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Existence and approximation of fixed points of generalized contractions in CAT(0) spaces

Javid Ali and Mohd Jubair

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Corresponding Author: Javid Ali

Abstract *Let M be a nonempty subset of a metric space B. A mapping $T : M \to M$ is said to be generalized contraction of Suzuki type if there exist $\beta \in (0,1)$ and $a, b, c \in [0,1]$ where $a + 2b + 2c = 1$ such that for all $x, y \in M$*
$\beta d(x, Tx) \leq d(x, y)$ implies

$$d(Tx, Ty) \leq ad(x, y) + b\Big(d(x, Tx) + d(y, Ty)\Big) + c\Big(d(x, Ty) + d(y, Tx)\Big).$$

In this paper, we obtain fixed point and convergence theorems for such mappings in a CAT(0) space. We also study the convergence behavior of JF-iterative scheme for generalized contraction of Suzuki type in CAT(0) space. Further, we give a non-trivial numerical example which shows that JF-iterative scheme converges faster than some leading iterative schemes. The results of the paper are new and generalize various pertinent results, in particular results of Atailia et al. [Some fixed point results for generalized contractions of Suzuki type in Banach spaces, J. Fixed Point Theory Appl. 21:78, (2019)].

1 Introduction

Throughout this paper, we assume that M is a nonempty subset of a CAT(0) space B, $T : M \to M$ a mapping and $F(T) = \{t \in M : Tt = t\}$ denotes the set of fixed points of the mapping T while $\mathbb{Z}_+$ denotes the set of all nonnegative integers. A mapping $T : M \to M$ is called non-expansive if $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in M$. It is said to be quasi non-expansive if $F(T) \neq \emptyset$ and $d(Tx, t) \leq d(x, t)$ for all $x \in M$ and $t \in F(T)$.

The concept of generalized non-expansive mappings was coined by Hardy and Rogers [1] which is defined as follows:
A mapping $T : M \to M$ is called generalized non-expansive if there exist real numbers $b_1, ..., b_5 \geq 0$ with $b_1 + b_2 + b_3 + b_4 + b_5 \leq 1$ such that for all $x, y \in M$

$$d(Tx, Ty) \leq b_1 d(x, y) + b_2 d(x, Tx) + b_3 d(y, Ty) + b_4 d(x, Ty) + b_5 d(y, Tx). \tag{1.1}$$

If $a = b_1$, $b = \frac{b_2+b_3}{2}$ and $c = \frac{b_4+b_5}{2}$, then (1.1) is equivalent to the following condition with $a, b, c \geq 0$ and $a + 2b + 2c \leq 1$,

$$d(Tx, Ty) \leq ad(x, y) + b\big(d(x, Tx) + d(y, Ty)\big) + c\big(d(x, Ty) + d(y, Tx)\big), \tag{1.2}$$

for all $x, y \in M$.

Further, Suzuki [2] gave the notion of generalized non-expansive mappings (also called condition (C)), and is defined as follows: A mapping $T : M \to M$ is said to satisfy condition (C)

if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \implies d(Tx,Ty) \leq d(x,y), \ \forall x,y \in M. \tag{1.3}$$

In 2011, Karapinar and Taş [3] introduced the following two generalized non-expansive mappings. A mapping $T : M \to M$ is said to satisfy:

(i) Kannan-Suzuki-(C) condition (in short, (KSC)-condition) if for all $x, y \in M$,

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \implies d(Tx,Ty) \leq \frac{1}{2}\Big(d(x,Tx) + d(y,Ty)\Big). \tag{1.4}$$

(ii) Chatterjea-Suzuki-(C) condition (in short, (CSC)-condition) if for all $x, y \in M$,

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \implies d(Tx,Ty) \leq \frac{1}{2}\Big(d(x,Ty) + d(y,Tx)\Big). \tag{1.5}$$

In the same year, Popescu [4] generalized two classical results given by Bogins [5] and Gregus [6]. He introduced the generalized contractions and proved the following two results.

Theorem 1.1. *[4] Let B be a Banach space and $T : B \to B$ be a mapping satisfying, $\frac{1}{2}d\big(x,Tx\big) \leq d(x,y)$ implies*

$$d(Tx,Ty) \leq ad(x,y) + b\big(d(x,Tx) + d(y,Ty)\big) + c\big(d(x,Ty) + d(y,Tx)\big) \tag{1.6}$$

for all $x, y \in B$, where $a \geq 0, b > 0, c > 0$ and $a+2b+2c = 1$. Then T has a unique fixed point.

Theorem 1.2. *[4] Let B be a Banach space and $T : B \to B$ be a mapping satisfying, $\frac{1}{2}d\big(x,Tx\big) \leq d(x,y)$ implies*

$$d(Tx,Ty) \leq ad(x,y) + b\big(d(x,Tx) + d(y,Ty)\big) \tag{1.7}$$

for all $x, y \in B$, where $a > 0, b > 0$ and $a + 2b = 1$. Then T has a unique fixed point.

Now, we have the following observation.

Remark 1.3. We observe that the conditions $a + 2b + 2c = 1$ and $a + 2b = 1$ are more complicated for the study in comparison of the conditions $a+2b+2c \leq 1$ and $a+2b \leq 1$, respectively. Moreover, the mappings satisfying conditions (1.6) and (1.7) are only generalizations of the contractive type mappings but not the generalization of the mappings satisfying Suzuki's condition (C) (1.3).

