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Commutativity of prime rings and Banach algebras with generalized (β, α)-derivations

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Abstract The commutativity of prime associative rings and prime Banach algebras with generalized (β, α)-derivations satisfying certain differential identities is investigated in this study. Consequently, we use continuous linear generalized (β, α)-derivations to extend our theoretical results on rings to unital Banach algebras. Examples are also given to prove the necessity of imposing restrictions on the hypotheses of our various theorems.

1 Introduction

In ring theory, numerous research in ring theory have demonstrated that certain rings must be commutative under certain conditions. Our major objective is here is to use this idea to investigate the commutativity property of Banach algebras. In all that follows, as usual, the symbols $Z(\mathcal{R})$ is the multiplicative center of a ring $\mathcal{R}$, $[s,t]$ will represent the commutator $st - ts$, $I_{\mathcal{R}}$ will represent the identity mapping of $\mathcal{R}$, and $0_{\mathcal{R}}$ will represent the null application of $\mathcal{R}$. For any subset $A \subseteq \mathcal{R}$, $\mathcal{C}_{\mathcal{R}}(A) = \{x \in \mathcal{R} \mid xa = ax \quad \forall\, a \in A\}$ is said to be the centralizer of $\mathcal{A}$, and it is obviously a subring of $\mathcal{R}$. Recalling that a ring $\mathcal{S}$ is called prime if $x\mathcal{S}y = \{0\}$ gives $x = 0$ or $y = 0$, and $\mathcal{S}$ is called semiprime if $x\mathcal{S}x = \{0\}$ implies $x = 0$. An additive mapping $H : \mathcal{R} \longrightarrow \mathcal{R}$ is said to be a left multiplier if $H(rs) = H(r)s \;\; \forall\, r, s \in \mathcal{R}$. Let φ and ψ are automorphisms of $\mathcal{R}$. An additive mapping $d : \mathcal{R} \longrightarrow \mathcal{R}$ is called a (φ,ψ)-derivation if $d(xy) = d(x)\varphi(y) + \psi(x)d(y) \;\; \forall\;\, x, y \in \mathcal{R}$. Furthermore, a mapping F on $\mathcal{R}$ is called a generalized (φ,ψ)-derivation if there exists a (φ,ψ)-derivation d on $\mathcal{R}$ such that $F(xy) = F(x)\varphi(y) + \psi(x)d(y) \;\forall\; x, y \in \mathcal{R}$.
The commutativity of prime rings admitting derivations and generalized derivations satisfying suitable algebraic criteria has been studied by a number of authors. In this context, we recall an important result due to Howard E. Bell [3, Theorem 4.1], where he showed that a prime ring R with nonzero center and $char(\mathcal{R}) = 0$ or $char(\mathcal{R}) > n > 1$ must be a commutative ring if $\mathcal{R}$ admits a nonzero derivation d satisfying $d([x^n, y]) - [x, y^n] \in Z(\mathcal{R}) \;\; \forall\, x, y \in \mathcal{R}$. Recently, in [2, Theorem 2.4], Ashraf et al. proved that if a prime ring $\mathcal{R}$ admits a generalized derivation f with associated with a derivation d such that $d(Z(\mathcal{R})) \neq \{0\}$ satisfying either $f([r^m, s^n]) + [r^m, s^n] \in Z(R)$ or $f([r^m, s^n]) - [r^m, s^n] \in Z(\mathcal{R}) \;\; \forall\, r, s \in \mathcal{R}$, where $m, n \in \mathbb{N}$, then $\mathcal{R}$ is commutative. As an application, in [1] S. Ali et al. S. Ali et al. extended the above theorems to the area of Banach algebras, proving the following: Let $d : \mathcal{A} \longrightarrow \mathcal{A}$ be a nonzero continuous linear derivation of a prime Banach algebra $\mathcal{A}$. If there exist open subsets G_1 and G_2 of $\mathcal{A}$ such that either $d((xy)^m) - x^m y^m \in Z(\mathcal{A})$ or $d((xy)^m) - y^m x^m \in Z(\mathcal{A})$ for each $x \in G_1$, $y \in G_2$ and $m = m(x, y) \in \mathbb{N}^*$. Then $\mathcal{A}$ is commutative. Motivated by the above-mentioned results, our objective is to investigate the commutativity of Banach algebras via (β, α)-derivations satisfying some algebraic identities. Moreover, we connected these identities to a left multiplier H for further generalization.
Note that The following commutator identities will be used without special mention: For every

$u, v, w \in \mathcal{R}$, we get $[u, vw] = v[u, w] + [u, v]w$ and $[uv, w] = u[v, w] + [u, w]v$.

