilon_j(h) = \lim\limits_{h\to\infty}\lim\limits_{n\to\infty} \varepsilon_j(n,h) = 0.$
In this step, we will show that

$$\lim_{h\to\infty}\limsup_{n\to\infty} \frac{1}{h}\sum_{i=1}^N \int_{\{|u_n|\le h\}} a_i(x,T_n(u_n),\nabla u_n)\cdot D^i u_n\,dx = 0. \tag{4.31}$$

Indeed, let $h \ge \max(\|\psi^+\|_\infty, 1)$ we set

$$v = u_n - \eta\frac{T_h(u_n-\psi^+)}{h}e^{B(|u_n|)} \in W_0^{1,\vec{p}}(\Omega) \qquad \text{where} \qquad B(s) = \int_0^s \frac{d\tau}{(1+|\tau|)^\theta},$$

with $1 < \theta < \min\left\{p_0 - \lambda, \sigma_i p_i' + 1\right\}$, for $i = 1, \ldots, N$. By taking η small enough, we can insert v as a test function in (4.1), and we obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \cdot D^i \left(\frac{T_h(u_n - \psi^+)}{h} e^{B(|u_n|)}\right) dx \\ + \int_\Omega \Gamma_n(x, \nabla u_n) \frac{T_h(u_n - \psi^+)}{h} e^{B(|u_n|)} dx \\ \leq \int_\Omega f_n(x, u_n) \frac{T_h(u_n - \psi^+)}{h} e^{B(|u_n|)} dx.\end{aligned} \tag{4.32}$$

From the condition (2.6), we have

$$\begin{aligned}&\frac{1}{h} \sum_{i=1}^{N} \int_{\{|u_n - \psi^+| \leq h\}} a_i(x, T_n(u_n), \nabla u_n) \cdot D^i(u_n - \psi^+) e^{B(|u_n|)} dx \\ &+ \frac{1}{h} \sum_{i=1}^{N} \int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n}{(1 + |u_n|)^\theta} |T_h(u_n - \psi^+)| e^{B(|u_n|)} dx \\ &\leq \frac{1}{h} \int_\Omega |f_n(x, u_n)|\, |T_h(u_n - \psi^+)| e^{B(|u_n|)} dx \\ &+ \frac{1}{h} \sum_{i=1}^{N} \int_\Omega c_i(x) |D^i u_n|^{q_i} |T_h(u_n - \psi^+)| e^{B(|u_n|)} dx,\end{aligned} \tag{4.33}$$

using (2.4) and (2.5), we conclude that

$$\begin{aligned}&\frac{1}{2h} \sum_{i=1}^{N} \int_{\{|u_n - \psi^+| \leq h\}} a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n \, dx + \frac{b_0}{2h} \sum_{i=1}^{N} \int_{\{|u_n - \psi^+| \leq h\}} \frac{|D^i u_n|^{p_i}}{(1 + |u_n|)^\lambda} dx \\ &+ \frac{b_0}{h} \sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i}}{(1 + |u_n|)^{\theta + \lambda}} |T_h(u_n - \psi^+)| dx \\ &\leq \frac{e^{B(\infty)}}{h} \int_\Omega |f_n(x, u_n)|\, |T_h(u_n - \psi^+)| dx + \frac{e^{B(\infty)}}{h} \sum_{i=1}^{N} \int_\Omega c_i(x) |D^i u_n|^{q_i} |T_h(u_n - \psi^+)| dx \\ &+ \frac{e^{B(\infty)}}{h} \sum_{i=1}^{N} \int_{\{|u_n - \psi^+| \leq h\}} |a_i(x, T_n(u_n), \nabla u_n)|\, |D^i \psi^+| \, dx.\end{aligned} \tag{4.34}$$

We have $\lim_{h \to \infty}$ meas $\{|u_n| > h\} = 0$ and $\psi^+ \in L^\infty(\Omega)$, it easy to see $\dfrac{|T_h(u_n - \psi^+)|}{h} \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$, moreover according to (4.27) we conclude that

$$\varepsilon_1(n, h) = \int_\Omega |f_n(x, u_n)| \frac{|T_h(u_n - \psi^+)|}{h} dx \longrightarrow 0 \qquad \text{as} \quad h \to \infty. \tag{4.35}$$