It is well known that non-expansive mappings are continuous but the Suzuki-type generalized non-expansive mappings need not be continuous. Therefore, these mappings are more important in theoretical and application point of view. In the sequel, many authors gave the generalizations of non-expansive mapping and proved existence, and convergence results in linear and non-linear spaces e.g. see [5, 8, 7, 9, 10, 11].

2 CAT(0) spaces

A metric space (B, d) is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in B is at least as '$thin$' as its comparison triangle in Euclidean plane $\mathbb{E}^2$. Examples of CAT(0) spaces include pre-Hilbert spaces, $\mathbb{R}$-trees and Euclidean buildings.

The study of fixed point theory in the setup of CAT(0) spaces was initiated by Kirk [12, 13]. The notion of $\triangle$-convergence in general metric spaces was introduced by Lim [14] in 1976. Kirk and Panyanak [15] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogous in this setting.
Let $\{\tau_n\}$ be a bounded sequence in a complete CAT(0) space B. For $x \in B$, we set

$$r(x, \{\tau_n\}) = \limsup_{n\to\infty} d(x, \tau_n).$$

The asymptotic radius of $r(\{\tau_n\})$ of $\{\tau_n\}$ is given by

$$r(\{\tau_n\}) = \inf\{r(x, \{\tau_n\}) : x \in B\}.$$

The asymptotic center $A(\{\tau_n\})$ *of* $\{\tau_n\}$ *is the set*

$$A(\{\tau_n\}) = \{x \in M : r(x, \{\tau_n\}) = r(\{\tau_n\})\}.$$

It is well known that in a complete CAT(0) space, $A(\{\tau_n\})$ *consists of exactly one point [16]. A sequence* $\{\tau_n\}$ *in* B *is called* $\triangle$*-convergent to* $x \in B$*, denoted by* $\triangle - \lim_{n\to\infty} \tau_n = x$ *if* x *is the unique asymptotic center of* $\{u_n\}$*, for every subsequence* $\{u_n\}$ *of* $\{\tau_n\}$*.*

Lemma 2.1. (i) *If* M *is a closed convex subset of* B *and if* $\{\tau_n\}$ *is a bounded sequence in* M*, then the asymptotic center of* $\{\tau_n\}$ *is in* M *(see [[17], Proposition 2.1]).*
(ii) *Every bounded sequence in* B *has a* $\triangle$*-convergent subsequence (see [[15], p. 3690]).*

Lemma 2.2. *[18] Let* B *be a CAT(0) space. Then*

$$d((1-\theta)x \oplus \theta y, \varsigma) \leq (1-\theta)d(x,\varsigma) + \theta d(y,\varsigma) \tag{2.1}$$

for all $x, y, \varsigma \in B$ *and* $\theta \in [0,1]$.

Lemma 2.3. *[18] Let* (B,d) *be a CAT(0) space. Then*

$$d((1-\theta)x \oplus \theta y, \varsigma)^2 \leq (1-\theta)d(x,\varsigma)^2 + \theta d(y,\varsigma)^2 - \theta(1-\theta)d(x,y)^2 \tag{2.2}$$

for all $x, y, \varsigma \in B$ *and* $\theta \in [0,1]$.

Lemma 2.4. *[18] Let* (B,d) *be a CAT(0) space. For* $x, y \in B$ *and* $\theta \in [0,1]$*, there exists a unique* $\varsigma \in [x,y]$ *such that*

$$d(\varsigma, x) = \theta d(x,y) \qquad \text{and} \qquad d(\varsigma, y) = (1-\theta)d(x,y).$$

In above Lemma 2.4, we use the notation $(1-\theta)x \oplus \theta y$ *for the unique point* ς*.*

Notice that for a given $\{\tau_n\} \subset B$ *which* $\triangle$*-converges to* x *and for any* $y \in B$ *with* $y \neq x$ *(owing to uniqueness of asymptotic center), we have*

$$\limsup_{n\to\infty} d(\tau_n, x) < \limsup_{n\to\infty} d(\tau_n, y).$$

Thus, every CAT(0) space satisfies the Opial's property.

Lemma 2.5. *[19] Let* B *be a complete CAT(0) space and* $x \in B$*. Suppose* $\{\tau_n\}$*,* $\{\sigma_n\}$ *are sequences in* B *such that* $\limsup_{n\to\infty} d(\tau_n, x) \leq a$*,* $\limsup_{n\to\infty} d(\sigma_n, x) \leq a$ *and* $\lim_{n\to\infty} d((1-t_n)\tau_n \oplus t_n\sigma_n, x) = a$ *for some* $a \geq 0$*, where* $\{t_n\}$ *is a sequence in* $[b,c]$ *for some* $b, c \in (0,1)$*. Then*

$$\lim_{n\to\infty} d(\tau_n, \sigma_n) = 0.$$

3 Generalized contraction of Suzuki type in CAT(0) spaces

In this section, we state generalized contraction of Suzuki type in CAT(0) spaces, and prove some basic properties and results for such mapping.

Definition 3.1. [20] Let M be a nonempty subset of a CAT(0) space B. A mapping $T : M \to M$ is said to be a generalized contraction of Suzuki type if there exist $\beta \in (0,1)$ and $a, b, c \in [0,1]$ where $a + 2b + 2c = 1$ such that for all $x, y \in M$
$\beta d(x, Tx) \leq d(x,y)$ implies

$$d(Tx, Ty) \leq ad(x,y) + b\Big(d(x,Tx) + d(y,Ty)\Big) + c\Big(d(x,Ty) + d(y,Tx)\Big). \tag{3.1}$$

We now mention some basic properties of generalized contraction of Suzuki type mapping.

