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Quadrics in a finite projective space

M.A. Rakdi and J.W.P. Hirschfeld

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Corresponding Author: M.A. Rakdi

Abstract *In* $\mathrm{PG}(n,q)$, *properties of non-singular quadrics have been frequently studied. In this paper, the number of points on a singular quadric is found, and some properties that are related to these quadrics are investigated. The weights of codewords in some error-correcting codes are found.*

1 Introduction

Let $V = V(n+1,q)$ *be an* $(n+1)$*-dimensional vector space over the field* K *with zero element 0. Consider the equivalence relation on the elements of* $V\backslash\{0\}$ *whose equivalence classes are the one-dimensional subspaces of* V *with the zero deleted. Thus, if* $X, Y \in V\backslash\{0\}$, *then* $X = (x_0,\dots,x_n)$ *is equivalent to* $Y = (y_0,\dots,y_n)$ *if* $Y = tX$ *for some* t *in* $K_0 = K\backslash\{0\}$*; that is,* $y_i = tx_i$ *for all* i*. Then the set of equivalence classes is the* n-dimensional projective space over K *and is denoted by* $\mathrm{PG}(n,K)$*; when* $K = \mathbf{F}_q$*, it is* $\mathrm{PG}(n,q)$ *or* $\mathbf{P}^n$.

A point with coordinate vector $X = (x_0,\dots,x_n)$ *is denoted by* $P(X) = P(x_0,x_1,\dots,x_n)$, *where* $x_0,x_1,\dots,x_n \in \mathbf{F}_q$ *and not all zero.*

A subspace of dimension m*, or* m*-space, of* $\mathrm{PG}(n,q)$ *is a set of points, all of whose representing vectors form, together with the origin, a subspace of dimension* $m+1$ *of* $V(n+1,q)$*; it is denoted by* Π_m*. Subspaces of dimension zero, one, two and three are called a point, a line, a plane, and a solid. A subspace of dimension* $n-1$ *is called a prime or a hyperplane.*

The set of projectivities fixing $\mathrm{PG}(n,q)$ *is the group* $\mathrm{PGL}(n+1,q)$.

The theory of quadrics and their number of points in $\mathrm{PG}(n,q)$ *is motivated by many important applications, especially in the theory of error-correcting codes; see [3, 9, 2].*

The theory of non-singular quadrics and their number of points was studied by Segre [10]. A comprehensive summary may be found in [7, Chapter 1]. In this paper, on the other hand, the number of points of singular quadrics is determined, as well as some related properties. [11].

Some undefined terms can be found in references [1, 11].

2 Quadrics in $\mathrm{PG}(n,q)$

A hypersurface of the second degree in $\mathrm{PG}(n,q)$ *is a* quadric.

A quadric $\mathcal{Q}$ *is given by a quadratic form; that is,*

$$\mathcal{Q} = V(F) = \{P(X) \mid F(X) = 0\},$$

where $F \in \mathbf{F}_q[X_0, \ldots, X_n]$, *and*

$$\begin{aligned} F(X) &= \sum_{i,j=0}^{n} a_{ij} X_i X_j = a_{00} X_0^2 + a_{01} X_0 X_1 + \cdots \\ &= \sum_{i=0}^{n} a_{ii} X_i^2 + \sum_{i \neq j, i,j=0}^{n} a_{ij} X_i X_j. \end{aligned}$$

When $p \neq 2$, $t_{ij} = t_{ji} = (a_{ij} + a_{ji})/2$. *Therefore, with* $T = (t_{ij}),\ X = (X_0, \ldots, X_n)$ *and* X^* *its transpose,*

$$\begin{aligned} F(X) &= \sum_{i,j=0}^{n} t_{ij} X_i X_j \\ &= X T X^*. \end{aligned}$$

For a quadric $\mathcal{Q}$*, the* rank *of* $\mathcal{Q}$*, denoted* $r(\mathcal{Q})$*, is the smallest number of indeterminates appearing in* F *under any change of coordinate system. The quadric* $\mathcal{Q}$ *is* degenerate *or* singular *if* $r(Q) < n+1$*; otherwise, it is* non-degenerate *or* non-singular.