We can show similarly to (4.10) that

$$\begin{aligned}&\frac{e^{B(\infty)}}{h} \int_\Omega c_i(x) |D^i u_n|^{q_i} |T_h(u_n - \psi^+)| dx \\ &\leq \frac{C_{17}}{h} \int_\Omega |c_i(x)|^{\frac{p_i(p_0 - \theta - \lambda)}{p_i(p_0 - \theta - \lambda) - q_i p_0}} |T_h(u_n - \psi^+)| dx + \frac{1}{h} \int_\Omega |u_n|^{p_0 - \theta - \lambda} |T_h(u_n - \psi^+)| dx \\ &\quad + \frac{1}{4h} \int_\Omega \frac{b_0 |D^i u_n|^{p_i}}{(1 + |u_n|)^{\theta + \lambda}} |T_h(u_n - \psi^+)| dx.\end{aligned} \tag{4.36}$$

Thanks to (4.30) we have $|u_n|^{p_0 - \theta - \lambda} \to |u|^{p_0 - \theta - \lambda}$ strongly in $L^1(\Omega)$, and since $|c_i|^{\frac{p_i(p_0 - \theta - \lambda)}{p_i(p_0 - \theta - \lambda) - q_i p_0}}$ belong to $L^1(\Omega)$, it follows that

$$\varepsilon_2(n, h) = \frac{C_{17}}{h} \int_\Omega |c_i(x)|^{\frac{p_i(p_0 - \theta - \lambda)}{p_i(p_0 - \theta - \lambda) - q_i p_0}} |T_h(u_n - \psi^+)| dx + \frac{1}{h} \int_\Omega |u_n|^{p_0 - \theta - \lambda} |T_h(u_n - \psi^+)| dx \longrightarrow 0, \tag{4.37}$$

as h tends to infinity. It follows that

$$\frac{e^{B(\infty)}}{h}\int_\Omega |c_i(x)|\,|D^iu_n|^{q_i}|T_h(u_n-\psi^+)|dx \le \varepsilon_2(h)+\frac{1}{4h}\int_\Omega \frac{b_0|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}|T_h(u_n-\psi^+)|dx. \tag{4.38}$$

Now, we treat the last behavior on the right-hand side of (4.34). Applying Young's inequality and the growth hypothesis (2.2), we can write

$$\begin{aligned}
&\frac{e^{B(\infty)}}{h}\int_{\{|u_n-\psi^+|\le h\}}|a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx\\
&\le \frac{b_0\varepsilon}{2^{p_i}\beta^{p_i'}h}\int_{\{|u_n-\psi^+|\le h\}}\frac{|a_i(x,T_n(u_n),\nabla u_n)|^{p_i'}}{(1+|u_n|)^\lambda}\,dx+\frac{C_{18}}{h}\int_{\{|u_n-\psi^+|\le h\}}|D^i\psi^+|^{p_i}(1+|u_n|)^{\frac{p_i\lambda}{p_i'}}\,dx\\
&\le \frac{b_0\varepsilon}{2h}\int_{\{|u_n-\psi^+|\le h\}}\frac{|a_0|^{p_i'}}{(1+|u_n|)^\lambda}\,dx+\frac{b_0\varepsilon}{2h}\int_{\{|u_n-\psi^+|\le h\}}\frac{|u_n|^{p_i-\sigma_ip_i'}}{(1+|u_n|)^\lambda}\,dx\\
&\quad+\frac{b_0\varepsilon}{2h}\int_{\{|u_n-\psi^+|\le h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\lambda}\,dx+\frac{C_{19}h^{\frac{\lambda p_i}{p_i'}}}{h}\int_{\{|u_n-\psi^+|\le h\}}|D^i\psi^+|^{p_i}dx.
\end{aligned}$$

Let $M=h+\|\psi^+\|_\infty$, and since $1<\theta\le\sigma_ip_i'+1$ for $i=1,\dots,N$. In view of Poincaré's inequality, we have