2 Commutativity of prime rings with generalized (β, α)-derivations

In this section, we will study the commutativity of prime rings using generalized (β, α)-derivations that satisfy some algebraic identities. The following results, will serve as a starting point for our discussion.

Lemma 2.1. [4, Lemma 4] *If $aUb = \{0\}$ and $U \nsubseteq Z(\mathcal{R})$ is a Lie ideal of $\mathcal{R}$, then $a = 0$ or $b = 0$.*

Lemma 2.2. [7, lemma 1] *Let $\mathcal{R}$ be a semiprime, 2-torsion-free ring and let U be a Lie ideal of $\mathcal{R}$. If $[U, U] \subseteq Z(\mathcal{R})$, then $Z(\mathcal{R})$ contains U.*

Lemma 2.3. [4, Lemma 2] *If U is a Lie ideal of a prime ring $\mathcal{R}$ which is not included in $Z(\mathcal{R})$, then $\mathcal{C}_{\mathcal{R}}(U) = Z(\mathcal{R})$.*

Lemma 2.4. [3, Lemma 1.2] *Let $\mathcal{R}$ be a prime ring that satisfies an identity $q(X) = 0$, where $q(X)$ is a polynomial in a finite number of non-commuting indeterminate, its coefficients being integers with the highest common factor 1. If there exists no prime p for which the ring of 2×2 matrices over $GF(p)$ satisfies $q(X) = 0$, then $\mathcal{R}$ is commutative.*

In the rest of this section, d represents a (β, α)-derivation of $\mathcal{R}$ satisfying the assumption $d(Z(\mathcal{R})) \neq \{0\}$. Noting this, we call a Lie ideal U proper if there exists a non-zero ideal I of $\mathcal{R}$ such that $[I, \mathcal{R}] \subseteq U$. As a property of a Lie ideal U we recall the following property: U is proper if and only if $[U, U] \neq \{0\}$ (see, [6], page 2).

Theorem 2.5. *Let $\mathcal{R}$ be a 2-torsion free prime ring, U be a Lie ideal of $\mathcal{R}$, and F be a generalized (β, α)-derivation associated with a map d. If $F([u, v]) \in Z(\mathcal{R})\ \forall\, u, v \in U$, then U is contained in the center of $\mathcal{R}$.*

Proof. Supposing that
$$F([u, v]) \in Z(\mathcal{R})\ \forall\ u, v \in U. \tag{2.1}$$
Taking vz instead of v in (2.1), where $z \in Z(\mathcal{R})$ and using it, we get
$$\alpha([u, v])d(z) \in Z(\mathcal{R})\ \forall\ u, v \in U. \tag{2.2}$$
Which equivalent to
$$[u, v]\alpha^{-1}(d(z)) \in Z(\mathcal{R})\ \forall\ u, v \in U,\ z \in Z(\mathcal{R}). \tag{2.3}$$
Replacing u by $[s, v]$ in (2.3), we get
$$[[s, v], v]\alpha^{-1}(d(z)) \in Z(\mathcal{R})\ \forall\ v \in U,\ s \in R,\ z \in Z(\mathcal{R}). \tag{2.4}$$
Taking vs in place of s in (2.4), we find
$$v[[s, v], v]\alpha^{-1}(d(z)) \in Z(\mathcal{R})\ \forall\ v \in U,\ s \in \mathcal{R},\ z \in Z(\mathcal{R}). \tag{2.5}$$
Using (2.4), we may easily arrive at
$$v \in Z(R)\ \text{ or }\ [[s, v], v]\alpha^{-1}(d(z)) = 0\ \forall\ s \in \mathcal{R},\ v \in U,\ z \in Z(\mathcal{R}).$$
The last two expressions force
$$[[s, v], v]\alpha^{-1}(d(z)) = 0\ \forall\ v \in U,\ s \in \mathcal{R},\ z \in Z(\mathcal{R}). \tag{2.6}$$
Replacing s by st in (2.6) and using it, we infer that
$$2[s, v][t, v]\alpha^{-1}(d(z)) + [[s, v], v]t\alpha^{-1}(d(z)) = 0\ \forall\ v \in U,\ s, t \in \mathcal{R},\ z \in Z(\mathcal{R}). \tag{2.7}$$