2.1 Quadrics in $\mathrm{PG}(2, q)$ and $\mathrm{PG}(3, q)$

The numbers of points of the quadrics $\mathcal{Q}$ *in* $\mathrm{PG}(n, q)$*, for* $n = 2$ *and* $n = 3$*, are given in Tables 1 and 2.*

The quadrics of $\mathrm{PG}(2, q)$ *fall in four orbits under* $\mathrm{PGL}(3, q)$*, as in Table 1. The quadrics of* $\mathrm{PG}(3, q)$ *fall into six orbits under* $\mathrm{PGL}(4, q)$*, as in Table 2.*

Rank	Description	$\|\mathcal{Q}\|$
1	repeated line	$q+1$
2	pair of distinct line	$2q+1$
2	point	1
3	conic	$q+1$

Table 1. Quadrics in $\mathrm{PG}(2, q)$

Rank	Description	$\|\mathcal{Q}\|$
1	repeated plane	q^2+q+1
2	pair of distinct planes	$2q^2+q+1$
2	line	$q+1$
3	quadric cone	q^2+q+1
4	hyperbolic quadric	$(q+1)^2$
4	elliptic quadric	q^2+1

Table 2. Quadrics in $\mathrm{PG}(3, q)$

3 Non-singular quadrics in $\mathrm{PG}(n, q)$

Theorem 3.1. ([7], Section 1.1) *The number of projectively distinct non-singular quadrics in* $\mathrm{PG}(n, q)$ *is one or two as* n *is even or odd. They have the following canonical forms.*

(1) $n = 2m,\ m \geq 0$,

$$\mathcal{Q} = \mathcal{P}_n = V(X_0^2 + X_1 X_2 + \cdots + X_{2m-1} X_{2m});$$

the number of points is

$$\frac{q^{2m}-1}{q-1}.$$

(2) $n = 2m - 1,\ m \geq 1,$

(a)

$$\mathcal{Q} = \mathcal{H}_n = V(X_0X_1 + X_2X_3 + \cdots + X_{2m-2}X_{2m-1});$$

the number of points is

$$\frac{(q^{m-1}+1)(q^m-1)}{q-1},$$

(b)

$$\mathcal{Q} = \mathcal{E}_n = V(f(X_0, X_1) + X_2X_3 + \cdots + X_{2m-2}X_{2m-1}),$$

where f is any irreducible binary quadratic form; the number of points is

$$\frac{(q^m+1)(q^{m-1}-1)}{q-1}.$$

The quadric symbols are $\mathcal{P}$ for parabolic, $\mathcal{H}$ for hyperbolic and $\mathcal{E}$ for elliptic.

4 Singular quadrics in $\mathrm{PG}(n, q)$

Theorem 4.1. *As for non-singular quadrics, singular quadrics are of three types.*

(1) *Let* $2m < n,\ m \geq 0$.

A singular parabolic quadric has the standard form,

$$\mathcal{P} = V(X_0^2 + X_1X_2 + \cdots + X_{2m-1}X_{2m}).$$

Its number of points is

$$q^{n-1} + q^{n-2} + \cdots + \cdots + q + 1 = |\mathbf{P}^{n-1}|.$$

(2) *Let* $2m - 1 < n,\ m \geq 1$.

(a) *A singular hyperbolic quadric has the standard form,*

$$\mathcal{H} = V(X_0X_1 + X_2X_3 + \cdots + X_{2m-2}X_{2m-1}).$$

Its number of points is

$$q^{n-1} + q^{n-2} + \cdots + 2q^{n-m} + q^{n-m-1} + \cdots + 1 = |\mathbf{P}^{n-1}| + q^{n-m}.$$

(b) *A singular elliptic quadric has the standard form,*

$$V(f(X_0, X_1) + X_2X_3 + \cdots + X_{2m-2}x_{2m-1}),$$

with f irreducible over $\mathbf{F}_q$. Its number of points is

$$q^{n-1} + q^{n-2} + \cdots + q^{n-m+1} + q^{n-m-1} + \cdots + 1 = |\mathbf{P}^{n-1}| - q^{n-m}.$$

Proof. For $2m < n$, $m \geq 0$, the number of points on a singular parabolic quadric is

$$\frac{q^{2m}-1}{q-1} \times q^{n-2m} + |\mathbf{P}^{n-2m-1}|$$
$$= q^{n-1} + \cdots + q + 1 = |\mathbf{P}^{n-1}|.$$