$$\begin{aligned}
&\frac{1}{h}\int_{\{|u_n-\psi^+|\le h\}}\frac{|u_n|^{p_i-\sigma_ip_i'}}{(1+|u_n|)^\lambda}\,dx\le\frac{2^{p_i-1}}{h}\int_{\{|u_n|\le M\}}\frac{|u_n|^{p_i}}{(1+|u_n|)^{\lambda+\sigma_ip_i'}}\,dx+\varepsilon_3(h)\\
&\le\frac{2^{p_i-1}}{h}\int_\Omega\Big|\int_0^{|T_M(u_n)|}\frac{ds}{(1+s)^{\frac{\lambda+\sigma_ip_i'}{p_i}}}\Big|^{p_i}\,dx+\varepsilon_3(h)\\
&\le\frac{2^{p_i-1}C_{p_i}^{p_i}}{h}\int_\Omega\frac{|D^iT_M(u_n)|^{p_i}}{(1+|u_n|)^{\lambda+\sigma_ip_i'}}\,dx+\varepsilon_3(h)\\
&\le\frac{2^{p_i-1}C_{p_i}^{p_i}}{h}\int_{\{|u_n-\psi^+|\le h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\lambda}\,dx\\
&\qquad+2^{p_i+1}C_{p_i}^{p_i}\int_{\{|u_n-\psi^+|>h\}\cap\{|u_n|\le M\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\lambda+\sigma_ip_i'+1}}\,dx+\varepsilon_3(h)\\
&\le\frac{2^{p_i-1}C_{p_i}^{p_i}}{h}\int_{\{|u_n-\psi^+|\le h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\lambda}\,dx+2^{p_i+1}C_{p_i}^{p_i}\int_{\{|u_n-\psi^+|>h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\lambda+\theta}}\,dx+\varepsilon_3(h).
\end{aligned}$$

Thus, by taking $\varepsilon>0$ small enough, we obtain

$$\begin{aligned}
&\frac{e^{B(\infty)}}{h}\int_{\{|u_n-\psi^+|\le h\}}|a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx\\
&\le\frac{b_0}{2h}\int_{\{|u_n-\psi^+|\le h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\lambda}\,dx+\frac{b_0}{2h}\sum_{i=1}^N\int_\Omega\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}|T_h(u_n-\psi^+)|dx+\varepsilon_4(n,h).
\end{aligned}\tag{4.39}$$

By combining (4.34) and (4.35) – (4.39) we deduce that

$$\begin{aligned}
&\frac{1}{2h}\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le h\}}a_i(x,T_n(u_n),\nabla u_n)\cdot D^iu_n\,dx\\
&+\frac{b_0}{4h}\sum_{i=1}^N\int_\Omega\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}|T_h(u_n-\psi^+)|dx\le\varepsilon_5(n,h).
\end{aligned}\tag{4.40}$$

By letting h goes to infinity in (4.40), we get

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^N\int_{\{|u_n|\le h\}}a_i(x,T_n(u_n),\nabla u_n)\cdot D^iu_n\,dx=0. \tag{4.41}$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^N\int_{\{|u_n|>h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx=0. \tag{4.42}$$

Step 5 : Almost everywhere convergence of the gradients

We set

$$S_h(s) = 1 - \frac{|T_{2h}(s) - T_h(s)|}{h} \qquad \text{and} \qquad \varphi(s) = s.\exp(s^2) \quad \text{for any} \quad s \in \mathbb{R},$$

where $h > k \geq \max\{1, \|\psi^+\|_\infty\}$. Evidently, $\varphi'(s) - 2|\varphi(s)| \geq \frac{1}{2}$ for any $s \in \mathbb{R}$.
Let η small enough such that $v = u_n - \eta\varphi(T_k(u_n) - T_k(u))S_h(u_n)e^{B(|u_n|)}$ belong to K_ψ, taking v as test function in (4.3), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x, T_n(u_n), \nabla u_n)\cdot D^i\Big(\varphi(T_k(u_n) - T_k(u))S_h(u_n)e^{B(|u_n|)}\Big)\,dx \\
&\qquad + \int_\Omega \Gamma_n(x, \nabla u_n)\varphi(T_k(u_n) - T_k(u))S_h(u_n)e^{B(|u_n|)}\,dx \\
&\leq \int_\Omega f_n(x, u_n)\varphi(T_k(u_n) - T_k(u))S_h(u_n)e^{B(|u_n|)}\,dx.
\end{aligned} \tag{4.43}$$