Proposition 3.2. *(i) If* $a = 1, \beta = \frac{1}{2}$ *and* $b = c = 0$ *in the condition* (3.1)*, then it reduces to the Suzuki condition (C)* (1.3)*.*

(ii) If $a = c = 0$ and $\beta = b = \frac{1}{2}$ in the condition (3.1), then it reduces to the (KSC)-condition (1.4).

(iii) If $a = b = 0$ and $\beta = c = \frac{1}{2}$ in the condition (3.1), then it reduces to the (CSC)-condition (1.5).

(iv) If $a \geq 0,\ b > 0,\ c > 0$ and $\beta = \frac{1}{2}$ in the condition (3.1), then it reduces to the condition (1.6).

(v) If $a > 0,\ b > 0,\ c = 0$ and $\beta = \frac{1}{2}$ in the condition (3.1), then it reduces to the condition (1.7).

The following two propositions are very easy to verify.

Proposition 3.3. *Let M be a nonempty subset of a CAT(0) space B and $T : M \to M$ is a non-expansive mapping. Then T satisfies condition (3.1).*

Proposition 3.4. *Let M be a nonempty subset of a CAT(0) space B. Suppose $T : M \to M$ is a generalized contraction of Suzuki type and has a fixed point, then T is quasi non-expansive mapping.*

Lemma 3.5. *Let M be a nonempty subset of a CAT(0) space B and let $T : M \to M$ be a generalized contraction of Suzuki type for $\beta \in (0, 1)$. Then, for all $x, y \in M$, the following holds:*
(1) $d(Tx, T^2x) \leq d(x, Tx)$.
(2) Either $\beta d(x, Tx) \leq d(x, y)$ or $(1-\beta)d(Tx, T^2x) \leq d(Tx, y)$.
(3) Furthermore, for $\beta \in [\frac{1}{2}, 1)$ and T is a generalized contraction of Suzuki type for $1-\beta$, then

$$d(Tx, Ty) \leq ad(x, y) + b\Big(d(x, Tx) + d(y, Ty)\Big) + c\Big(d(x, Ty) + d(y, Tx)\Big),$$

or

$$d(T^2x, Ty) \leq ad(Tx, y) + b\Big(d(Tx, T^2x) + d(y, Ty)\Big) + c\Big(d(Tx, Ty) + d(y, T^2x)\Big).$$

Proof. One can prove it by follow the lines of proof of Theorem 2.3 [20]. □

Lemma 3.6. *Let M be a nonempty subset of a CAT(0) space B and let $\beta \in [\frac{1}{2}, 1)$. Assume that $T : M \to M$ is a generalized contraction of Suzuki type for β and $(1-\beta)$. Then for all $x, y \in M$, the following holds:*

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y),$$

where $\mu = \frac{2+a+b+3c}{1-b-c}$.

Proof. One can prove it by follow the lines of proof of Theorem 2.5 [20]. □

Remark 3.7. We have the following facts:

(i) If T satisfies Suzuki condition (C) ($b = c = 0, a = 1$ and $\beta = \frac{1}{2}$), then we obtain $\mu = 3$.

(ii) If T satisfies (KSC)-condition ($a = c = 0$ and $\beta = b = \frac{1}{2}$), then $\mu = 5$.

(iii) If T satisfies (CSC)-condition ($a = b = 0$ and $\beta = c = \frac{1}{2}$), then $\mu = 7$.

Now, we prove demiclosedness principle for generalized contraction of Suzuki type which is used to prove convergence results.

Lemma 3.8. *Let M be a nonempty closed convex subset of a complete CAT(0) space B and suppose $T : M \to M$ generalized contraction of Suzuki type. If $\{\tau_n\}$ is a sequence in M such that $d(T\tau_n, \tau_n) \to 0$ and $\triangle$-$\lim_n\{\tau_n\} = t$ for some $t \in B$. Then $Tt = t$.*

Proof. Since B is a complete CAT(0) space, then $A(\{\tau_n\})$ consists only one element. Let $t \in A(\{\tau_n\})$, by Lemma 3.6, we have

$$\begin{aligned} r(Tt, \{\tau_n\}) = \limsup_{n\to\infty} d(\tau_n, Tt) &\leq \limsup_{n\to\infty} \big(d(\tau_n, t) + \mu d(T\tau_n, \tau_n)\big) \\ &= \limsup_{n\to\infty} d(\tau_n, t) \\ &= r(t, \{\tau_n\}) = r(\{\tau_n\}). \end{aligned}$$

By uniqueness of asymptotic center, we have $Tt = t$. □

4 Existence theorem

In this section, we prove existence result for generalized contraction of Suzuki type in CAT(0) space by employing a weaker condition.

Theorem 4.1. *Let M be a nonempty closed and convex subset of a complete CAT(0) space B. Let $T : M \to M$ be a generalized contraction of Suzuki type (3.1) for β and $(1-\beta)$ where $\beta \in [\frac{1}{2}, 1)$. Assume that $\{T^n\tau\}$ is a bounded sequence for some $\tau \in M$ and $\inf\{d(\tau_n, T\tau_n) : n \in \mathbb{Z}_+\} = 0$. Then T has a fixed point in M. Moreover, if $a, b, c \in [0, 1]$ with $a + 2b + 2c < 1$, then T has a unique fixed point.*