The variety $V(X_0X_1 + X_2X_3 + \cdots + X_{2m-2}X_{2m-1})$ is the disjoint union of the subvariety for which at least one of $x_0, \ldots, x_{2m-1}$ is non-zero. Then

$$|V(X_0X_1 + X_2X_3 + \cdots + X_{2m-2}X_{2m-1})| = \frac{(q^{m-1}+1)(q^m-1)}{q-1},$$

with the subvariety for which $X_0, \ldots, X_{2m-1}$ are all zero. As for the first subvariety if $2m-1 < n$, then there are $n-2m+1$ free variables; therefore, the number of points in these cases is,

$$|V(X_0X_1 + X_2X_3 + \cdots + X_{2m-2}X_{2m-1})| \times |\mathbf{F}_q^{n-2m+1}| = \frac{(q^{m-1}+1)(q^m-1)}{q-1} \times q^{n-2m+1}.$$

For the second subvariety,

$$\mathbf{P}\{X_0 = X_1 = \cdots = X_{2m-1} = 0\} = \mathbf{P}\{e_{2m}, \ldots, e_n\} \simeq \mathbf{P}^{n-2m}.$$

Therefore, the number of points on a singular quadric is

$$\begin{aligned} & \frac{(q^{m-1}+1)(q^m-1)}{q-1} \times q^{n-2m+1} + |\mathbf{P}^{n-2m}| \\ = \; & q^{n-1} + q^{n-2} + \cdots + 2q^{n-m} + q^{n-m-1} + \cdots + 1 \\ = \; & |\mathbf{P}^{n-1}| + q^{n-m}. \end{aligned}$$

Similarly, the number of points on a singular elliptic quadric is

$$\begin{aligned} & \frac{(q^{m-1}-1)(q^m+1)}{q-1} \times q^{n-2m+1} + |\mathbf{P}^{n-2m}| \\ = \; & q^{n-1} + q^{n-2} + \cdots + q^{n-m+1} + q^{n-m-1} + \cdots + 1 \\ = \; & |\mathbf{P}^{n-1}| - q^{n-m}. \end{aligned}$$

□

Example 4.2. (1) In PG$(6, q)$, the singular hyperbolic quadric

$$V(X_1X_5 - X_2X_6),$$

for $n = 6,\ m = 2$, contains $q^5 + 2q^4 + q^3 + q^2 + q + 1$ points.

(2) In PG$(7, q)$, the singular parabolic quadric

$$V(X_1^2 - X_2X_3)$$

contains $q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$ points.

Now, sections of a singular quadric by a subspace are considered.

Proposition 4.3. *The number of points on a section of a quadric with base in* Π_{2m-1} *or* Π_{2m} *in* PG(n, q) *by a subspace of dimension* $n - (m + m')$ *is shown in Table* 3.

Equation of quadric and conditions	Number of points
If $2m-1+m' < n$, $m \geq 1$, $m' \geq 1$, $\begin{cases} x_0x_1 + x_2x_3 + \cdots + x_{2m-2}x_{2m-1} = 0 \\ x_{2m} = x_{2m+1} = \cdots = x_{2m-1+m'} = 0. \end{cases}$	$\dfrac{q^{n-m'}-1}{q-1} + q^{n-m'-m}$
If $2m-1+m' < n$, $m \geq 1$, $m' \geq 1$, $\begin{cases} f(x_0, x_1) + x_2x_3 + \cdots + x_{2m-2}x_{2m-1} = 0 \\ x_{2m} = x_{2m+1} = \cdots = x_{2m-1+m'} = 0. \end{cases}$	$\dfrac{q^{n-m'}-1}{q-1} - q^{n-m'-m}$
If $2m+m' < n$, $m \geq 0$, $m' \geq 1$, $\begin{cases} x_0^2 + x_1x_2 + \cdots + x_{2m-1}x_{2m} = 0 \\ x_{2m+1} = x_{2m+2} = \cdots = x_{2m+m'} = 0. \end{cases}$	$\dfrac{q^{n-m'}-1}{q-1}$

Table 3. The number of points on a singular quadric

Proof. The same idea as in the previous proof is applied. If $2m - 1 + m' < n$, $m \geq 1$, $m' \geq 1$, then the number of points in the hyperbolic case is

$$\begin{aligned}&\frac{(q^{m-1}+1)(q^m-1)}{q-1} \times q^{n-2m+1-m'} + |\mathbf{P}^{n-2m-m'}|\\&= q^{n-m'-1} + q^{n-m'-2} + \cdots + 2q^{n-m'-m} + q^{n-m'-m-1} + \cdots + 1\\&= |\mathbf{P}^{n-m'-1}| + q^{n-m'-m}.\end{aligned}$$