Substituting tr instead of t in (2.7), then for all $v \in U,\ s,t \in \mathcal{R},\ z \in Z(\mathcal{R})$, we get

$$2[s,v]t[r,v]\alpha^{-1}(d(z)) + 2[s,v][t,v]r\alpha^{-1}(d(z)) + \big[[s,v],v\big]tr\alpha^{-1}(d(z)) = 0 \qquad (2.8)$$

Right-multiplying (2.7) by r and comparing it with (2.8), we find that

$$2[s,v]t[r,v]\alpha^{-1}(d(z)) + 2[s,v][t,v][r,\alpha^{-1}(d(z))] + \big[[s,v],v\big]t[r,\alpha^{-1}(d(z))] = 0 \qquad (2.9)$$

$\forall\, v \in U,\ r,s,\ t \in \mathcal{R},\ z \in Z(\mathcal{R})$. For $r = \alpha^{-1}(d(z))$ in (2.9) with the 2-torsion freeness of $\mathcal{R}$, we remark that

$$[s,v]t[\alpha^{-1}(d(z)),v]\alpha^{-1}(d(z)) = 0 \ \forall\, v \in U,\ s,t \in \mathcal{R},\ z \in Z(\mathcal{R}). \qquad (2.10)$$

Since $\mathcal{R}$ is prime, we get

$$[s,v] = 0 \ \text{ or } \ [\alpha^{-1}(d(z)),v]\alpha^{-1}(d(z)) = 0 \ \forall\, s \in \mathcal{R},\ v \in U,\ z \in Z(\mathcal{R}).$$

The both equations give

$$[\alpha^{-1}(d(z)),v]\alpha^{-1}(d(z)) = 0 \ \forall\, v \in U,\ z \in Z(\mathcal{R}). \qquad (2.11)$$

For $v = [s,v]$ in (2.11), we get

$$\Big[\alpha^{-1}(d(z)),[s,v]\Big]\alpha^{-1}(d(z)) = 0 \ \forall\ s \in \mathcal{R},\ u,v \in U,\ z \in Z(\mathcal{R}). \qquad (2.12)$$

Taking $\alpha^{-1}(d(z))s$ in place of s in (2.12) and using it again, we get

$$[\alpha^{-1}(d(z)),v][\alpha^{-1}(d(z)),s]\alpha^{-1}(d(z)) + \Big[\alpha^{-1}(d(z)),[\alpha^{-1}(d(z)),v]\Big]s\alpha^{-1}(d(z)) = 0. \quad (2.13)$$

Replacing s by su in (2.13), where $u \in U$, and using (2.11), it follows that

$$\left([\alpha^{-1}(d(z)),v][\alpha^{-1}(d(z)),s] + \Big[\alpha^{-1}(d(z)),[\alpha^{-1}(d(z)),v]\Big]s\right)u\alpha^{-1}(d(z)) = 0. \qquad (2.14)$$

Which implies that

$$\left([\alpha^{-1}(d(z)),v][\alpha^{-1}(d(z)),s] + \Big[\alpha^{-1}(d(z)),[\alpha^{-1}(d(z)),v]\Big]s\right)U\alpha^{-1}(d(z)) = \{0\}$$

$\forall\, v \in U,\ s \in \mathcal{R},\ z \in Z(\mathcal{R})$. From Lemma 2.1, we find

$$[\alpha^{-1}(d(z)),v][\alpha^{-1}(d(z)),s] + \Big[\alpha^{-1}(d(z)),[\alpha^{-1}(d(z)),v]\Big]s = 0 \text{ or } \alpha^{-1}(d(z)) = 0.$$

So, we can combine the last two expressions into the following expression

$$[\alpha^{-1}(d(z)),v][\alpha^{-1}(d(z)),s] + \Big[\alpha^{-1}(d(z)),[\alpha^{-1}(d(z)),v]\Big]s = 0 \ \forall\ v \in U,\ s \in \mathcal{R},\ z \in Z(\mathcal{R}). \qquad (2.15)$$