Proof. Define $\{\tau_n\} = \{T^n\tau\}$ for some $\tau \in M$ and for all $n \in \mathbb{Z}_+$. Since, B is complete CAT(0) space, $A(\{\tau_n\})$ consists only one element. Now, our claim is that $d(\tau_n, T\tau_n)$ a non-increasing sequence. Since, $\beta d(\tau_n, T\tau_n) = \beta d(\tau_n, \tau_{n+1}) \leq d(\tau_n, \tau_{n+1})$, by (3.1), we get

$$\begin{aligned}
d(T\tau_n, T\tau_{n+1}) &\leq ad(\tau_n, \tau_{n+1}) + b\big(d(\tau_n, T\tau_n) + d(\tau_{n+1}, T\tau_{n+1})\big) \\
&\quad + c\big(d(\tau_n, T\tau_{n+1}) + d(\tau_{n+1}, T\tau_n)\big) \\
&\leq ad(\tau_n, T\tau_n) + b\big(d(\tau_n, T\tau_n) + d(T\tau_n, T\tau_{n+1})\big) \\
&\quad + c\big(d(\tau_n, T\tau_n) + d(T\tau_n, T\tau_{n+1})\big) \\
(1 - b - c)d(T\tau_n, T\tau_{n+1}) &\leq (a + b + c)d(\tau_n, T\tau_n) \\
d(T\tau_n, T\tau_{n+1}) &\leq \frac{a + b + c}{1 - b - c} d(\tau_n, T\tau_n) \\
&\leq d(\tau_n, T\tau_n).
\end{aligned}$$

Thus the sequence $\{d(\tau_n, T\tau_n)\}$ is non-increasing and bounded below. Hence

$$\lim_{n\to\infty} d(\tau_n, T\tau_n) = 0.$$

Now, let $z \in A(\{\tau_n\})$, then by Lemma 3.6, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} d(\tau_n, Tz) &\leq \limsup_{n\to\infty} \big(d(\tau_n, z) + \mu d(\tau_n, T\tau_n)\big) \\
&\leq \limsup_{n\to\infty} d(\tau_n, z).
\end{aligned}$$

Consequently, $Tz \in A(\{\tau_n\})$, ensuring that $Tz = z$.

Next, if $a + 2b + 2c < 1$, then we prove uniqueness of the fixed point. Presume that w is another fixed point of T such that $w \neq z$. Then, $\beta d(w, Tw) = 0 \leq d(w, z)$ and we have

$$\begin{aligned}
d(w, z) &= d(Tw, Tz) \\
&\leq ad(w, z) + b\big(d(w, Tw) + d(z, Tz)\big) + c\big(d(w, Tz) + d(z, Tw)\big) \\
&= ad(w, z) + c\big(d(w, z) + d(z, w)\big) \\
&= (a + 2c)d(w, z) \\
&< d(w, z),
\end{aligned}$$

which is a contradiction. Hence $w = z$. □

Remark 4.2. One can also prove Theorem 4.1 via JF iteration process by using Lemma 3.6.

By using this result along with Proposition 3.4 and ([21], Theorem 1.3), we can obtain the following corollary.

Corollary 4.3. *Let M be a nonempty closed convex subset of a CAT(0) space B. Suppose $T : M \to M$ be a generalized contraction of Suzuki type whose fixed point set is nonempty. Then $F(T)$ is closed and convex.*

5 Convergence theorems

For approximation of fixed point, the well known Banach contraction theorem uses the Picard iterative scheme. Many iterative schemes have been developed to approximate fixed points of different non-linear mappings. Some of the well-known iterative schemes are those of Ishikawa [22], Thakur et al. (Thakur-New) [23], Agrawal et al. (S) [24], Sahu et al. [25], Thakur et al. [26], JF [27] and so on. In [29] Okeke proved convergence results for G-nonexpansive mappings in convex metric spaces with a directed graph while Bera et al. [30] proved convergence results for a general class of non-expansive mappings in hyperbolic metric spaces. In this section, we prove some strong and $\triangle$-convergence theorems of a sequence generated by JF iterative scheme for generalized contraction of Suzuki type in the setting of CAT(0) spaces. The JF iterative scheme in CAT(0) spaces is given by

$$\begin{cases} \tau_0 \in M, \\ \xi_n = T((1-\eta_n)\tau_n \oplus \eta_n T\tau_n), \\ \sigma_n = T\xi_n, \\ \tau_{n+1} = T((1-\mu_n)\sigma_n \oplus \mu_n T\sigma_n), \ n \in \mathbb{Z}_+, \end{cases} \tag{5.1}$$

where $\{\mu_n\}$ and $\{\eta_n\}$ are sequences in $(0,1)$ satisfying appropriate conditions and T is a self mapping on M.
Throughout this section, we presume that $T : M \to M$ is a generalized contraction of Suzuki type and M is a nonempty bounded convex and closed subset of a complete CAT(0) space B. Following useful lemmas will be used in proving the main results.