In the elliptic case, with $2m - 1 + m' < n,\ m \geq 1,\ m' \geq 1$, then the number of points is

$$\begin{aligned}&\frac{(q^{m-1}-1)(q^m+1)}{q-1} \times q^{n-2m+1-m'} + |\mathbf{P}^{n-2m-m'}|\\&= q^{n-m'-1} + q^{n-m'-2} + \cdots + q^{n-m'-m+1} + q^{n-m'-m-1} + \cdots + 1\\&= |\mathbf{P}^{n-m'-1}| - q^{n-m'-m}.\end{aligned}$$

In the parabolic case, with $2m + m' < n$, $m \geq 0$, $m' \geq 1$, the number of points is

$$\begin{aligned}&\frac{q^{2m}-1}{q-1} \times q^{n-2m-m'} + |\mathbf{P}^{n-2m-1-m'}|\\&= q^{n-m'-1} + \cdots + q + 1 = |\mathbf{P}^{n-m'-1}|.\end{aligned}$$

□

Remark 4.4. In particular, when $m' = 0$, Theorem 4.1 is obtained.

Example 4.5. Let V be an 8-dimensional vector space, where

$$x_6^2 - x_4x_8 = 0, \quad x_1 = x_2 = x_5 = x_7 = 0.$$

This corresponds, in PG$(7, q)$, with $m' = 4$, to a singular parabolic quadric with $q^2 + q + 1$ points.

Theorem 4.6. *In* PG(n, q)*, let* Π_r *be a subspace of dimension* r*. If* $\mathcal{Q} = V(F^s)$ *is a quadric given by* F *in* PG(s, q)*, then*

$$|V(F^n)| = |V(F^r)| \cdot q^{n-r} + |\mathrm{PG}(n - r - 1, q)|.$$

Proof. If there exists $i \leq r$ with $x_i \neq 0$, for $r < n$, then write $|V(F^s)|$ for the the number of points of $V(F)$ in PG(s, q). With $n - r$ new variables and so q^{n-r} possibilities, in the number of points is

$$|V(F^r)| \cdot q^{n-r}.$$

If $x_i = 0$ for all $i \leq r$, the number of points is $|\mathrm{PG}(n - r - 1, q)|$. So, the number of points of $V(F)$ in PG(n, q) is

$$|V(F^n)| = |V(F^r)| \cdot q^{n-r} + |\mathrm{PG}(n - r - 1)|.$$

□

Remark 4.7. This formula is equivalent to that of Corollary 1.42 in [7].

Corollary 4.8. *Since the quadrics* $\mathcal{Q}$ *in* $PG(3, q)$ *fall into six classes under projective equivalence, the numbers of points on a singular quadric in* PG$(7, q)$ *with base* Π_3 *is given in Table* 4.

Rank	F	$\lvert V(F^7)\rvert$
1	x_0^2	$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$
2	x_0x_1	$2q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$
2	$f(x_0, x_1)$	$q^5 + q^4 + q^3 + q^2 + q + 1$
3	$x_0^2 + x_1x_2$	$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$
4	$x_0x_1 + x_2x_3$	$q^6 + 2q^5 + q^4 + q^3 + q^2 + q + 1$
4	$f(x_0, x_1) + x_2x_3$	$q^6 + q^4 + q^3 + q^2 + q + 1$

Table 4. The number of points on a singular quadric of rank ≤ 4 in PG$(7, q)$

Proof. From Theorem 4.6, for $n = 7,\ r = 3$, the number of points on a singular quadric of rank < 7 is $q^4 \times |V(F^3)|$. Hence, the number of points on a singular quadric in $\mathrm{PG}(7, q)$ is

$$|V(F^7)| = |V(F^3)| \times q^4 + (q^3 + q^2 + q + 1).$$ □

5 An application in Coding Theory

An example clarifies the importance of the number of points on a quadric as applied to the weight of codewords in certain error-correcting codes.

Let V be an 8-dimensional vector space over an arbitrary field K. Consider the mapping $\phi_w : V \longrightarrow \wedge^2 V^*$ sending $v \longmapsto \iota_v \omega$, where ι_v is the operation of the interior multiplication defined by

$$\langle \iota_v \omega, \beta \rangle = \langle \omega, v \wedge \beta \rangle, \qquad \textit{for all } \beta \in \wedge^2 V.$$

Here, $\langle , \rangle$ is the pairing between $\wedge^j V^*$ and $\wedge^j V$ for each j.