Then

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n))\cdot(D^iT_k(u_n) - D^iT_k(u))\varphi'(T_k(u_n) - T_k(u))S_h(u_n)e^{B(|u_n|)}\,dx \\
&+\sum_{i=1}^{N}\int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n)\cdot D^iu_n}{(1+|u_n|)^\theta}\varphi(T_k(u_n) - T_k(u))\mathrm{sign}(u_n)S_h(u_n)e^{B(|u_n|)}\,dx \\
&-\frac{1}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|<2h\}} a_i(x, T_n(u_n), \nabla u_n)\cdot D^iu_n|\varphi(T_k(u_n) - T_k(u))|e^{B(|u_n|)}\,dx \\
&+\int_\Omega \Gamma_n(x, \nabla u_n)\varphi(T_k(u_n) - T_k(u))S_h(u_n)e^{B(|u_n|)}\,dx \\
&\leq \int_\Omega f_n(x, u_n)\varphi(T_k(u_n) - T_k(u))S_h(u_n)e^{B(|u_n|)}\,dx.
\end{aligned} \tag{4.44}$$

It's easy to remark that $\varphi(T_k(u_n) - T_k(u))$ has the same sign as u_n on the set $\{|u_n| > k\}$. Thus, using (2.6) we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n))\cdot(D^iT_k(u_n) - D^iT_k(u))\varphi'(T_k(u_n) - T_k(u))\,dx \\
&-\sum_{i=1}^{N}\int_{\{k<|u_n|\leq 2h\}} |a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))|\,|D^iT_k(u)|\varphi'(T_k(u_n) - T_k(u))\,dx \\
&-\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}} \frac{a_i(x, T_k(u_n), \nabla T_k(u_n))\cdot D^iT_k(u_n)}{(1+|u_n|)^\theta}|\varphi(T_k(u_n) - T_k(u))|S_h(u_n)\,dx \\
&+\sum_{i=1}^{N}\int_{\{k<|u_n|\leq 2h\}} \frac{a_i(x, T_n(u_n), \nabla u_n)\cdot D^iu_n}{(1+|u_n|)^\theta}|\varphi(T_k(u_n) - T_k(u))|S_h(u_n)\,dx \\
&\leq e^{B(\infty)}\int_\Omega |f_n(x, u_n)|\,|\varphi(T_k(u_n) - T_k(u))|\,dx \\
&\qquad +\frac{e^{B(\infty)}}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|<2h\}} a_i(x, T_n(u_n), \nabla u_n)\cdot D^iu_n|\varphi(T_k(u_n) - T_k(u))|\,dx \\
&\qquad +e^{B(\infty)}\sum_{i=1}^{N}\int_\Omega c_i(x)|D^iu_n|^{q_i}\,|\varphi(T_k(u_n) - T_k(u))|S_h(u_n)\,dx.
\end{aligned} \tag{4.45}$$

Look to the first behavior on the right-hand side of (4.45). We have $\varphi(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ as n goes to infinity and according to (4.27), we get

$$\varepsilon_5(n) = e^{B(\infty)}\int_\Omega |f_n(x, u_n)|\,|\varphi(T_k(u_n) - T_k(u))|\,dx \longrightarrow 0 \qquad \text{as} \qquad n \to \infty. \tag{4.46}$$

In view of (4.41) we obtain

$$\begin{aligned}\varepsilon_6(h) &= \frac{e^{B(\infty)}}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|<2h\}} a_i(x,T_n(u_n),\nabla u_n)\cdot D^i u_n|\varphi(T_k(u_n)-T_k(u))|\,dx \\ &\leq \frac{\varphi(2k)e^{B(\infty)}}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|<2h\}} a_i(x,T_n(u_n),\nabla u_n)\cdot D^i u_n\,dx \longrightarrow 0 \quad \text{as} \quad h\to\infty.\end{aligned} \tag{4.47}$$

On the one hand, applying Young's inequality and presumptions (2.4) we have

$$\begin{aligned}&e^{B(\infty)}\sum_{i=1}^{N}\int_\Omega c_i(x)|D^i u_n|^{q_i}\,|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\,dx \\ &\leq C\sum_{i=1}^{N}\int_\Omega |c_i(x)|^{\frac{p_i(p_0-\theta-\lambda)}{p_i(p_0-\theta-\lambda)-q_ip_0}}|\varphi(T_k(u_n)-T_k(u))|\,dx+\int_\Omega |u_n|^{p_0-\theta-\lambda}|\varphi(T_k(u_n)-T_k(u))|\,dx \\ &\quad+\sum_{i=1}^{N}\int_\Omega \frac{b_0|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\,dx \\ &\leq C\sum_{i=1}^{N}\int_\Omega |c_i(x)|^{\frac{p_i(p_0-\theta-\lambda)}{p_i(p_0-\theta-\lambda)-q_ip_0}}|\varphi(T_k(u_n)-T_k(u))|\,dx+\int_\Omega |u_n|^{p_0-\theta-\lambda}|\varphi(T_k(u_n)-T_k(u))|\,dx \\ &+\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\frac{a_i(x,T_k(u_n),\nabla T_k(u_n))\cdot D^iT_k(u_n)}{(1+|u_n|)^\theta}|\varphi(T_k(u_n)-T_k(u))|\,dx \\ &+\sum_{i=1}^{N}\int_{\{k<|u_n|\leq 2h\}}\frac{a_i(x,T_n(u_n),\nabla u_n)\cdot D^iu_n}{(1+|u_n|)^\theta}|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\,dx,\end{aligned} \tag{4.48}$$