Lemma 5.1. *Let $\{\tau_n\}$ be a sequence developed by the iteration process (5.1), then $\lim_{n\to\infty} d(\tau_n, t)$ exists for all $t \in F(T)$.*

Proof. As T is a generalized contraction of Suzuki type, so for all $t \in F(T)$ and $\{\tau_n\} \in M$, we can easily obtain that

$$d(T\tau_n, t) \le d(\tau_n, t).$$

By JF iterative scheme (5.1), we have

$$\begin{aligned} d(\xi_n, t) &= d(T((1-\eta_n)\tau_n \oplus \eta_n T\tau_n), t) \\ &\le d((1-\eta_n)\tau_n \oplus \eta_n T\tau_n, t) \\ &\le (1-\eta_n) d(\tau_n, t) + \eta_n d(\tau_n, t) \\ &= d(\tau_n, t). \end{aligned} \tag{5.2}$$

Using equation (5.2), we have

$$\begin{aligned} d(\sigma_n, t) &= d(T\xi_n, t) \\ &\le d(\xi_n, t) \\ &\le d(\tau_n, t). \end{aligned} \tag{5.3}$$

Using equation (5.3), we have

$$\begin{aligned} d(\tau_{n+1}, t) &= d(T((1-\mu_n)\sigma_n \oplus \mu_n T\sigma_n), t) \\ &\le (1-\mu_n) d(\sigma_n, t) + \mu_n d(T\sigma_n, t) \\ &\le (1-\mu_n) d(\tau_n, t) + \mu_n d(\tau_n, t) \\ &= d(\tau_n, t). \end{aligned} \tag{5.4}$$

Thus the sequence $\{d(\tau_n, t)\}$ is non-increasing and bounded below, for all $t \in F(T)$. Hence the result. □

Lemma 5.2. *Let $\{\tau_n\}$ be defined by (5.1). Then, $\{\tau_n\}$ is bounded and $\lim_{n\to\infty} d(\tau_n, T\tau_n) = 0$ if and only if $F(T) \neq \emptyset$.*

Proof. Suppose $F(T) \neq \emptyset$ and $t \in F(T)$. Then by Lemma 5.1, it follows that $\lim_{n\to\infty} d(\tau_n, t)$ exists. So one can presume that $\lim_{n\to\infty} d(\tau_n, t) = c$.
By inequalities (5.2) and (5.3), we get

$$\limsup_{n\to\infty} d(\xi_n, t) \leq c \tag{5.5}$$

and

$$\limsup_{n\to\infty} d(\sigma_n, t) \leq c, \tag{5.6}$$

respectively. Since, T satisfies condition (3.1), we have

$$d(T\tau_n, t) \leq d(\tau_n, t), \;\; d(T\sigma_n, t) \leq d(\sigma_n, t) \; and \; d(T\xi_n, t) \leq d(\xi_n, t).$$

$$\limsup_{n\to\infty} d(T\tau_n, t) \leq c, \tag{5.7}$$

$$\limsup_{n\to\infty} d(T\sigma_n, t) \leq c \tag{5.8}$$

and

$$\limsup_{n\to\infty} d(T\xi_n, t) \leq c. \tag{5.9}$$

Since

$$\begin{aligned} d(\tau_{n+1}, t) &= d(T((1-\mu_n)\sigma_n \oplus \mu_n T\sigma_n), t) \\ &\leq d((1-\mu_n)\sigma_n \oplus \mu_n T\sigma_n, t) \\ &\leq (1-\mu_n)d(\sigma_n, t) + \mu_n d(\sigma_n, t) \\ &= d(\sigma_n, t). \end{aligned}$$

Taking the lim inf on both sides, we obtain

$$c = \liminf_{n\to\infty} d(\tau_{n+1}, t) \leq \liminf_{n\to\infty} d(\sigma_n, t). \tag{5.10}$$

So that (5.6) and (5.10) give,

$$\lim_{n\to\infty} d(\sigma_n, t) = c.$$

And

$$\begin{aligned} c = \liminf_{n\to\infty} d(\sigma_n, t) &= \liminf_{n\to\infty} d(T\xi_n, t) \\ &\leq \liminf_{n\to\infty} d(\xi_n, t). \end{aligned} \tag{5.11}$$

By (5.5) and (5.11), we have

$$\lim_{n\to\infty} d(\xi_n, t) = c.$$

So,

$$\begin{aligned} c &= \lim_{n\to\infty} d(\xi_n, t) \\ &= \lim_{n\to\infty} d(T((1-\eta_n)\tau_n \oplus \eta_n T\tau_n), t) \\ &\leq \lim_{n\to\infty} d((1-\eta_n)\tau_n \oplus \eta_n T\tau_n, t) \\ &\leq \lim_{n\to\infty} (1-\eta_n)d(\tau_n, t) + \eta_n d(T\tau_n, t) \\ &\leq \lim_{n\to\infty} (1-\eta_n)d(\tau_n, t) + \eta_n d(\tau_n, t) \\ &\leq \lim_{n\to\infty} d(\tau_n, t) = c. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} d((1-\eta_n)\tau_n \oplus \eta_n T\tau_n, t) = c. \tag{5.12}$$

By (5.12) and Lemma 2.5, we have

$$\lim_{n\to\infty} d(\tau_n, T\tau_n) = 0.$$

Conversely, presume that $\{\tau_n\}$ is bounded and $\lim_{n\to\infty} d(\tau_n, T\tau_n) = 0$. Suppose $\eta \in A(\{\tau_n\})$ so by Lemma 3.6, we have

$$\begin{aligned} r(T\eta, \{\tau_n\}) &= \limsup_{n\to\infty} d(\tau_n, T\eta) \\ &\le \limsup_{n\to\infty} \left(d(\tau_n, \eta) + \mu d(T\tau_n, \tau_n)\right) \\ &= \limsup_{n\to\infty} d(\tau_n, \eta) \\ &= r(\eta, \{\tau_n\}). \end{aligned}$$

This implies that $T\eta \in A(\{\tau_n\})$. Since B is a CAT(0) space which implies $A(\{\tau_n\})$ consists only one element, hence we have $T\eta = \eta$ which shows $F(T) \neq \emptyset$. □

Theorem 5.3. *Presume that M is a compact subset of B. Then the sequence $\{\tau_n\}$ defined by (5.1) converges strongly to an element of $F(T)$.*