Definition 5.1. Given a 2-form $\lambda \in \wedge^2 V^*$, the *k-th Pfaffian of* λ, $\mathbf{Pf}_k(\lambda) \in \wedge^{2k} V^*$ is defined inductively as follows for each $k \geq 1$:

$$\begin{aligned} \mathbf{Pf}_0(\lambda) &= 1; \\ \iota_v \lambda \wedge \mathbf{Pf}_{k-1}(\lambda) &= \iota_v \mathbf{Pf}_k(\lambda), \qquad \text{for all } v \in V. \end{aligned}$$

This $\mathbf{Pf}_k(\lambda)$ generalizes the form $\lambda^k/k! = (\lambda \wedge \cdots \wedge \lambda)/k!$.

Proposition 5.2. [3] *(k-th Pfaffian of a 2-form)*

(1) *Given $\lambda \in \wedge^2 V^*, \lambda \neq 0$, for each $k \geq 1$ there is a unique element $\mathbf{Pf}_k(\lambda) \in \wedge^{2k} V^*$.*

(2) $\mathbf{Pf}_k(\lambda_1 + \lambda_2) = \sum_{j=0}^{k} \mathbf{Pf}_j(\lambda_1) \wedge \mathbf{Pf}_{k-j}(\lambda_2)$.

(3) *The* rank *of λ is the unique integer $2r$ such that $\mathbf{Pf}_r(\lambda) \neq 0$ and $\mathbf{Pf}_{r+1}(\lambda) = 0$.*

Definition 5.3. [3] Given a non-degenerate trivector ω on $\mathbf{F}_q^8$, the k-th weight variety of ω is the subvariety of $\mathbf{P}^7$ given by

$$X_k(\omega) = \mathbf{P}\{x \in \mathbf{F}_q^8 \backslash \{0\} \mid \mathbf{Pf}_{k+1}(\iota_x \omega) = 0\}.$$

Then

$$\varnothing = X_0(\omega) \subset X_1(\omega) \subset X_2(\omega) \subset X_{\lfloor \frac{8-1}{2} \rfloor = 3}(\omega) = \mathbf{P}^7.$$

Lemma 5.4. [9] *Given a non-degenerate trivector ω on $\mathbf{F}_q^8$, let*

$$n_i := |X_i(\omega)| - |X_{i-1}(\omega)|.$$

The weight $\mathrm{wt}(\omega)$ is given by

$$\mathrm{wt}(\omega) = q^{15} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8 + q^6 - q^6 \left(\frac{|X_2(\omega)| + q^2 |X_1(\omega)|}{1 + q + q^2} \right), \tag{5.1}$$

Some undefined terms can be found in references [3] and [9].

To calculate the weight $\mathrm{wt}(\omega_{8,7})$, where $\omega_{8,7} = e_1e_2e_3 + e_1e_4e_5 + e_1e_6e_7 + e_2e_5e_6 + e_2e_7e_8$ in [8]*, let $x = \sum_{j=1}^{8} x_j e_j$. Then*

$$\mathbf{Pf}_2(\iota_x \omega_{8,7}) = \sum_{j=1}^{8} x_j^2 \mathbf{Pf}_2(\iota_{e_j} \omega_{8,7}) + \sum_{i<j} x_i x_j (\iota_{e_i} \omega_{8,7}) \wedge (\iota_{e_j} \omega_{8,7}). \tag{5.2}$$

Now, $\mathbf{Pf}_2(\iota_x \omega_i)$ can be calculated using the above formula (5.2): set it equal to zero to determine the varieties $X_1(\omega_{8,7})$. The forms $\iota_{e_j} \omega_{8,7}$ and $\mathbf{Pf}_2(\iota_{e_j} \omega_{8,7})$ for $j = 1 \cdots 8$ are as *follows:*