since $|c_i(x)|^{\frac{p_i(p_0-\theta-\lambda)}{p_i(p_0-\theta-\lambda)-q_ip_0}}$ belongs to $L^1(\Omega)$, and thanks to (4.30) we deduce that

$$\begin{aligned}\varepsilon_7(n) &= C\sum_{i=1}^{N}\int_\Omega |c_i(x)|^{\frac{p_i(p_0-\theta-\lambda)}{p_i(p_0-\theta-\lambda)-q_ip_0}}|\varphi(T_k(u_n)-T_k(u))|\,dx \\ &\quad+\int_\Omega |u_n|^{p_0-\theta-\lambda}|\varphi(T_k(u_n)-T_k(u))|\,dx \longrightarrow 0 \qquad \text{as} \qquad n\to\infty.\end{aligned} \tag{4.49}$$

Therefore

$$\begin{aligned}&e^{B(\infty)}\sum_{i=1}^{N}\int_\Omega c_i(x)|D^i u_n|^{q_i}\,|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\,dx \\ &\leq \varepsilon_7(n)+\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\frac{a_i(x,T_k(u_n),\nabla T_k(u_n))\cdot D^iT_k(u_n)}{(1+|u_n|)^\theta}|\varphi(T_k(u_n)-T_k(u))|\,dx \\ &+\sum_{i=1}^{N}\int_{\{k<|u_n|\leq 2h\}}\frac{a_i(x,T_n(u_n),\nabla u_n)\cdot D^iu_n}{(1+|u_n|)^\theta}|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\,dx.\end{aligned} \tag{4.50}$$

By combining (4.45) – (4.47) and (4.50), we conclude that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))\,dx \\ &-2\sum_{i=1}^{N}\int_\Omega \frac{a_i(x,T_k(u_n),\nabla T_k(u_n))\cdot D^iT_k(u_n)}{(1+|u_n|)^\theta}|\varphi(T_k(u_n)-T_k(u))|\,dx \\ &\leq \varepsilon_8(n,h)+(\varphi(2k)+\varphi'(2k))\sum_{i=1}^{N}\int_{\{k<|u_n|\leq 2h\}}|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\,|D^iT_k(u)|\,dx.\end{aligned} \tag{4.51}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u))\\
&\qquad\qquad\times\Big(\varphi'(T_k(u_n)-T_k(u))-2|\varphi(T_k(u_n)-T_k(u))|\Big)\,dx\\
&\le(\varphi(2k)+\varphi'(2k))\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}}|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\,|D^iT_k(u)|\,dx\\
&+(\varphi'(2k)+2\varphi(2k))\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))|\,|D^iT_k(u_n)-D^iT_k(u)|\,dx\\
&+2\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u))|\varphi(T_k(u_n)-T_k(u))|\,dx+\varepsilon_8(n,h).
\end{aligned}\tag{4.52}$$

For the first term on the right-hand side of (4.52). Thanks to (2.2), we have $(|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|)_{n\in\mathbb{N}}$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\xi_{i,h}\in L^{p_i'}(\Omega)$ such that $|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\rightharpoonup\xi_{i,h}$ weakly in $L^{p_i'}(\Omega)$. Therefore,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}}|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\,|D^iT_k(u)|\,dx\\
&\longrightarrow\sum_{i=1}^{N}\int_{\{k<|u|\le 2h\}}\xi_{i,h}|D^iT_k(u)|\,dx=0\qquad\text{as}\quad n\to\infty.
\end{aligned}\tag{4.53}$$