Proof. Let $F(T) \neq \emptyset$, then by Lemma 5.2, $\lim_{n\to\infty} d(T\tau_n, \tau_n) = 0$. Since M is compact, there is a subsequence $\{\tau_{n_j}\}$ of $\{\tau_n\}$ such that $\tau_{n_j} \to t$ strongly for some $t \in M$. Now, it is enough to show that $'t'$ is a fixed point of the mapping T. By using Lemma 3.6, we get

$$d(\tau_{n_j}, Tt) \le \mu d(\tau_{n_j}, T\tau_{n_j}) + d(\tau_{n_j}, t), \ \forall j \ge 1.$$

This implies that $\tau_{n_j} \to Tt$ as $j \to \infty$. Thus, $Tt = t$. □

Theorem 5.4. *Let $\{\tau_n\}$ be the sequence developed by equation (5.1). Then $\liminf_{n\to\infty} d(\tau_n, F(T)) = 0$ if and only if $\{\tau_n\}$ converges strongly to a fixed point of T, where $d(\tau_n, F(T)) = \inf\{d(\tau_n, t) : t \in F(T)\}$.*

Proof. If the sequence $\{\tau_n\}$ converges to a point $\ell \in F(T)$, then $\liminf_{n\to\infty} d(\tau_n, F(T)) = 0$. Now, we prove the direct part. Presume that $\liminf_{n\to\infty} d(\tau_n, F(T)) = 0$. From Lemma 5.1, $\lim_{n\to\infty} d(\tau_n, t)$ exists $\forall t \in F(T)$, therefore $\lim_{n\to\infty} d(\tau_n, F(T)) = 0$ by assumption.

Now our assertion is that $\{\tau_n\}$ a Cauchy sequence in M. Since $\lim_{n\to\infty} d(\tau_n, F(T)) = 0$, for a given $\alpha > 0$, there exists $w_0 \in \mathbb{N}$ such that for all $n \ge w_0$,

$$d(\tau_n, F(T)) < \frac{\alpha}{2}$$

implies that

$$\inf\{d(\tau_n, t) : t \in F(T)\} < \frac{\alpha}{2}.$$

Precisely, $\inf\{d(\tau_{w_0}, t) : t \in F(T)\} < \frac{\alpha}{2}$. So, there exists $t \in F(T)$ such that

$$d(\tau_{w_0}, t) < \frac{\alpha}{2}.$$

Now, for $\iota, \eta \ge w_0$,

$$\begin{aligned} d(\tau_{\eta+\iota}, \tau_\eta) &\le d(\tau_{\eta+\iota}, t) + d(\tau_\eta, t) \\ &\le d(\tau_{w_0}, t) + d(\tau_{w_0}, t) \\ &= 2d(\tau_{w_0}, t) < \alpha. \end{aligned}$$

Thus, $\{\tau_n\}$ is a Cauchy sequence in M. Since M is closed, therefore $\lim_{n\to\infty} \tau_n = \ell$ for some $\ell \in M$. Now, $\lim_{n\to\infty} d(\tau_n, F(T)) = 0$ implies $d(\ell, F(T)) = 0$, hence we get $\ell \in F(T)$. □

Definition 5.5. [28] A mapping $T : M \to M$ is said to enjoy property (I), if for a nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ and $\psi(z) > 0$, $\forall z > 0$ satisfy $d(x, T(x)) \geq \psi(d(x, F(T)))$, $\forall x \in M$.

Theorem 5.6. *Presume that the mapping T satisfies condition (I). Then $\{\tau_n\}$ defined by (5.1) converges strongly to a fixed point of T.*

Proof. We proved in Lemma 5.2, that

$$\lim_{n\to\infty} d(\tau_n, T\tau_n) = 0. \tag{5.13}$$

By (5.13) and applying condition (I), we get

$$0 = \lim_{n\to\infty} d(\tau_n, T\tau_n) \quad \geq \quad \lim_{n\to\infty} \psi(d(\tau_n, F(T))) \geq 0$$

hence

$$\lim_{n\to\infty} \psi(d(\tau_n, F(T))) = 0.$$

Therefore,

$$\lim_{n\to\infty} d(\tau_n, F(T)) = 0.$$

We obtain the desired result by Theorem 5.4. □

Now, we prove $\triangle$-convergence theorem for JF-iterative scheme (5.1) in CAT(0) space.

Theorem 5.7. *Let $\{\tau_n\}$ be a sequence defined by (5.1). Then the sequence $\{\tau_n\}$ $\triangle$-converges to a fixed point of T.*

Proof. From Lemma 5.2, we have $\lim_{n\to\infty} d(\tau_n, T\tau_n) = 0$. Also, $\lim_{n\to\infty} d(\tau_n, t)$ exists for all $t \in F(T)$. Thus $\{\tau_n\}$ is bounded. Let $W_\triangle(\{\tau_n\}) := \cup A(\{u_n\})$, where union is taken over all subsequence $\{u_n\}$ of $\{\tau_n\}$. In order to prove that $\triangle$-convergence of $\{\tau_n\}$ to a fixed point of T, firstly we will prove $W_\triangle(\{\tau_n\}) \subset F(T)$ and thereafter argue that $W_\triangle(\{\tau_n\})$ is singleton set. To show $W_\triangle(\{\tau_n\}) \subset F(T)$, let $\rho \in W_\triangle(\{\tau_n\})$. Then, there exists a subsequence $\{u_n\}$ of $\{\tau_n\}$ such that $A(\{u_n\}) = \{\rho\}$. By Lemma 2.1(i) and (ii) there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle$- $\lim_{n\to\infty} v_n = \varrho \in M$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$, then $\varrho \in F(T)$ by Lemma 3.8 and $d(\tau_n, \varrho)$ exists by Lemma 5.1. We claim that $\rho = \varrho$. Suppose not, by the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, \varrho) < \limsup_{n\to\infty} d(v_n, \rho) \leq \limsup_{n\to\infty} d(u_n, \rho) \quad &< \quad \limsup_{n\to\infty} d(v_n, \varrho) \\
&= \quad \limsup_{n\to\infty} d(\tau_n, \varrho) \\
&= \quad \limsup_{n\to\infty} d(v_n, \varrho)
\end{aligned}$$