$$\begin{array}{ll}
\iota_{e_1\omega_{8,7}} = e_2e_3 + e_4e_5 + e_6e_7, & \mathbf{Pf}_2(\iota_{e_1}\omega_{8,7}) = e_2e_3e_4e_5 + e_2e_3e_6e_7 + e_4e_5e_6e_7; \\
\iota_{e_2}\omega_{8,7} = e_3e_1 + e_5e_6 + e_7e_8, & \mathbf{Pf}_2(\iota_{e_2}\omega_{8,7}) = e_3e_1e_5e_6 + e_3e_1e_7e_8 + e_5e_6e_7e_8; \\
\iota_{e_3}\omega_{8,7} = e_1e_2, & \mathbf{Pf}_2(\iota_{e_3}\omega_{8,7}) = 0; \\
\iota_{e_4}\omega_{8,7} = e_5e_1, & \mathbf{Pf}_2(\iota_{e_4}\omega_{8,7}) = 0; \\
\iota_{e_5}\omega_{8,7} = e_1e_4 + e_6e_2, & \mathbf{Pf}_2(\iota_{e_5}\omega_{8,7}) = e_1e_4e_6e_2; \\
\iota_{e_6}\omega_{8,7} = e_7e_1 + e_2e_5, & \mathbf{Pf}_2(\iota_{e_6}\omega_{8,7}) = e_7e_1e_2e_5; \\
\iota_{e_7}\omega_{8,7} = e_1e_6 + e_8e_2, & \mathbf{Pf}_2(\iota_{e_7}\omega_{8,7}) = e_1e_6e_8e_2; \\
\iota_{e_8}\omega_{8,7} = e_2e_7, & \mathbf{Pf}_2(\iota_{e_8}\omega_{8,7}) = 0.
\end{array} \tag{5.3}$$

If $\mathbf{Pf}_2(\iota_x\omega_{8,7}) = 0$*, then*

$$X_1(\omega_{8,7}) = \{x \in \mathbf{F}_q^8 \backslash \{0\} \mid x_1 = x_2 = x_5 = x_7 = x_6^2 + x_4x_8 = 0\},$$

by Proposition 4.3; then

$$|X_1(\omega_{8,7})| = q^2 + q + 1.$$

Now, the varieties $X_2(\omega_{8,7})$ *and their cardinality are computed. Let* $x = \sum_{j=1}^{8} x_je_j$*. Then*

$$\begin{aligned}
\mathbf{Pf}_3(\iota_x\omega_{8,7}) &= \sum_{j=1}^{8} x_j^3 \mathbf{Pf}_3(\iota_{e_j}\omega_{8,7}) + \\
&\quad \sum_{i<j} [x_i^2 x_j \mathbf{Pf}_2(\iota_{e_i}\omega_{8,7}) \wedge (\iota_{e_j}\omega_{8,7}) + x_i x_j^2 (\iota_{e_i}\omega_{8,7}) \wedge \mathbf{Pf}_2(\iota_{e_j}\omega_{8,7})].
\end{aligned} \tag{5.4}$$

Now, $\mathbf{Pf}_3(\iota_x\omega_{8,i})$ *is calculated using the above formula (5.4): set it equal to zero to determine the variety* $X_2(\omega_{8,7})$*. Then*

$$\mathbf{Pf}_3(\iota_{e_j}\omega_{8,7}) = \begin{cases} e_2e_3e_4e_5e_6e_7 & \text{for } j = 1, \\ e_3e_1e_5e_6e_7e_8 & \text{for } j = 2, \\ 0 & \text{for } j = 3, 4, 5, 6, 7, 8. \end{cases}$$

When $\mathbf{Pf}_3(\iota_x\omega_{8,7}) = 0$*, this implies that* $x_1 = x_2 = 0$*. Therefore*

$$\begin{aligned}
X_2(\omega_{8,7}) &= \{x \mid x_1 = x_2 = 0\} \\
&= \mathbf{P}\{e_3, e_4, e_5, e_6, e_7, e_8\} \simeq \mathbf{P}^5; \\
|X_2(\omega_{8,7})| &= |\mathbf{P}^5| = q^5 + q^4 + q^3 + q^2 + q + 1.
\end{aligned}$$

Also, using (5.1), the weight of $\omega_{8,7}$ *is*

$$\mathrm{wt}(\omega_{8,7}) = q^{15} + q^{13} + q^{12} + q^{11} + q^{10}.$$

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Author information

M.A. Rakdi, Department of Mathematics, Higher Normal School of Kouba, Laboratory of Pure and Applied Mathematics, Algeria.
E-mail: mohamedanouar.rakdi@g.ens-kouba.dz

J.W.P. Hirschfeld, Department of Mathematics, University of Sussex Brighton BN1 9QH, United Kingdom.
E-mail: jwph@sussex.ac.uk

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