Replacing s by sv in (2.15) and using it again, we find that

$$[\alpha^{-1}(d(z)),v]s[\alpha^{-1}(d(z)),v] = 0 \ \forall\ v \in U,\ s \in \mathcal{R},\ z \in Z(\mathcal{R}). \qquad (2.16)$$

Since $\mathcal{R}$ is prime, we infer that

$$[\alpha^{-1}(d(z)),v] = 0 \ \forall\, v \in U,\ z \in Z(\mathcal{R}).$$

This lead to $\alpha^{-1}(d(z)) \in Z(\mathcal{R})$ for all $z \in Z(\mathcal{R})$ by Lemma 2.3. Since $d(Z(\mathcal{R})) \neq \{0\}$, there exists $z_0 \in Z(\mathcal{R})$ such that $d(z_0) \neq 0$. So, $\alpha^{-1}(d(z_0)) \neq 0$ and from equation (2.3), we get $[U,U] \subseteq Z(\mathcal{R})$, which forces that $U \subseteq Z(\mathcal{R})$ by Lemma 2.2.

□

Theorem 2.6. *Let F be a generalized (β, α)-derivation of a 2-torsion free prime ring associated with a map d. If $F([u^n, v^m]) \in Z(\mathcal{R})$ for all $u, v \in \mathcal{R}$, where $m, n \in \mathbb{N}^*$, then $\mathcal{R}$ is an integral domain.*

Proof. We consider that

$$F([u^n, v^m]) \text{ in the center of } \mathcal{R} \text{ for all } u, v \in \mathcal{R}. \tag{2.17}$$

Let A_1 and A_2 be the additive subgroups generated by $\{r^n \mid r \in \mathcal{R}\}$ and $\{r^m \mid r \in \mathcal{R}\}$ respectively. Clearly, (2.17) implies $\forall u \in A_1, v \in A_2$, we obtain $F([u, v])$ in the center of $\mathcal{R}$. Using the main Theorem in [5], we get either A_1 contains a proper Lie ideal L_1 or $r^n \in Z(\mathcal{R})\ \ \forall\ r \in \mathcal{R}$. Similarly, there exists a proper Lie ideal $L_2 \subseteq A_2$ or $r^m \in Z(\mathcal{R})\ \forall\, r \in \mathcal{R}$. If $r^n \in Z(\mathcal{R})$ or $r^m \in Z(\mathcal{R})\ \forall\, r \in \mathcal{R}$, then $\mathcal{R}$ is commutative by Lemma 2.4. Now, supposing there are two nonzero ideals I_1 and I_2 of $\mathcal{R}$, such that $[I_1, \mathcal{R}] \subseteq L_1$ and $[I_2, \mathcal{R}] \subseteq L_2$. So, we have

$$F([u, v]) \in Z(\mathcal{R})\ \ \forall\, u \in [I_1, I_1], v \in [I_2, I_2].$$

In view of [12, Theorem 3], then I_1, I_2 and $\mathcal{R}$ satisfy the same differential identities, so that $F([u, v]) \in Z(R)\ \ \forall\, u, v \in [\mathcal{R}, \mathcal{R}]$. Setting $U = [\mathcal{R}, \mathcal{R}]$ and invoking the theorem 2.5, we conclude that $\mathcal{R}$ is commutative. Since $\mathcal{R}$ is prime, this all boils down to $\mathcal{R}$ being an integral domain. So the theorem is proved. □

Note that if F is a generalized skew derivation (i.e., a generalized $(I_{\mathcal{R}}, \alpha)$-derivation) associated with a $(I_{\mathcal{R}}, \alpha)$-derivation δ, where $I_{\mathcal{R}}$ is the identity map of $\mathcal{R}$, then $F \pm I_{\mathcal{R}}$ is also a generalized skew derivation associated with the same $(I_{\mathcal{R}}, \alpha)$-derivation δ. Consequently, if we take $F \pm I_{\mathcal{R}}$ instead of F in the theorem 2.6, we derive the following corollaries, which generalize some theorems due to [2], Theorem 2.5 and Theorem 2.4, respectively, which deal with the case of generalized derivation in a more general situation.