a contradiction, and hence $\rho = \varrho \in F(T)$. To show that $\{\tau_n\}$ $\triangle$-converges to a fixed point of T, it suffices to show that $W_\triangle(\{\tau_n\})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{\tau_n\}$. By Lemma 2.1(i) and (ii) there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle$- $\lim_{n\to\infty} v_n = \varrho \in M$. Let $A\{u_n\} = \{\rho\}$ and $A\{\tau_n\} = \{x\}$. We have seen that $\rho = \varrho$ and $\varrho \in F(T)$. We can complete the proof by showing that $x = \varrho$. Suppose not, since $\{d(\tau_n, \varrho)\}$ is convergent, then by uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, \varrho) < \lim_{n\to\infty} d(v_n, x) \leq \limsup_{n\to\infty} d(\tau_n, x) \quad &< \quad \limsup_{n\to\infty} d(\tau_n, \varrho) \\
&= \quad \limsup_{n\to\infty} d(v_n, \varrho)
\end{aligned}$$

a contradiction, and hence the conclusion follows. □

Now, we furnish a numerical example in the support of our result.

Example 5.8. Let $M = [-1, 1] \subset \mathbb{R}$ endowed with $d(x, y) = |x - y|$. Define $T : M \to M$ by

$$T(x) = \begin{cases} -x, & \text{if } x \in [0, \frac{3}{4}) \bigcup (\frac{3}{4}, 1] = U, \\ \frac{1}{2}\sin x, & \text{if } x \in [-1, 0) = V, \\ 0, & \text{if } x = \frac{3}{4}. \end{cases}$$

We shall prove that T is a generalized contraction of Suzuki type with $a = \frac{1}{2}, b = c = \frac{1}{8}$ and $\beta \in (0, 1)$, i.e.,

$$\beta d(x, Tx) \le d(x, y) \implies d(Tx, Ty) \le M(x, y),$$

where $M(x, y) = \frac{1}{2}d(x, y) + \frac{1}{8}\big(d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)\big)$.

Verification. *Consider the following cases:*
(i) For $x, y \in U$, $M(x, y) = \frac{1}{2}\big|y - x\big| + \frac{1}{8}\big(2x + 2y + x + y + x + y\big) = \frac{1}{2}\big|y - x\big| + \frac{1}{2}(x + y) \ge |y - x| = d(Tx, Ty)$.

(ii) If $x, y \in V$, then $d(Tx, Ty) = \big|\frac{1}{2}\sin x - \frac{1}{2}\sin y\big| \le \frac{1}{2}\big|x - y\big|$.
And
$M(x, y) = \frac{1}{2}|x - y| + \frac{1}{8}\Big(|x - \frac{1}{2}\sin x| + |y - \frac{1}{2}\sin y| + |x - \frac{1}{2}\sin y| + |y - \frac{1}{2}\sin x|\Big) \ge \frac{1}{2}|x - y| = d(Tx, Ty)$.

(iii) If $x \in U$ and $y \in V$, then
$d(Tx, Ty) = |-x - \frac{1}{2}\sin y| = |x + \frac{1}{2}\sin y| \le x + \frac{|y|}{2} = x - \frac{y}{2}$.
And

$$\begin{aligned} M(x, y) &= \frac{1}{2}|x - y| + \frac{1}{8}\Big(2x + |y - \frac{1}{2}\sin y| + |x - \frac{1}{2}\sin y| + |y + x|\Big) \\ &= \frac{1}{2}\Big(x - y\Big) + \frac{1}{8}\Big(2x + \frac{1}{2}\sin y - y + x - \frac{1}{2}\sin y + |y + x|\Big) \\ &= \frac{7x}{8} - \frac{5y}{8} + \frac{1}{8}|y + x| \\ &\ge \frac{7x}{8} - \frac{5y}{8} + \frac{x + y}{8} \\ &= x - \frac{4y}{8} \\ &\ge x - \frac{y}{2} = d(Tx, Ty). \end{aligned}$$

(iv) If $x \in V, y \in U$, then $M(x, y) \ge d(Tx, Ty)$ like in (iii).

(v) If $x \in U$ and $y = \frac{3}{4}$, then

$$\begin{aligned} M(x, y) &= \frac{1}{2}\Big|x - \frac{3}{4}\Big| + \frac{1}{8}\Big(2x + \frac{3}{4} + x + \frac{3}{4} + x\Big) \\ &= \frac{1}{2}\Big|x - \frac{3}{4}\Big| + \frac{1}{2}x + \frac{3}{16}. \end{aligned}$$

And $d(Tx, Ty) = x$. Since $\frac{1}{2}d(Tx, Ty) \le d(x, y)$ we have $x \le |x - \frac{3}{4}|$, so $x \le \frac{3}{8}$. Therefore, $M(x, y) = \frac{3}{8} + \frac{3}{16} = \frac{9}{16} \ge x = d(Tx, Ty)$.