Concerning the second term on the right-hand side of (4.52), by applying the Lebesgue dominated convergence theorem, we have $T_k(u_n)\to T_k(u)$ strongly in $L^{p_i}(\Omega)$, then $|a_i(x,T_k(u_n),\nabla T_k(u))|\to|a_i(x,T_k(u),\nabla T_k(u))|$ strongly in $L^{p_i'}(\Omega)$, and since $D^iT_k(u_n)\rightharpoonup D^iT_k(u)$ weakly in $L^{p_i}(\Omega)$, we obtain

$$\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))|\,|D^iT_k(u_n)-D^iT_k(u)|\,dx\longrightarrow 0\qquad\text{as}\quad n\to\infty.\tag{4.54}$$

For the third term on the right-hand side of (4.52), we have $(a_i(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\xi_{i,k}\in L^{p_i'}(\Omega)$ such that $a_i(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup\xi_{i,k}$ weakly in $L^{p_i'}(\Omega)$, and since $D^iT_k(u)|\varphi(T_k(u_n)-T_k(u))|\to 0$ strongly in $L^{p_i}(\Omega)$, it follows that

$$\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)|\varphi(T_k(u_n)-T_k(u))|\,dx\longrightarrow 0\quad\text{as}\quad n\to\infty.\tag{4.55}$$

Thus, by combining (4.52) and (4.53) – (4.55), we arrive that

$$\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u))dx\le\varepsilon_9(n,h).\tag{4.56}$$

Therefore, by letting n and h tend to infinity in the previous inequality, and since $T_k(u_n)\to T_k(u)$ strongly in $L^{\underline{p}}(\Omega)$ we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\cdot(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\quad+\int_{\Omega}\left(|T_k(u_n)|^{\underline{p}-2}T_k(u_n)-|T_k(u)|^{\underline{p}-2}T_k(u_n)\right)(T_k(u_n)-T_k(u))\,dx\to 0\quad\text{as}\quad n\to\infty.
\end{aligned}\tag{4.57}$$

By using Lemma 2.3, we deduce that

$$T_k(u_n)\longrightarrow T_k(u)\qquad\text{strongly in}\quad W_0^{1,\vec{p}}(\Omega)\quad\text{and}\quad\nabla u_n\longrightarrow\nabla u\qquad\text{a.e. in}\quad\Omega.\tag{4.58}$$

Therefore, for $i = 1, \cdots, N$ we have $a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n$ tends to $a_i(x, u, \nabla u) \cdot D^i u$ almost everywhere in Ω. Thanks to Fatou's Lemma and (4.41), we conclude that

$$\begin{aligned} &\lim_{h\to\infty} \frac{1}{h} \int_{\{|u|\le h\}} a_i(x, u, \nabla u) \cdot D^i u \, dx \\ &\le \lim_{h\to\infty} \liminf_{n\to\infty} \frac{1}{h} \int_{\{|u_n|\le h\}} a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n \, dx \\ &\le \lim_{h\to\infty} \limsup_{n\to\infty} \frac{1}{h} \int_{\{|u_n|\le h\}} a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n \, dx = 0. \end{aligned} \tag{4.59}$$

Step 6 : Equi-integrability of the sequence $(\Gamma_n(x, \nabla u_n))_n$.

In this part, we show that

$$\Gamma_n(x, \nabla u_n) \longrightarrow \Gamma(x, \nabla u) \qquad \text{strongly in} \quad L^1(\Omega). \tag{4.60}$$

In view of (4.58), we have

$$\Gamma_n(x, \nabla u_n) \longrightarrow \Gamma(x, \nabla u) \qquad \text{a.e. in} \quad \Omega. \tag{4.61}$$

Then, to show that (4.60), by using Vitali's theorem, it is sufficient to prove that the sequence $(\Gamma_n(x, \nabla u_n))_n$ is uniformly equi-integrable.
In view of Young's inequality, for any measurable subset $E \subset \Omega$ and for all $h > 0$, we have

$$\begin{aligned} \int_E |\Gamma_n(x, \nabla u_n)| \, dx &\le \sum_{i=1}^N \int_E c_i(x) |D^i u_n|^{q_i} \, dx \\ &\le C_{20} \sum_{i=1}^N \int_E |c_i(x)|^{\frac{p_i(p_0-\theta-\lambda)}{p_i(p_0-\theta-\lambda)-q_i p_0}} \, dx + \int_E |u_n|^{p_0-\theta-\lambda} \, dx \\ &\quad + \sum_{i=1}^N \int_E \frac{|D^i T_h(u_n)|^{p_i}}{(1+|T_h(u_n)|)^{\lambda+\theta}} \, dx + \sum_{i=1}^N \int_{E\cap\{|u_n|>h\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+\theta}} \, dx. \end{aligned} \tag{4.62}$$