Corollary 2.7. *Let f be a non-zero generalized skew derivation of a 2-torsion free prime ring $\mathcal{R}$ which is associated with a $(I_{\mathcal{R}}, \alpha)$-derivation δ such that $\delta(Z(\mathcal{R})) \neq \{0\}$. Then the following assertions are equivalent:*

i) $f([u^n, v^m]) \in Z(\mathcal{R}),\ \ \forall\, u, v \in \mathcal{R}$, *where* $m, n \in \mathbb{N}^*$,

ii) $f([u^l, v^k]) + [u^l, v^k] \in Z(\mathcal{R}),\ \ \forall\, u, v \in \mathcal{R}$, *where* $l, k \in \mathbb{N}^*$,

iii) $f([u^r, v^s]) - [u^s, v^s] \in Z(\mathcal{R}),\ \ \forall\, u, v \in \mathcal{R}$, *where* $r, s \in \mathbb{N}^*$,

iii) $\mathcal{R}$ *is an integral domain.*

3 Results concerning Banach algebras

In the following, the study of identities in Banach algebras with derivations will be investigated using the previous ring theoretic results. We will use $\mathcal{A}$ to represent a unital Banach algebra with center $Z(\mathcal{A})$ and identity e. We use the following crucial Lemma to support our results.

Lemma 3.1. [*14*, *page*1] *If $p(t) = \sum_{i=1}^{n} a_i t^i$ is a polynomial in the real variable t in the center of $\mathcal{A}$ for an infinite set of real values t, with coefficients in $\mathcal{A}$, then $a_i \in Z(\mathcal{A})$.*

Theorem 3.2. *Let A and B be nonvoid open subsets of a prime Banach algebra $\mathcal{A}$, and F be a continuous linear generalized (β, α)-derivation of $\mathcal{A}$ associated with a (β, α)-derivation d such that $d(Z(\mathcal{A})) \neq \{0\}$. If $F([u^n, v^m]) \in Z(\mathcal{A})\ \ \forall\ \ u \in A, v \in B$, where $m = m(u, v) \in \mathbb{N}^*$, $n = n(u, v) \in \mathbb{N}^*$, then $\mathcal{A}$ is commutative.*

Proof. Fix $v \in B$, and let

$$S_{n,m} = \{u \in \mathcal{A} \mid F([u^n, v^m]) \text{ is not in the center of } \mathcal{A}\}.$$

We assert that $S_{n,m}$ is open. For this we show that its complement, given by $S^c_{n,m}$, is closed. Indeed, given a sequence $\{h_k\} \in S^c_{n,v}$ such that $h_k \longrightarrow h$ as $k \longrightarrow \infty$; it follows that

$F([h_k^n, v^m]) \in Z(\mathcal{A})$. Using the facts that F is a continuous map and that $Z(\mathcal{A})$ is closed, we obtain

$$\lim_{k\longrightarrow\infty} F([h_k^n, v^m]) = F([\lim_{k\longrightarrow\infty}(h_k^n), v^m]) = F([h^n, v^m]) \text{ is in the center of } \mathcal{A}.$$

So, $h \in S_{n,m}^c$ and therefore, $S_{n,m}^c$ is closed, which means that $S_{n,m}$ is open. According to the Baire category theorem, an open set of $\mathcal{A}$ cannot exist if every $S_{n,m}$ is dense. Then $\exists r, s \in \mathbb{N}^*$ such that $S_{r,s}$ is strictly contained in $\mathcal{A}$. So, $\exists\, \theta_1 \neq \emptyset$ an open set contained in $\overline{S_{r,s}}^c$, for which we have

$$F([u^r, v^s]) \in Z(\mathcal{A}) \text{ for all } u \in \theta_1.$$

Let $u_0 \in \theta_1$ and $w \in \mathcal{A}$, then $u_0 + tw$ is in θ_1 for any sufficiently small real t. This means that

$$F([(u_0 + tw)^r, v^s]) \in Z(\mathcal{A}). \tag{3.1}$$

On the other hand, we get

$$(u_0 + tw)^r = P_{r,0}(u_0, w) + P_{r-1,1}(u_0, w)t + P_{r-2,2}(u_0, w)t^2 + ... + P_{0,r}(u_0, w)t^r,$$

where $P_{i,j}(u_0, w)$ is the sum of all terms in which u_0 occurs precisely i times and w occurs precisely j times, such that $i + j = r$. Then (3.1) may be true

$$\begin{aligned} F([(P_{r,0}(u_0, w), v^s]) + (F([P_{r-1,1}(u_0, w), v^s])t \\ + (F([P_{r-2,2}(u_0, w), v^s])t^2 + ... \\ ... \\ + (F([P_{0,r}(u_0, w), v^s])t^r. \end{aligned}$$