(vi) $x \in V$ and $y = \frac{3}{4}$. Then $d(Tx, Ty) = |\frac{1}{2}\sin x - 0| \le \frac{1}{2}|x| = -\frac{x}{2}$.

And

$$\begin{aligned} M(x,y) &= \frac{1}{2}\Big(\frac{3}{4}-x\Big)+\frac{1}{8}\Big(-x+\frac{1}{2}\sin x+\frac{3}{4}-x+\frac{3}{4}-\frac{1}{2}\sin x\Big) \\ &= \frac{3}{8}-\frac{1}{2}x-\frac{x}{4}+\frac{3}{4} \\ &= \frac{-2x-x}{4}+\frac{15}{8} \\ &= \frac{-3x}{4}+\frac{15}{8}\geq\frac{-x}{2}=d(Tx,Ty). \end{aligned}$$

(vii) If $x=\frac{3}{4}$ and $y\in U$. Then

$$\begin{aligned} M(x,y) &= \frac{1}{2}\Big|\frac{3}{4}-y\Big|+\frac{1}{8}\Big(\frac{3}{4}+2y+y+\frac{3}{4}+y\Big) \\ &= \frac{1}{2}\Big|\frac{3}{4}-y\Big|+\frac{y}{2}+\frac{3}{16} \end{aligned}$$

and $d(Tx,Ty)=y$. By $\frac{1}{2}d(x,Ty)\leq d(x,y)$, we have $\frac{3}{8}\leq|\frac{3}{4}-y|$, so $y\leq\frac{3}{8}$. Therefore, $M(x,y)=\frac{9}{16}\geq y=d(Tx,Ty)$.

(viii) If $x=\frac{3}{4}$ and $y\in V$, then

$$\begin{aligned} M(x,y) &= \frac{1}{2}\Big(\frac{3}{4}-y\Big)+\frac{1}{8}\Big(\frac{3}{4}-y+\frac{1}{2}\sin y+\frac{3}{4}-\frac{1}{2}\sin y-y\Big) \\ &= \frac{3}{8}-\frac{y}{2}+\frac{3}{16}-\frac{y}{4} \\ &= \frac{9}{16}-\frac{3y}{4}\geq\frac{9}{16}, \end{aligned}$$

and $d(Tx,Ty)=|0-\frac{1}{2}\sin y|=\frac{1}{2}\sin y\leq\frac{1}{2}y\leq\frac{9}{16}\leq M(x,y)$.

(ix) If $x=y$, then verification is obvious.

It can be easily seen in Figure 1 and Table 1 that JF iterative scheme converges faster than leading iterative schemes to the fixed point $t=0$ with control sequences $\mu_n=0.55$, $\theta_n=0.65$, $\eta_n=0.22$ and initial guess $\tau_0=0.5$, $n\in\mathbb{Z}_+$.

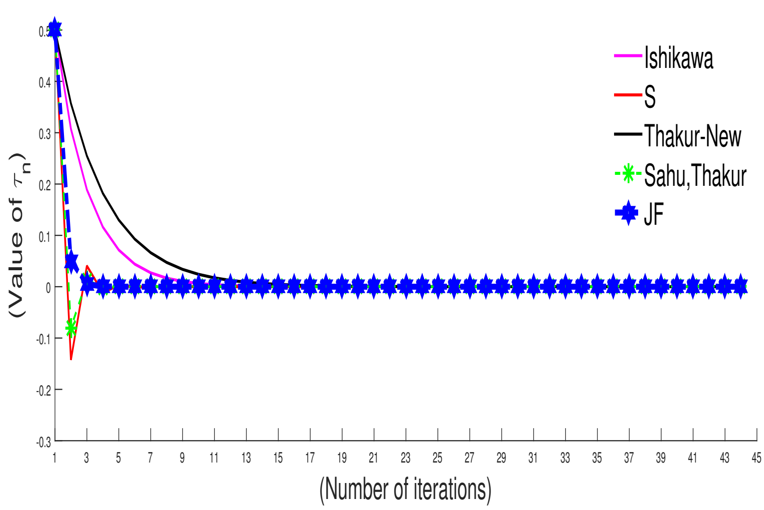

Figure 1. Graphical representation of iterative schemes.

Iter.	Ishikawa	S	Thakur-New	Sahu, Thakur	JF
1	0.500000	0.500000	0.500000	0.500000	0.500000
2	0.307500	-0.142500	0.357000	-0.079800	0.050000
⋮	⋮	⋮	⋮	⋮	⋮
7	0.027053	0.000268	0.066246	0.000008	0.000001
8	0.016638	0.000076	0.047300	0.000001	0.000000
9	0.010232	0.000022	0.033772	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮
12	0.002380	0.000001	0.012293	0.000000	0.000000
13	0.001464	0.000000	0.008777	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮
29	0.000001	0.000000	0.000040	0.000000	0.000000
30	0.000000	0.000000	0.000029	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮
42	0.000000	0.000000	0.000001	0.000000	0.000000
43	0.000000	0.000000	0.000000	0.000000	0.000000

Table 1. Computational table of the rate of convergence.

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Author information

Javid Ali, Department of Mathematics, Aligarh Muslim University, Aligarh, India.
E-mail: `javid.mm@amu.ac.in`

Mohd Jubair, Department of Mathematics, Aligarh Muslim University, Aligarh, India.
E-mail: `jubairnnn@gmail.com`

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