We have $|c_i(x)|^{\frac{p_i(p_0-\theta-\lambda)}{p_i(p_0-\theta-\lambda)-q_i p_0}} \in L^1(\Omega)$, in addition to (4.12) and (4.30), we conclude that : for any $\varepsilon > 0$, there exists $\beta(\varepsilon, h) > 0$ such that: for $meas(E) \le \beta(\varepsilon)$ we have

$$C_{20} \sum_{i=1}^N \int_E |c_i(x)|^{\frac{p_i(p_0-\theta-\lambda)}{p_i(p_0-\theta-\lambda)-q_i p_0}} \, dx + \int_E |u_n|^{p_0-\theta-\lambda} \, dx + \sum_{i=1}^N \int_E \frac{|D^i T_h(u_n)|^{p_i}}{(1+|T_h(u_n)|)^{\lambda+\theta}} \, dx \le \frac{\varepsilon}{2}. \tag{4.63}$$

On the other hand, in view of (4.42) we have : for all $\varepsilon > 0$, there exists h_0 such that

$$\sum_{i=1}^N \int_{E\cap\{|u_n|>h\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+\theta}} \, dx \le \frac{\varepsilon}{3} \qquad \forall h \ge h_0. \tag{4.64}$$

By combining (4.62) and (4.63) – (4.64), we conclude that : for any $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that: for all $E \subset \Omega$ with $meas(E) \le \beta(\varepsilon)$, we have

$$\int_E |\Gamma_n(x, \nabla u_n)| \, dx \le \varepsilon. \tag{4.65}$$

Then, $(\Gamma_n(x, \nabla u_n))_n$ is uniformly equi-integrable, consequently

$$\Gamma_n(x, \nabla u_n) \longrightarrow \Gamma(x, \nabla u) \qquad \text{strongly in} \quad L^1(\Omega). \tag{4.66}$$

Step 7 : Passage to the limit.

Now, we will passe to the limit in our approximate problem (4.3).

Indeed, let $\varphi \in K_\psi \cap L^\infty(\Omega)$ and $M = k + \|\varphi\|_\infty$. By taking $v = u_n - \eta T_k(u_n - \varphi)$ as a test function in (4.3), we obtain

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \cdot D^i T_k(u_n - \varphi)\, dx + \int_\Omega \Gamma_n(x, \nabla u_n) T_k(u_n - \varphi)\, dx \\ \leq \int_\Omega f_n(x, u_n) T_k(u_n - \varphi)\, dx. \tag{4.67}$$

Firstly, we have $\{|u_n - \varphi| \leq k\} \subseteq \{|u_n| \leq M\}$, In view of Fatou's Lemma, the first term on the left-hand side of (4.67) can be written as

$$\begin{aligned} &\liminf_{n\to\infty} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \cdot D^i T_k(u_n - \varphi)\, dx \\ &= \liminf_{n\to\infty} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \cdot (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}}\, dx \\ &= \liminf_{n\to\infty} \int_\Omega (a_i(x, T_M(u), \nabla T_M(u)) - a_i(x, T_M(u), \nabla \varphi)) \cdot (D^i T_M(u) - D^i \varphi) \chi_{\{|u - \varphi| \leq k\}}\, dx \\ &\quad + \lim_{n\to\infty} \int_\Omega a_i(x, T_M(u_n), \nabla \varphi) \cdot (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}}\, dx \\ &\geq \int_\Omega a_i(x, T_M(u), \nabla T_M(u)) \cdot (D^i T_M(u) - D^i \varphi) \chi_{\{|u - \varphi| \leq k\}}\, dx \\ &= \int_\Omega a_i(x, u, \nabla u) \cdot D^i T_k(u - \varphi)\, dx. \end{aligned} \tag{4.68}$$

Secondly, we have $T_k(u_n - \varphi) \rightharpoonup T_k(u - \varphi)$ weak–$\star$ in $L^\infty(\Omega)$. We conclude from (4.66) and (4.27) that

$$\int_\Omega \Gamma_n(x, \nabla u_n) T_k(u_n - \varphi)\, dx \longrightarrow \int_\Omega \Gamma(x, \nabla u) T_k(u - \varphi)\, dx, \tag{4.69}$$

and

$$\int_\Omega f_n(x, u_n) T_k(u_n - \varphi)\, dx \longrightarrow \int_\Omega f(x, u) T_k(u - \varphi)\, dx. \tag{4.70}$$

By combining (4.67), and (4.68) – (4.70), we deduce that

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \cdot D^i T_k(u - \varphi)\, dx + \int_\Omega \Gamma(x, \nabla u) T_k(u - \varphi)\, dx \leq \int_\Omega f(x, u) T_k(u - \varphi)\, dx, \tag{4.71}$$

for all $\varphi \in K_\psi \cap L^\infty(\Omega)$, which complete the proof of theorem 3.2.