The last expression is a polynomial in the center of $\mathcal{A}$ with the variable t, and its coefficient of t^r is $F([w^r, v^s])$. By Lemma 3.1, we now have $F([w^r, v^s])$ in the center of $\mathcal{A}$.
Consequently, for given $v \in B$, there exist $r, s \in \mathbb{N}$ related on w such that for every $w \in \theta_1$, $F([w^r, v^s]) \in Z(\mathcal{A})$. Let $u_0 \in \theta_2$ and $y \in \mathcal{A}$, then $u_0 + ty \in \theta_2$ for sufficiently small t. For these t, we have $F([(u_0 + ty)^k, v^l])$ belongs to the center of $\mathcal{A}$. This can be expressed as a polynomial in t, where the coefficient of t^k is $F([y^k, v^l])$, and therefore, $F([y^k, v^l]) \in Z(\mathcal{A})\ \forall\ y \in \mathcal{A}$. Consequently, given $v \in B$, then there exist $k, l \in \mathbb{N}$ such that either $F([y^k, v^l]) \in Z(\mathcal{A})\, \forall\, y \in \mathcal{A}$.

Now, we swap roles of A and B and using the same techniques, we infer that $F([y^k, v^l]) \in Z(\mathcal{A})\ \forall\, y, v \in \mathcal{A}$. Hence, the commutativity of $\mathcal{A}$ by using Theorem 2.6. □

Let F be a generalized skew derivation of $\mathcal{A}$ associated with a $(I_{\mathcal{R}}, \alpha)$ derivation δ and let H be a left multiplier of $\mathcal{A}$. It is clear that the sum $F + H$ is also a generalized skew derivation of $\mathcal{A}$ associated with the same $(I_{\mathcal{R}}, \alpha)$ derivation δ. Consequently, if we apply the previous theorem, we have the following corollary:

Corollary 3.3. *Let $\mathcal{A}$ be a prime Banach algebra, and let A and B be nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous linear generalized skew derivation F associated with a $(I_{\mathcal{R}}, \alpha)$-derivation δ and a continuous linear left multiplier H such that $\delta(Z(\mathcal{A})) \neq \{0\}$ and $F([u^n, v^m]) + H([u^n, v^m]) \in Z(\mathcal{A})\ \ \forall\ u \in A, v \in B$, where $m = m(u, v) \in \mathbb{N}^*$ and $n = n(u, v) \in \mathbb{N}^*$, then $\mathcal{A}$ is commutative.*

Once we know that $I_{\mathcal{A}}$ and $-I_{\mathcal{A}}$ are special cases of the left multiplier of $\mathcal{A}$, then as an immediate consequence established from the previous corollary, if we take $H = \pm I_{\mathcal{A}}$, we get exactly Theorem 3.4 due to [2].

Theorem 3.4. *Let A and B be nonvoid open subsets of a prime Banach algebra $\mathcal{A}$. If $\mathcal{A}$ admits a continuous linear mapp L and a continuous linear generalized (β, α)- derivation F satisfying $F((uv)^n) + L(u^n v^n) \in Z(\mathcal{A})\ \forall\, u \in A$, $v \in B$, where $n = n(u, v) \in \mathbb{N}^*$, then $\mathcal{A}$ is commutative.*

Proof. Let $v \in B$, and

$$S_n = \{u \in \mathcal{A} \mid F((uv)^n) + L(u^n v^n) \notin Z(\mathcal{A})\}.$$

Every S_n is open. To justify this, it suffices to show that S_n^c is closed. Let $\{h_k\} \in S_{n,v}^c$ be a sequence such that $h_k \longrightarrow h$ as $k \longrightarrow \infty$; that is, $F((h_k v)^n) + L(h_k^n v^n) \in Z(\mathcal{A})$. Since F and L are continuous and $Z(\mathcal{A})$ is closed, we have

$$\begin{aligned}\lim_{k\longrightarrow\infty}(F((h_k v)^n) + L(h_k^n v^n)) &= F(\lim_{k\longrightarrow\infty}((h_k v)^n) + L(\lim_{k\longrightarrow\infty}(h_k^n)v^n)) \\ &= F((hv)^n) + L(h^n v^n) \in Z(\mathcal{A}),\end{aligned}$$

which means that $h \in S_n^c$. So S_n^c is closed and so S_n is open. Using Baire's theorem and reasoning similar to the previous theorem, we conclude that $\exists\, r \in \mathbb{N}^*$ such that S_r is not dense in $\mathcal{A}$. Again, $\exists\, \Omega_1 \neq \emptyset$ an open of $\overline{S_r}^c$ which satisfies

$$F((uv)^r) + L(u^r v^r) \in Z(\mathcal{A}) \;\; \forall\, u \in \Omega_1.$$

For $u_0 \in \Omega_1$ and $w \in \mathcal{A}$, $u_0 + tw \in \Omega_1$ for all sufficiently small real t and therefore,

$$F(((u_0 + tw)v)^r) + L((u_0 + tw)^r v^r) \in Z(\mathcal{A}). \tag{3.2}$$

This can be expressed as

$$\begin{aligned}F((P_{r,0}(u_0,w,v)) + L(Q_{r,0}(u_0,w)v^r) + (F(P_{r-1,1}(u_0,w,v)) + L(Q_{r-1,1}(u_0,w)v^r))t \\ + (F(P_{r-2,2}(u_0,w,v)) + L(Q_{r-2,2}(u_0,w)v^r))t^2 + ... \\ + (F(P_{0,r}(u_0,w,v)) + L(Q_{0,r}(u_0,w)v^r))t^r \in Z(\mathcal{A}),\end{aligned}$$

where $P_{i,j}(u_0,w,v)$ is the sum of all terms in which $u_0 v$ occurs exactly i times and wv occurs exactly j times, such that $i + j = r$. Similarly, $Q_{i,j}(u_0,w)$ is the sum of all terms in which u_0 occurs exactly $itimesandwoccursexactly$j times, such that $i + j = r$.
The last expression is a polynomial in t, and its coefficient for t^r is $F((wv)^r) + L(w^r v^r)$. In light of Lemma 3.1, we conclude that $F((wv)^r) + L(w^r v^r) \in Z(\mathcal{A})$. Therefore, we proved the existence of r in $\mathbb{N}$ such that for every w in $\mathcal{A}$, $F((wv)^r) + L(w^r v^r)$ in $Z(\mathcal{A})$. Let

$$\Phi_r = \{w \in \mathcal{A} \mid F((wv)^r) + L(w^r v^r) \in Z(\mathcal{A})\}.$$

Then the union of the sets Φ_r is $\mathcal{A}$ and every Φ_r is closed. Thus, by the Baire category theorem, some Φ_m must have a nonempty open subset Ω_2 of $\mathcal{A}$. Let $x_0 \in \Omega_2$ and $y \in \mathcal{A}$, then $x_0 + ty \in \Omega_2$ for sufficiently small t. For these t we have

$$F(((x_0 + ty)v)^m) + L((u_0 + ty)^m v^m) \in Z(\mathcal{A}).$$

As before, this expression can be rewritten as a polynomial in t, where the coefficient of t^m is $F((yv)^m) + L(y^m v^m)$ and thus $F((yv)^m) + L(y^m v^m) \in Z(\mathcal{A})$ for all $y \in \mathcal{A}$. Accordingly, for given $v \in B$, there exists $m \in \mathbb{N}$ such that $F((yv)^m) + L(y^m v^m) \in Z(\mathcal{A}) \;\forall\, y \in \mathcal{A}$.

Now we reverse the roles of A and B, and using the same arguments as above, we get

$$F((yv)^m) + L(y^m v^m) \in Z(\mathcal{A}) \;\; \forall\, y, v \in \mathcal{A}. \tag{3.3}$$

Since $\mathcal{A}$ is unital, for a real t, replacing y by $e + tu$ in (3.3), where $u \in \mathcal{A}$, we get

$$F\Big(((e + tu)v)^m\Big) + L\Big((e + tu)^m v^m\Big) \in Z(\mathcal{A}) \;\; \forall\, u, v \in \mathcal{A}.$$

Taking the coefficient of t in the development of the last expression and using Lemma 3.1, we get

$$F\Big(uv^m + \sum_{k=1}^{m-1} v^k u v^{m-k}\Big) + mL(uv^m) \in Z(\mathcal{A}). \tag{3.4}$$