5 Appendix

Lemma 5.1. *The bounded operator $B_n = A_n + G_n$, acted from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$, is pseudo-monotone. Moreover, B_n is coercive in the following sense : there exists $v_0 \in K_\psi$ such that*

$$\frac{\langle B_n v, v - v_0 \rangle}{\|v\|_{1,\vec{p}}} \longrightarrow \infty \qquad \text{as} \quad \|v\|_{1,\vec{p}} \to \infty \quad \text{for} \quad v \in K_\psi.$$

Proof of Lemma 5.1

Evidently, the operator A_n is bounded, and by using (4.2), we may infer that B_n is also bounded. For the coercivity, let $v_0 \in K_\psi$, we have for any $v \in K_\psi$

$$\begin{aligned}
|\langle A_n v, v_0\rangle| &= \Big|\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(v),\nabla v)\cdot D^i v_0\,dx\Big| \\
&\leq \sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(v),\nabla v)||D^i v_0|\,dx \\
&\leq \beta\sum_{i=1}^{N}\int_\Omega (K_i(x)+|T_n(v)|^{p_i-1-\sigma_i}+|D^i v|^{p_i-1})|D^i v_0|\,dx \\
&\leq \sum_{i=1}^{N}\int_\Omega |K_i(x)|^{p_i'}\,dx+\sum_{i=1}^{N}\int_\Omega n^{p_i}\,dx+\frac{b_0}{2(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i v|^{p_i}\,dx \\
&\qquad +C_0(n)\sum_{i=1}^{N}\int_\Omega |D^i v_0|^{p_i}\,dx \\
&\leq C_1+\frac{b_0}{2(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i v|^{p_i}\,dx+C_0\sum_{i=1}^{N}\int_\Omega |D^i v_0|^{p_i}\,dx.
\end{aligned} \tag{5.1}$$

Using (2.4) and the Poincaré's inequality, we have

$$\begin{aligned}
|\langle A_n v, v\rangle| &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(v),\nabla v)\cdot D^i v\,dx \\
&\geq \frac{b_0}{(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i v|^{p_i}\,dx \\
&\geq C_2\|v\|_{1,\vec{p}}^{\underline{p}}+\frac{b_0}{2(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i v|^{p_i}\,dx - C_3.
\end{aligned} \tag{5.2}$$

It follows that

$$\frac{\langle A_n v, v-v_0\rangle}{\|v\|_{1,\vec{p}}} \geq C_2\frac{\|v\|_{1,\vec{p}}^{\underline{p}}}{\|v\|_{1,\vec{p}}}-\frac{C_1+C_3}{\|v\|_{1,\vec{p}}}-\frac{C_0}{\|v\|_{1,\vec{p}}}\sum_{i=1}^{N}\int_\Omega |D^i v_0|^{p_i}\,dx \longrightarrow \infty \quad \text{as} \quad \|v\|_{1,\vec{p}}\to\infty. \tag{5.3}$$

On the one hand, we have

$$|\langle G_n v, v-v_0\rangle| \leq 2n(\|v\|_{1,\vec{p}}+\|v_0\|_{1,\vec{p}}). \tag{5.4}$$

We conclude that

$$\frac{\langle B_n v, v-v_0\rangle}{\|v\|_{1,\vec{p}}} = \frac{\langle A_n v, v-v_0\rangle}{\|v\|_{1,\vec{p}}}+\frac{\langle G_n v, v-v_0\rangle}{\|v\|_{1,\vec{p}}} \to \infty \quad \text{as} \quad \|v\|_{1,\vec{p}}\to\infty, \tag{5.5}$$

and similarly as in [9], we can show that the operator B_n is pseudo-monotone, which completes the proof of lemma 5.1.

Conflict of interest

All authors declare that they have no conflicts of interest.

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