Once again, replacing y and v by v and $e + tu$ respectively in (3.3), the coefficient of t will be

$$F(v^m u + \sum_{k=1}^{m-1} v^k u v^{m-k}) + mL(v^m u) \in Z(\mathcal{A}). \tag{3.5}$$

Subtracting (3.4)from (3.5), we obtain

$$F([v^m,u]) + mL([v^m,u]) \in Z(\mathcal{A}) \ \forall\, u,v \in \mathcal{A}. \tag{3.6}$$

Taking $v = e$ in (3.4), we infer that $m(F(u) + L(u)) \in Z(\mathcal{A}) \ \forall\, u \in \mathcal{A}$. Which, since $m \in \mathbb{N}^*$, implies that $F(u) + L(u) \in Z(\mathcal{A})$ for all $u \in \mathcal{A}$. In particular, If we substitute $[v^m,u]$ instead of u in the latter result, we get

$$F([v^m,u]) + L([v^m,u]) \in Z(\mathcal{A}) \ \forall\, u,v \in \mathcal{A}. \tag{3.7}$$

Using (3.6) and (3.7), we get $(m-1)L([v^m,u]) \in Z(\mathcal{A})$. Since $m \in \mathbb{N}^*$, $L([v^m,u]) \in Z(\mathcal{A})$ and (3.7) gives $F([v^m,u]) \in Z(\mathcal{A}) \ \forall\, u,v \in \mathcal{A}$. Now, substituting $e + tv$ for v in the last relation, the coefficient of t will be $mF([v,u]) \in Z(\mathcal{A})$ by Lemma 3.1, it follows that $F([v,u]) \in Z(\mathcal{A}) \ \forall\, u,v \in \mathcal{A}$. Therefore, $\mathcal{A}$ is commutative by Theorem 2.5. □

The Theorem 3.4 immediately leads to the next corollary:

Corollary 3.5. [2, Theorem 3.1] Let G and H be nonvoid open subsets of a unital prime Banach algebra $\mathcal{A}$. If $\mathcal{A}$ admits a linear continuous map L and a continuous generalized skew derivation F associated with a map d satisfying $F((uv)^n) + L(u^nv^n) \in Z(\mathcal{A})$ or $F((uv)^n) - L(u^nv^n) \in Z(\mathcal{A}) \ \forall\, u \in G, \ \ \forall\, v \in H$, where $n = n(u,v) \in \mathbb{N}^*$, then $\mathcal{A}$ is commutative.

Lastly, in favor of our main theorems, we will enrich this paper with an example showing that the condition $d(Z(\mathcal{A})) \neq \{0\}$ cannot be deleted in this paper.

Example 3.6. Let $\mathcal{A} = M_2(\mathbb{C})$, be the set of 2×2 matrices with the usual addition and multiplication and Frobenius norm defined by: $\| \, . \, \|_F$ on $\mathcal{A}$ as follows: $\| \, A \, \|_F = \sqrt{\sum_{i,j=1}^{2} \mid a_{ij} \mid^2}$ for all $A = (a_{ij}) \in \mathcal{A}$. Then $\mathcal{A}$ is a non-commutative unital prime Banach algebra. Also define the maps α, β, d, F, L: $\mathcal{A} \longrightarrow \mathcal{A}$ by:

$$\beta\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & -y \\ -z & t \end{pmatrix}, \ \ \alpha = I_{\mathcal{A}}, \ \ d\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 0 & -y \\ -z & 0 \end{pmatrix},$$

$$F\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} -x & 0 \\ 0 & -t \end{pmatrix} \ \text{and} \ L\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}.$$

Let us define the subset G_1 and G_2 as follows:

$$G_1 = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix}, t \in \mathbb{R} \right\} \ \text{and} \ G_2 = \left\{ \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \ \theta \in \mathbb{R} \right\}.$$

By a simple calculation, we get:

(a) $F([X^n, Y^m]) \in Z(\mathcal{A}) \ \forall\, X \in G_1, Y \in G_2$,

(b) $F((XY)^n) + L(X^nY^n) \in Z(\mathcal{A}) \ \forall\, X \in G_1, Y \in G_2$, and $d(Z(\mathcal{A})) = \{0\}$.

But $\mathcal{A}$ is not commutative.

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